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ASYMPTOTIC BEHAVIOUR OF THE SOLUTIONS OF SCHRÖDINGER EQUATION WITH IMPULSE EFFECT IN A BANACH SPACE

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1 Introduction

Differential equations with impulse effect are the object of a new trend in the theory of ordinary differential equations. The first contributions in this respect is the paper by V.D. Milman and A.D. Myshkis [1]. In spite of their numerous publications in science and technology, the matematical theory of equations with impulse effect has been advancing rather slowly because considerable difficulties are involved in their research, mainly due to their specific features. We will note that in [2,3] the first results concerning linear diffe rential eqquations with impulse effect in a Banach space were obtained.

In the paper [4] has been used from the impulsive moving mirror model the Schrödinger equation with impulse effect

$$H_{n+1}(t)\Psi[\{a_\omega(t_n)\}, t] = d/dt\Psi[\{a_\omega(t_n)\}, t], \; t_n < t \leq t_{n+1}, \qquad (1)$$

$n = 0, 1, ...,$ with the free Hamiltonian:

$$H_{n+1} = -i1/2 \int_0^\infty d\omega\{-\delta^2/\delta a_\omega^2(t_n) + (m^2 + \omega^2(1 - \dot{x}_n^2))a_\omega^2(t_n)\}. \qquad (2)$$

$t_n < t \leq t_{n+1}, \; n = 0, 1, ...$ and the impulsive operator:

$$F_n = 1 - \dot{x}_n t_n \int_0^\infty d\omega \int_0^\infty d\omega' F(\omega, \omega') a_{\omega'}(t_n) \delta/\delta a_\omega(t_n), \; t = t_n, \qquad (3)$$

$n = 0, 1, ...$ with

$$\Psi_n^+[\{a_\omega^+(t_n + 0)\}, t_n + 0] = F_n\Psi[\{a_\omega(t_n)\}, t_n], \; t = t_n, \; n = 0, 1, ... \qquad (4)$$

So in this paper we have extended the covariant functional Schrödinger formalism to the treatment of non-separable spacetime foliations. As a special example of this class of system we discussed in some detail Minkowski space field theory with time dependent boundary conditions, i.e., those of impulsive moving mirror.

In these problems the fundamental degrees of freedom of the Schroedinger picture necessarilly acquire an implicit time dependence and we simplify the problem if we consider a mirror trajectory assumed that this change by jumps.

The aim of the present paper is to study the solution of a linear homogeneous Schröodinger equation with impulse effect at fixed moments and to establish a dependence between the asymptotic behaviour of its solutions and the spectrum of the linear Hamiltonian operator with impulse effet.

2 Problem statement

Consider the linear Schródinger equation with impulse effect

$$\begin{aligned} d/dt\Psi[\{a_\omega(t_n)\},t] &= H_{n+1}\Psi[\{a_\omega(t_n)\},t],\ t_n < t \leq t_{n+1}, \quad (5)\\ \Psi_n^+[\{a_\omega^+(t_n+0)\},t_n+0] &= F_n\Psi[\{a_\omega(t_n)\},t_n],\ t = t_n,\ n = 0,1,...(6) \end{aligned}$$

where the Hamiltonian H is a linear continuous operator with spectrum Sp(H), mapping the complex Banach space B into itself, the continuous mappings F_n, (n = 0,1,...) map B into B, while

$0 < t_1 < t_2 < ...,$

$\lim t_n = \infty, n \to \infty.$

Definition 1. The piecewise continuous function with first order discontinuities for $t = t_n$ and such that

$d/dt\Psi = H_{n+1}\Psi(t)$

for $t \neq t_n$, while for $t = t_n$

$\Psi_n^+(t_n+0) = F_n\Psi(t_n) = F_n\Psi_n$ will be called a solution of the equation with impulse effect (5),(6).

$\Psi(t)$ will be assumed to be right continuous at the discontinuity points t_n.

Consider the Cauchy problem with initial condition

$$\Psi(\{a_\omega(t_0)\},t_0) = \Psi_0,\ 0 < t_0 < t_1, \qquad (7)$$

where for the free Hamiltonian we have from [4] for $t = t_0$ the solution

$$\Psi(\{a_\omega(t_0)\},t_0) = \Psi_0(\{a_\omega\}) =$$
$$\exp(-1/2\int_0^\infty d\omega\sqrt{(}m^2+\omega^2)(a_\omega(t_0))^2 - iE_0t_0).$$

with E_0 the ground state energy, for the equation with impulse effect (5),(6). Its solution is given by the formula:

$$\Psi(t) = e^{H_{n+1}(t-t_n)}F_n e^{H_n(t_n-t_{n-1})}...F_1e^{H_1(t_1-t_0)}F_0\Psi_0, \qquad (8)$$

for $t_n < t \leq t_{n+1}$, n = 0,1,...

3 Main results

We will find sufficient conditions for boundedness of the Cauchy problem with initial condition (7) for the Schrödinger equation with inpulse effect (5),(6) for $t > t_0$.

