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RENORMALIZATIONS IN SUPERSYMMETRIC AND NONSUPERSYMMETRIC NON-ABELIAN CHERN-SIMONS FIELD THEORIES WITH MATTER

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1 Introduction

Recently, there has been a considerable interest in Chern–Simons gauge field theories in (2+1) dimensions. In contrast to the usual Yang–Mills gauge theories, they have some remarkable features and are intimately connected with three dimensions, owing to the structure of the kinetic term which contains the $\epsilon_{\mu\nu\rho}$ tensor. The main motivation for studying these theories, apart from general interest in a new class of gauge theories, was their resemblance to the topological field theory [1] and the high-temperature limit of four-dimensional field theories [2]. Among the possible physical applications of abelian models known so far, it appears to be the quantum Hall effect [3] and the high-T_c superconductivity [4]. At the same time, if one writes down the well-known θ term in a usual gauge theory in four dimensions

$$\mathrm{tr}(F_{\mu\nu}\tilde{F}_{\mu\nu}) = \ \mathrm{tr}\ \partial_\mu K_\mu,$$

where $K_\mu = \epsilon_{\mu\nu\rho\sigma}(A_\nu\partial_\rho A_\sigma + \frac{2}{3}\ g\ A_\nu A_\rho A_\sigma)$, one finds that the total derivative, after integration by parts in the presence of a non-vanishing boundary, gives us the Chern–Simons lagrangian. Thus, Chern–Simons field theories may arise from the models with non-trivial boundary conditions, which also reflects their topological nature [5].

In a recent paper [6], we have considered the renormalizations in abelian Chern–Simons theories with matter and found out that they have remarkable properties like the existence of various renormalization-group (RG) fixed points, supersymmetry and extended supersymmetry. The presence of a fermion-number violating interaction, specific of three dimensions, leads to an interesting opportunity to get fermion condensates, which may have an interpretation in the context of the high-T_c superconductivity [7].

In what follows we show that non-abelian Chern–Simons theories with matter exhibit the same properties. The gauge coupling is not infinitely renormalized, being protected by the gauge invariance [8], while the matter couplings have a complicated RG flow. The requirement of the closed renormalizability entails the fact that the number of different matter-interaction terms in the lagrangian, mutually generating one another in the course of ultraviolet (UV) renormalizations, depends on the gauge group and matter-field representations. For higher representations, rather many counterterms may arise. If we limit ourselves to a minimum number of matter-field couplings, only a few possibilities are left. The same is likewise true for the supersymmetric theories, where we present some general formulas, valid for $SU(n)$, $SO(n)$ and $Sp(n)$ gauge groups. Like in the abelian case, owing to the odd number of dimensions, there are no genuine UV divergencies in the one-loop approximation as well as for any odd number of loops. The first non-trivial contribution comes from the two-loop diagrams.

The paper is organized as follows. In sect.2 we consider the supersymmetric theories. We start with the N=2 supersymmetry and then, rewriting the model in terms of the N=1 superfields, let the couplings to be independent. All the two-loop RG functions are calculated without specifying a certain matter-field representation. Particular cases of $U(1)$, $SU(n)$, $Sp(n)$, and $SO(n)$ are discussed in more detail. It is shown that the N=2 supersymmetry manifests itself as an infrared (IR) stable fixed point or a saddle-point solution. Sect.3 contains the component-field formulation of the model. We allow a further freedom for the choice of the couplings and concentrate on the fundamental representation of $SU(n)$ for the matter multiplets. Again, all the two-loop calculations are performed and the RG fixed points are found. Their stability properties are examined, and the RG flow is studied. Sect.4 contains the discussion of a possible application of the fixed-point analysis to critical phenomena, and our conclusions.

