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IMPULSIVE MOVING MIRROR MODEL AND THE STABILITY OF LINEAR HOMOGENEOUS DIFFERENTIAL EQUATIONS WITH IMPULSE EFFECT IN A BANACH SPACE

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1 Introduction

The equations with impulse effect are models of real processes which are sub ject to disturbances in its evolution acting in time very short compared with the entire duration of the process.

The first paper in this direction is the paper by Milman and Myshkis [1](1960). The mathematical theory of ordinary differential equations with impulse effect has been further developed in the papers [2] - [4]. The paper [5] initiates the investigations of the equations with impulse effect in a Banach space: The first application of these equations in quantum physics is given in the papers [6] and [7].

The advantage of the local field theoretical approach for studing the properties of the vacuum is that it permits a different point of view and a deeper understanding of the nature of vacuum energy. With the application of the equations with impulse effect in the impulsive moving mirror model in Minkowski space has been given another point of view in the field theoretical approach for studing the properties of the vacuum.

2 Statement of the problem

Consider the differential equation with impulse effect of the form

$$d/dt X(t) = V(t).X(t)\,|t \neq t_n\,(n = 0, 1, \ldots) \quad (1)$$

$$X(t_n + 0) = F_n X(t_n),\ (n = 1, 2, \ldots) \quad (2)$$

where

$$V(t) = \begin{pmatrix} 0 & k(t) \\ v(t) & 0 \end{pmatrix}$$

with $v(t) = -V_{\omega\omega'}(t) - \dot{F}_{\omega\omega'}(t)$ or for antisymmetric $F_{\omega\omega'}$ is $v(t) = -V_{\omega\omega'}(t)$ and $k(t) = T^{-1}_{\omega\omega'}(t)$. For the impulse operator is

$$F_n = \begin{pmatrix} 1 & 0 \\ -F_{\omega\omega'}(t_n) & 1 \end{pmatrix}$$

and for

$$X(t) = \begin{pmatrix} A_\omega(t) \\ P_\omega(t) \end{pmatrix}$$

with $P_\omega(t) = T_{\omega\omega'}\dot{A}_{\omega'}(t)$ (see the paper [6]).

The operator $V(t)$, for $(t > t_0)$ is a continuous operator-function on t, the values of which are linear bounded operators mapping the complex Banach space B into itself. The operators $F_n : B \to B$, for $(n = 0, 1, \ldots)$ are continuous and the moments t_n, for $(n = 0, 1, \ldots)$ of the impulse effect satisfy the condition $0 < t_0 < t_1 < \ldots$. and $\lim_{n\to\infty} t_n = \infty$.

Definition 1. We say that $X(t)$ is a solution of the equation with impulse effect (1),(2) if X(t) is a piecewise right continuous function with first order discontinuities for $t = t_n$, $(n = 0, 1, \ldots)$, and such that

$$d/dt X(t) = V(t)X(t)$$

for $t \neq t_n$, $(t > t_0)$, $(n = 1, 2, \ldots)$ and

$$X(t_n + 0) = F_n X(t_n), \text{ for } t = t_n, \ (n = 1, 2, \ldots)$$

For $X_0 \in B$ and $t \geq t_0$ the equation with impulse effect (1), (2) has unique solution $X(t)$ which satisfies the condition

$$X(t_0) = X_0 \tag{3}$$

Then we can consider the family of operators W(t) defined by the formula

$$W(t)X_0 = X(t), \ (t_0 < t < \infty).$$

Denote by $U(t,s)$ the evolution operator of the equation

$$d/dt X(t) = V(t)X(t). \tag{4}$$

We note that for $t_n < t \leq t_{n+1}$, $(n = 0, 1, \ldots)$ the following equality holds

$$W(t) = U(t, t_n)F_n U(t_n, t_{n-1})F_{n-1} \ldots U(t_1, t_0).$$

Definition 2. The equation with impulse effect (1), (2) is called stable if

$$\sup_{t_0 \leq t < \infty} \| W(t) \| < \infty.$$

Definition 3. The equation with impulse effect (1), (2) is called asymptotically stable if