We will say that conditions (A) hold provided the following conditions are fulfilled:

A_1.

$$\| e^{Ht}\| \leq e^{\gamma t}$$

for $t > t_0$

where γ is a const.

A_2.

$$\| F_n\Psi\| \leq q_n \| \Psi\| + h_n$$

for $n = 1, 2, ...,$ and $\Psi \in B$, and

$$q_1q_2...q_{n(t)} \leq Le^{\delta t}$$

where $n(t) = n$ for $t_n < t \leq t_{n+1}$, while L and δ are constants.

Let conditions (A) hold. We will estimate the solution of the Cauchy problem with the initial condition (7) for the equation with impulse effect (5),(6):

$$\begin{aligned} \| \Psi(t)\| &\leq e^{\gamma(t-t_n)}\{q_n \| e^{H_n(t-t_{n-1})}F_{n-1}e^{H_{n-1}(t_{n-1}-t_{n-2})}F_{n-2}\cdots \\ &\quad F_1e^{H_1(t_1-t_0)}\Psi_0\| + h_n\} \\ &\leq e^{\gamma(t-t_{n-1})}q_n\{q_{n-1} \| e^{H_{n-1}(t_{n-1}-t_{n-2})}F_{n-2}\cdots \\ &\quad F_1e^{H_1(t_1-t_0)}\Psi_0\| + h_{n-1}\} + e^{\gamma(t-t_n)}h_n \\ &\leq e^{\gamma(t-t_{n-2})}q_nq_{n-1} \| F_{n-2}...F_1e^{H_1(t_1-t_0)}\Psi_0\| + \\ &\quad e^{\gamma(t-t_{n-1})}q_nh_{n-1} + e^{\gamma(t-t_n)}h_n \\ &\leq \ldots \qquad (9)\\ &\leq e^{\gamma(t-t_0)}q_nq_{n-1}...q_2q_1 \| \Psi_0\| + \{e^{\gamma(t-t_n)}h_n + e^{\gamma(t-t_{n-1})}q_nh_{n-1} + \\ &\quad e^{\gamma(t-t_{n-2})}q_nq_{n-1}h_{n-2} + ... + e^{\gamma(t-t_0)}q_nq_{n-1}...q_2h_1\} \end{aligned}$$

The estimate made allows to consider partial cases where the solutions of the Cauchy problem with initial conditions (7) for the Schrödinger equation with impulse effect (5),(6), are bounded. Here we will mention two of them.

Theorem 1. Let conditions (A) hold. Moreover, let $h_1 = h_2 = ... = 0$. Then for $\delta + \gamma = 0$ the solution $\Psi[\{a_\omega^+(t_0)\}, t], (t \geq t_0)$ is bounded, and for $\delta + \gamma < 0$, $\lim_{t \to \infty} \Psi(t) = 0$.

The proof of the theorem is implied immediately by (9).

Theorem 2. Let condition (A) be fulfilled. Let

$$q = q_1 = q_2 = ...; h = h_1 = h_2 = ...$$

and let

$$t_{2n-1} = t_0 + f(n, x_0)\kappa (n = 1, 2, ...).$$

From the paper [5] we have

$$t_{2n-1} = \cosh ns.t_0 - \sinh ns.x_0$$

$$t_{2n} = \cosh ns.t_o + \sinh ns.x_0$$

and then $t_{2n-1} < t \leq t_{2n}$, $t = t_0 + u(\tau, x_0)\kappa$.

where $n \leq \tau \leq n + 1$, while κ is a constant. Besides, let $qe^{\gamma\kappa} < 1$. Then for $\delta + \gamma \leq 0$ the solution $\Psi(t)$, $(t \geq t_0)$ is bounded.

Proof. By M denote the expression in braces in the right hand side in (9) and, having made certain transformations by $f(n, x_0) \approx n$ and $u(\tau, x_0) \approx \tau$,(from $\cosh(ns/2) = 1 + n^2/2(v^2/c^2) + ...$, $\sinh(ns/2) = nv/2 + (n/3 + n^3/6).v^3/c^3 + ...$), we obtain

$$M = e^{\gamma(\tau - n)\kappa}\{[q^n e^{n\gamma\kappa} - 1]/[qe^{\gamma\kappa} - 1]\}h.$$

Under the assumption made, M, is bounded. This fact and condition A_2 imply the assertion of Theorem 2.

Theorem 3. Let the following conditions be fulfilled:

1.

Positive constants $q_1, q_2, ...$ exist such that

$$\| F_n\Psi\| \geq q_n \| \Psi\| \text{ for } n = 1, 2, ..., \Psi \in B.$$

2. $$e^{H_{n+1}\tau}F_n = F_n e^{H_{n+1}\tau}, \text{ for } n = 1, 2, ..., \tau > 0 \text{ and}$$
$\| e^{(H_{n-1} - H_n)t_{n-1}}\| \geq e^{\gamma_{n-1}t_{n-1}}$ where γ_n, $(n = 1, ...)$ are a constants

3.