2 Supersymmetric non-abelian Chern–Simons field theories with matter

Recently, a completely covariant N=2 supersymmetric action in (2+1) dimensions has been constructed for the abelian and non-abelian Chern–Simons superfields interacting with chiral matter superfields [9]. The pure non-abelian part of the action can be written as

$$\mathcal{S} = \int d^3x \, d^4\theta \left[\mathcal{L}_{CS}^{N=2}(V) + \overline{\Psi} \exp(-gV)\Psi \right], \tag{1}$$

where the self-interaction of the general N=2 Chern–Simons superfield is an integral over an additional numeric parameter t

$$\mathcal{L}_{CS}^{N=2}(V) \sim \int_0^1 dt \;\; \mathrm{tr}\Big(\overline{D}^{\alpha} \{\exp[gV(t)]D_{\alpha} \exp[-gV(t)]\} \;\; \exp[gV(t)]\partial_t \exp[-gV(t)] \Big)$$

with the boundary conditions $V(0)$=0, $V(1)$=V. In spite of this integration, the action leads to local superfield equations of motion, like it occurs in the Wess–Zumino–Novikov–Witten σ model.

Retaining only one manifest supersymmetry (fixing a Wess–Zumino-type gauge and eliminating auxiliary fields of the second supersymmetry), one can derive a local action in terms of N=1 superfields [9]. Then, besides the self-interaction of the real (Majorana) spinor-connection Chern–Simons gauge superfield Γ_{α}^{a} and its minimal coupling to the complex scalar matter superfield Φ_j, there appears a matter self-interaction term $(\overline{\Phi}T^a\Phi)^2$ with the coefficient proportional to the square of the gauge charge g. If quantum corrections do not break this correlation between the couplings, we expect the model to have no UV divergencies since the Chern–Simons charge should not be infinitely renormalized, owing to the topological nature of that interaction.

It is interesting to investigate a somewhat more general model where the matter self-interaction is not restricted beforehand by the extended supersymmetry. Then, we would be able both to check the UV finiteness of the quantum N=2 theory and to study the RG behavior of N=1 models in a wider space of charges. However, in constructing the lagrangian, we encounter a problem that the form of the gauge-invariant self-interaction for the N=1 matter superfields is not unique. We can write renormalizable quartic interactions (with dimensionless coupling constants in three dimensions) of different ranks in group generators. Moreover, even if we do not introduce some structure from the very beginning, the corresponding UV divergencies may still be

present, and we shall have to add a new charge when renormalizing them. In fact, the number of independent terms in a consistently renormalizable lagrangian will depend on the gauge group and the particular representation that we choose for the matter fields. The only "miracle" that may prevent some possible counterterms from being generated is a symmetry (like the N=2 supersymmetry in the present case) which the action possesses only in the absence of these counterterms.

In our superfield calculations we introduce gauge-invariant quartic interaction terms of the two lowest ranks in the gauge-group generators. Without specifying a representation, we undoubtedly obtain counterterms of a more complicated structure, too. However, there are some important special cases when these counterterms can be reduced to the terms already present in the lagrangian. Into such cases fall the fundamental representations of the classical groups $SU(n)$, $SO(n)$, and $Sp(n)$. Proceeding in this way, we can also verify whether really the N=2 model is consistently renormalizable and finite for an arbitrary group and representation.

We write the action of the following form:

$$\begin{aligned} \mathcal{S} = \int d^3x \;\; d^2\theta \;\; \Big[& -\tfrac{1}{4}\,(D^{\alpha}\Gamma_{a}^{\beta})D_{\beta}\Gamma_{\alpha}^{a} - \tfrac{1}{6}\,g\,f^{abc}(D^{\alpha}\Gamma_{a}^{\beta})\Gamma_{\alpha}^{b}\Gamma_{\beta}^{c} - \tfrac{1}{24}\,g^2 f^{abc}f^{ade}\Gamma_{b}^{\alpha}\Gamma_{c}^{\beta}\Gamma_{\alpha}^{d}\Gamma_{\beta}^{e} \\ & -\tfrac{1}{2}\,(D^{\alpha}\overline{\Phi}_j + ig\,\overline{\Phi}_k T_{kj}^{a}\Gamma_{a}^{\alpha})(D_{\alpha}\Phi_j - ig\,\Gamma_{\alpha}^{b}T_{jl}^{b}\Phi_l) - m\,\overline{\Phi}_j\Phi_j \\ & +\tfrac{1}{4}\,\eta_0\,(\overline{\Phi}_j\Phi_j)^2 + \tfrac{1}{4}\,\eta_1\,(\overline{\Phi}_j T_{jk}^{a}\Phi_k)^2 \Big]. \end{aligned} \tag{2}$$