$$\lim_{t\to\infty} \| W(t)X \| = 0, \ (X \in B).$$

Definition 4. The equation with impulse effect (1), (2) is called uniformly stable if

$$\lim_{t\to\infty} \| W(t) \| = 0.$$

Lemma 1. Let the following condition hold

$$V(t) \int_{t_0}^{t} dsV(s) = \int_{t_0}^{t} dsV(s) \, . \, V(t) \ (t \geq t_0). \tag{5}$$

Then the solution X(t) of (4) for $t \geq t_0$ with initial condition has the form

$$X(t) = e^{\int_{t_0}^{t} dsV(s)} X_0.$$

The proof of Lemma 1. is standard.

Remark 1. For small $t \geq t_0$ the condition is sufficient as well for the solution of (4) to have the form

$$X(t) = e^{\int_{t_0}^{t} dsV(s)} X_0.$$

In fact, let $X(t) = e^{\int_{t_0}^{t} dsV(s)} X_0$. The equality

$$(e^{\int_{t_0}^{t} dsV(s)})' = V(t)e^{\int_{t_0}^{t} dsV(s)}$$

can be represented as

$\sum_{n=0}^{\infty}([(n+1)!]^{-1}((\int_{t_0}^{t} dsV(s))^{n+1})' - (n!)^{-1}V(s)(\int_{t_0}^{t} dsV(s))^n) = 0.$

We set $\Delta(t) = V(t)\int_{t_0}^{t} dsV(s) - \int_{t_0}^{t} dsV(s). V(t)$.

We can deform $V^{tr}(t).C. \int_{t_0}^{t} dsV(s) - \int_{t_0}^{t} dsV^{tr}(s).C.V(t) = \Delta^C(t)$ where $C = \begin{pmatrix} 0 & -q^{-1/2} \\ q^{1/2} & 0 \end{pmatrix}$ and $\Delta^C(t) = 0$ if the following conditions hold

$$k(t) \int_{t_0}^{t} dsv(s) = \int_{t_0}^{t} dsv(s). k(t) \tag{6}$$

$$v(t) \int_{t_0}^{t} dsk(s) = \int_{t_0}^{t} dsk(s). v(t) \tag{7}$$

Then the following equalities hold:

$$\sum_{n=0}^{\infty}([(n+1)!]^{-1}(\sum_{k=0}^{n}(\int_{t_0}^{t} dsV(s))^k V(t)(\int_{t_0}^{t} dsV(s))^{n-k} -$$
$$(n+1)V(t)(\int_{t_0}^{t} dsV(s))^n)) =$$
$$= \sum_{n=0}^{\infty}[(n+1)!]^{-1}\sum_{k=1}^{n}((\int_{t_0}^{t} dsV(s))^k V(t)(\int_{t_0}^{t} dsV(s))^{n-k} -$$
$$- V(t)(\int_{t_0}^{t} dsV(s))^n) = \sum_{n=0}^{\infty}[(n+1)!]^{-1}\sum_{k=1}^{n} k(\int_{t_0}^{t} dsV(s))^{n-k}\Delta(t).$$
$$(\int_{t_0}^{t} dsV(s))^{k-1} = 0, \text{i.e.}$$

$$1/2\Delta^C(t) = -\sum_{n=2}^{\infty}[(n+1)!]^{-1}\sum_{k=1}^{n} k(\int_{t_0}^{t} dsV(s))^{n-k}\Delta^C(t)(\int_{t_0}^{t} dsV(s))^{k-1}.$$

Finally we get $1/2 \parallel \Delta^C(t)\| \leq \sum_{n=2}^{\infty}[(n+1)!]^{-1} \parallel \Delta^C(t)\| k(k+1)/2$ $\parallel \int_{t_0}^{t} dsV(s)\|^{n-1} = \parallel \Delta^C(t)\|/2(e^{\|\int_{t_0}^{t} dsV(s)\|} - 1)$.