For all $\Psi_0 \in B$ and $t \geq t_0 = 0$ the solution of the Cauchy problem with initial condition (3) for the Schrödinger equation with impulse effect (5),(6), are bounded, i.e. for $t \geq 0$, $\| \Psi(t)\| \leq C(\Psi_0)$.

Then the spectrum Sp(H) of the operator H lies in the halfplane $\mathcal{R}\lambda \leq \mathcal{Q}$ where

$$\mathcal{Q} = \text{-}\lim_{t \to \infty} \inf 1/t \sum_{n=1}^{n(t)} \gamma_n t_n \ln q_n.$$

Proof. For k = 1,2,... introduce the notation $\Psi_k^+ = F_k\Psi(t_k)$. Let $t \in [t_n, t_{n+1}]$. For $\Psi_0 \in B$,

$$\begin{aligned} C(\Psi_0^+) &\geq \| e^{H_n(t - t_{n-1})}\Psi_{n-1}^+\| \\ &= \| e^{H_n(t - t_{n-1})}F_{n-1}e^{H_{n-1}(t_{n-1} - t_{n-2})}F_{n-2}\Psi_{n-2}\| \\ &= \| e^{H_n(t - t_{n-1})}e^{H_{n-1}(t_{n-1} - t_{n-2})}F_{n-1}\Psi_{n-2}^+\| \\ &= \| e^{H_n t + (H_{n-1} - H_n)t_{n-1} - H_{n-1}t_{n-2}}F_{n-1}\Psi_{n-2}^+\| \\ &= ... \\ &= \| e^{H_n t + (H_{n-1} - H_n)t_{n-1} + (H_{n-2} - H_{n-1})t_{n-2}} ... \\ &\quad e^{(H_1 - H_2)t_1 + (H_0 - H_1)t_0 - H_0 t_0}F_{n-1}F_{n-2}...F_1\Psi_0\| \\ &\geq e^{(\gamma_{n-1}t_{n-1} + ... + \gamma_0 t_0)} \| F_{n-1}F_{n-2}....F_1 e^{H_n t}\Psi_0\| \\ &\geq e^{(\gamma_{n-1}t_{n-1} + ... + \gamma_0 t_0)}q_{n-1}q_{n-2}...q_1 \| e^{H_n t}\Psi_0\| \end{aligned} \quad (10)$$

(5)yelds

$$\| e^{H_n t}\Psi\| \leq C(\Psi_0)/[e^{(\gamma_{n-1} + ... + \gamma_0 t_0)}q_1...q_{n-1}]$$

whence it follows that for $\epsilon > 0$ and for sufficiently large values of t,

$$\| e^{H_n t}\Psi_0\| \leq C(\Psi_0)e^{(\mathcal{Q} + \epsilon)t}$$

Hence, for $\lambda \in Sp(H)$, the inequality $\mathcal{R} \leq \mathcal{Q} + \epsilon$ holds, i.e. $\mathcal{R}\lambda \leq \mathcal{Q}$.

Thus, Theorem 3 is proved.

Remark 1. If the operators F_n are linears surjections then the condition 1 of Theorem 3 is equivalent to the condition for the operators F_n to have inverse ones. In this case for $n = 1, 2, ...$, $\| F_n^{-1}\| \leq q_n^{-1}$.

Theorem 4. Let the following conditions to be fulfilled:

1. $\| F_n\Psi\| \geq q_n \| \Psi\|$, for $n = 1, \ldots$,

where the numbers q_n satisfy the inequalities

$$q_1, q_2 \ldots q_m \geq \alpha,$$

for $m = 1, 2, \ldots$, and $\alpha > 0$ is a constant.

2. $F_n e^{H\tau} = e^{H\tau} F_n$, for $n = 1, 2, \ldots, \tau \in (0, \infty)$.

3. For all $\Psi_0 \in B$ and $t \in (0, \infty)$, the soulution of the Cauchy problem with initial condition (3) for the equation with impulse effect (5),(6), are bounded,i.e.$\| \Psi(t)\| \leq C(\Psi_0)$.

Then Sp(H) lies on the imaginery axis.

Proof. With arguments analogous to those employed in the proof of Theorem 3, we find the estemate

$$\| e^{H_n t}\Psi_0\| \leq C(\Psi)/\alpha \text{ for } t \in (0, \infty).$$

Hence a constant K exists for which the inequality

$$\| e^{Ht}\| \leq K \, for \, t \in (0, \infty) \qquad (11)$$

holds. (11) yields that Sp(H) lies on the imaginary axis.

This completes the proof of Theorem 4.

Remark 2. The assertion of Theorem 4 still remains true if in condition 3 the requirement $\Psi_0 \in B$ is replaced by the condition $\Psi_0 \in \delta\Omega$, where $\delta\Omega$ is the boundary of a bounded subset Ω of B with nonempty interior.

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