Here $a, b, \ldots$ are indices of the adjoint representation of a non-abelian gauge group; $j, k, \ldots = 1, \ldots, n$ are indices of the matter-field representation; T^a are gauge-group generators in this representation; f^{abc} are antisymmetric structure constants of the group, $[T^a, T^b]$=$if^{abc}T^c$; the Greek indices, taking on two values, are the spinor indices in three dimensions. At η_0=0, η_1=g^2, eq.(2) describes the N=2 theory (1).

The N=1 action (2) is invariant under infinitesimal gauge transformations of the form

$$\delta\Phi_j = ig\;\Lambda^a T_{jk}^{a}\Phi_k, \qquad \delta\overline{\Phi}_k = -ig\;\overline{\Phi}_j T_{jk}^{a}\Lambda^a, \qquad \delta\Gamma_{\alpha}^{a} = D_{\alpha}\Lambda^a + g\;f^{abc}\Gamma_{\alpha}^{b}\Lambda^c,$$

with a real scalar-superfield parameter Λ^a in the adjoint representation.

For the gauge fixing, we use the operator $D^{\alpha}\Gamma_{\alpha}^{a}$, thus, we add the following terms to the gauge-invariant action (2)

$$\mathcal{S}_G = \int d^3x \, d^2\theta \left[(4\xi)^{-1}\,(D^{\alpha}\Gamma_{\alpha}^{a})^2 + \overline{\Lambda}^a D^2\Lambda^a - \tfrac{1}{2}\,g\;f^{abc}(D^{\alpha}\overline{\Lambda}^a)\Gamma_{\alpha}^{b}\Lambda^c \right], \tag{3}$$

where Λ^a and $\overline{\Lambda}^a$ are mutually anticommuting scalar ghost superfields. The corresponding gauge-field propagator is

$$\langle \Gamma_{\alpha}(-p,\theta')\;\Gamma^{\beta}(p,\theta) \rangle = -\frac{i}{p^2 + i0} \left(\frac{1+\xi}{2}\,p_{\alpha}^{\;\beta} + \frac{1-\xi}{2}\,\delta_{\alpha}^{\beta}\;D^2 \right) \delta^2(\theta - \theta').$$

The most convenient choice of the gauge parameter for our calculations proved to be the supersymmetric Feynman gauge $\xi=1$.

Our spinor notation and the rules of superfield calculations are the same as described in detail in ref.[6] for the abelian case. We apply the recipe of the regularization by dimensional reduction [10]. First, all the supersymmetry algebra is done in three dimensions, and then the resulting momentum integrals are taken in $3-2\varepsilon$ dimensions. In the leading two-loop approximation for a three-dimensional theory (where all the one-loop momentum integrals contain no genuine UV divergencies), the global inconsistency of such an approach does not show up. The supergraph D algebra is performed as usual [11] by integrating the spinor derivatives by parts from a line of a graph to other lines and by reducing their number on any line to at most two. Thus, we remove all the derivatives from some lines, leaving the pure δ functions, so that the θ integration is easily performed. In this way, the higher powers of the derivatives generate additional momentum factors in numerators or cancel some denominators.