Then for sufficiently small t

$$|e^{\|\int_{\tau}^{t} dsV(s)\|} - 1| < 1$$

and there exists an interval $[t_0, t^*]$ such that $\Delta^C(t) = 0$. ($t \in [t_0, t^*]$).

Applying the step method we come to the conclusion that the equality $\Delta^C(t) = 0$ holds as well if t belongs to an interval containing $[t_0, t^*]$. So the equations (6) and (7) holds.

3 Main result

We say that the condition (A) holds if the following condition is fulfilled:

(A) $\int_{\tau}^{t} dsV(s).C.V^{tr}(t) = V^{tr}(t).C.\int_{\tau}^{t} dsV(s)$, $(t_0 < \tau < t < \infty)$. with

$$V^{tr}(t) = \begin{pmatrix} 0 & -v(t) \\ k(t) & 0 \end{pmatrix}$$

and $\int_{\tau}^{t} dsk(s).v(t) = v(t).\int_{\tau}^{t} dsk(s)$; $\int_{\tau}^{t} dsv(s).k(t) = k(t)\int_{\tau}^{t} dsv(s)$

Remark 2. Let the condition (A) hold. Then for $t_0 < \tau < t < \infty$ the following equality holds

$$V^{tr}(t).C.V(\tau) = V(\tau).C.V^{tr}(t) \text{ with}$$
$$k(t)v(\tau) = v(\tau)k(t);\ v(t)k(\tau) = k(\tau)v(t)$$

We say that conditions (B) are satisfied if:

B1: Condition (A) holds.

B2: There exists a constant $T > 0$ for which $t_{n+1} - t_n \leq T$ for n = 1,2,...

B3: $e^{\|\int_{\tau}^{t} dsV(s)\|} \leq e^{\gamma(t-\tau)}$ for $0 \leq t - \tau \leq T$ where γ is a constant.

B4. $\parallel F_n X\| \leq q_n \parallel X\| + h_n$ for n = 1,2,... and $X \in B$ so that $q_1 q_2 ... q_{n(t)} \leq Le^{\delta.t}$ where $n(t) = n$ for $t_n < t \leq t_{n+1}$, while L and δ are constants.

If the conditions (B) hold, for the solution X(t) of the equation with impulse effect (1),(2) with the initial condition (3) we obtain the following estimation

$$\begin{aligned}
\parallel X(t)\| &\leq e^{\gamma(t-t_n)}\{q_n e^{\int_{t_{n-1}}^{t_n} dsV(s)} F_{n-1} e^{\int_{t_{n-2}}^{t_{n-1}} dsV(s)} F_{n-2}...F_1 e^{\int_{t_0}^{t_1} dsV(s)} X_0\| + h_n\} \\
&\leq e^{\gamma(t-t_{n-1})} q_n\{q_{n-1} \parallel e^{\int_{t_{n-2}}^{t_{n-1}} dsV(s)} F_{n-2}...F_1 e^{\int_{t_0}^{t_1} dsV(s)} X_0\| + h_{n-1}\} + \\
&\quad + e^{\gamma(t-t_n)} h_n \leq \\
&\leq e^{(t-t_{n-2})} q_n q_{n-1} \parallel F_{n-2}...F_1 e^{\int_{t_0}^{t_1} dsV(s)} X_0\| + e^{\gamma(t-t_{n-1})} q_n h_{n-1} + \\
&\quad + e^{\gamma(t-t_n)} h_n \leq \\
&\leq ... \leq \\
&\leq e^{\gamma(t-t_0)} q_n q_{n-1}...q_1 \parallel X_0\| + \{e^{\gamma(t-t_n)} h_n + e^{\gamma(t-t_{n-1})} q_n \\
&\quad h_{n-1} + e^{\gamma(t-t_{n-2})} q_n q_{n-1} h_{n-2} + ... + e^{\gamma(t-t_1)} q_n q_{n-1}...q_2 h_1\}. \qquad (8)
\end{aligned}$$

Using this estimation we will find sufficient conditions under which the solution of the equation with the impulse effect are bounded.

Theorem 1. Let the conditions (B) hold. Let $h_1 = h_2 = ... = 0$.

Then for $\delta + \gamma < 0$, $\lim_{t\to\infty} X(t) = 0$ and for $\delta + \gamma = 0$ the solution X(t) is bounded for $t \geq t_0$.

The proof of the theorem 1 follows immediately from the estimation (8).

Theorem 2. Let the condition (B) be satisfied. Suppose additionally that

$$q = q_1 = q_2 = ...; h = h_1 = h_2 = ...$$

and let

$$t_n = t_0 + nk (n = 1, 2, ...), t = t_0 + \tau k,$$

where $n \leq \tau \leq n+1$, $k \leq T$ is a constant and the inequality

$$qe^{\gamma k} < 1$$

holds. Then for $\delta+\gamma \leq 0$ the solution X(t) is bounded on the half axis $t \geq t_0$.

Proof. We can denote by M the expression in braces in the right hand side in (8) and, having made certain transformations for nonrelativistic case, we obtain

$$M = e^{\gamma(\tau-n)k}(qe^{\gamma k}-1)^{-1}.(q^n e^{\gamma kn}-1)h.$$

Under the assumption made, M is bounded. This fact and the condition A_2 imply the assertion of the Theorem 2.

Theorem 3. Suppose that the following conditions hold:

1. The condition (A) is fulfilled.

2.There exist constants $p \geq 0$ such that $0 \leq i(t_0, t) - p(t-t_0) \leq \sigma$ for $0 \leq t_0 \leq t < \infty$ where i(a, b) is the number of points t_n which lie in the interval $(a, b)\,(t_0 < a < b < \infty)$.

3. $F = F_1 = F_2 = \ldots$ where the operator F is linear and commutes with $\int_\tau^t dsV(s)$ for $t_0 \leq \tau \leq t < \infty$.

4. The operator F has a logarithm, i.e. there exists a linear and bounded operator $\ln F : B \to B$ such that $F = e^{\ln F}$.

Then the equation with impulse effect is bounded, asymptotycally bounded, uniformly asymptotically bounded if and only if respectively

$$\text{a) } \sup_{t\geq t_0} \| e^{\int_{t_0}^t ds(V(s)+p\ln F)} \| < \infty,$$

$$\text{b) } \lim_{t\to\infty} \| e^{\int_{t_0}^t ds(V(s)+\ln F)} X \| = 0 \; (X \in B),$$

$$\text{c) } \lim_{t\to\infty} \| e^{\int_{t_0}^t ds(V(s)+\ln F)} \| = 0.$$

Proof. The operator-function $W(t)$ has the form

$$W(t) = e^{\int_{t_0}^t dsV(s)} F^{i(t_0,t)} = e^{\int_{t_0}^t dsV(s)} F^{[p(t-t_0)]} F^{i(t_0-t)-[p(t-t_0)]} =$$

$$= e^{\int_{t_0}^t dsV(s)} e^{\ln F[p(t-t_0)]} F^{i(t_0,t)-[p(t-t_0)]} = \qquad (9)$$

$$= e^{\int_{t_0}^t dsV(s)} e^{\ln Fp(t-t_0)} e^{\ln F[p(t-t_0)] - \ln Fp(t-t_0)} F^{i(t_0,t)-[p(t-t_0)]} =$$

$$= e^{\int_{t_0}^t ds(V(s)+\ln Fp)} e^{\ln F\{[p(t-t_0)]-p(t-t_0)\}} F^{i(t_0,t)-[p(t-t_0)]}.$$

Taking into account the represontation (9) the assertions a), b) and c) follows immediately.

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