In the minimal subtraction (MS) scheme, the RG β functions are gauge invariant. They can be calculated through the anomalous dimensions of the propagator- and vertex-type Green functions in the following way

$$\beta_{g^2} = g^2(2\gamma_{\bar{\Lambda}\Gamma\Lambda} + 2\gamma_\Lambda + \gamma_\Gamma), \qquad \beta_{\eta_j} = \eta_j(\gamma^{(j)}_{\bar{\Phi}^2\Phi^2} + 2\gamma_\Phi),$$

where the matter-field vertex contributions should be reduced to the structures present in the lagrangian (2). In the massless renormalization scheme, which we use, the mass is renormalized multiplicatively through itself, and its anomalous dimension depends only on dimensionless coupling constants. When computing its UV renormalization, we prefer to treat the mass term of eq.(2) as a two-point interaction in the massless theory. Then, by the supergraph power-counting rules, a UV-divergent two-point diagram can involve only one insertion of this vertex, which leads to the logarithmic divergency.

Having evaluated all the relevant two-loop supergraphs, we found that, in the supersymmetric Feynman gauge, there are nonzero contributions only from 8 ghost-field vertex, 1 ghost-field propagator, 4 gauge-field propagator, 4 matter-field propagator, 5 mass renormalization, and 23 matter-field vertex graphs. About thrice as many supergraphs turn out to equal zero separately (sometimes, very nontrivially) or sum up to zero.

Below we describe our results for different special cases separately.

2.1 The general formulas and the N=2 finiteness

Without specifying a particular group and representation for the matter superfields in the action (2), we obtain UV-divergent counterterms of the following operator types (in the matrix notation for brevity)

$$(\bar{\Phi}\,\Phi)^2, \qquad (\bar{\Phi}\,T^a\,\Phi)^2, \tag{4}$$

$$(\bar{\Phi}\,T^aT^b\,\Phi)^2, \tag{5}$$

$$(\bar{\Phi}\,T^aT^bT^c\,\Phi)^2, \tag{6}$$

$$(\bar{\Phi}\,T^a\,\Phi)\,(\bar{\Phi}\,T^bT^c\,\Phi)\;\mathrm{tr}(T^a\{T^b,T^c\}), \tag{7}$$

$$(\bar{\Phi}\,T^aT^b\,\Phi)\,(\bar{\Phi}\,T^cT^d\,\Phi)\;f^{eac}f^{edb}. \tag{8}$$

We assume that, using special relations for a particular group, we can reduce the new counterterms (5)–(8) to linear combinations of the original terms (4) with the coefficients that we denote by R_{2j} , R_{3j} , R_{tj} , and R_{fj} , respectively, with j=0 and 1, according to the rank of the resulting structure. In the most general case, the basis of independent counterterms (and of the terms in the lagrangian) should be extended, and j will take on more values. Besides the "reduction coefficients" R_{*j}, our expressions involve the usual Casimir operators

$$C_R\,\mathbf{1} = T^aT^a, \qquad T_R\,\delta^{ab} = \mathrm{tr}(T^aT^b), \qquad C_A\,\delta^{ab} = f^{acd}f^{bcd}.$$

Here are the net results [the anomalous dimensions of the superfields (9), (10), and (11) are given in the supersymmetric Feynman gauge; we everywhere omit the $(64\pi^2)^{-1}$ factor from the two-loop momentum integration]

$$\gamma_\Phi = \tfrac14\,(C_R+T_R-\tfrac12\,C_A)\,C_R\,\eta_1^2 + \tfrac12\,C_R\,\eta_1\eta_0 + \tfrac14\,(n+1)\,\eta_0^2 - \tfrac12\,(\tfrac32\,C_R+T_R-C_A)\,C_R\,g^4, \tag{9}$$

$$\gamma_\Gamma = \tfrac18\,T_R\,C_A\,g^4, \tag{10}$$

$$\gamma_\Lambda = -\tfrac18\,T_R\,C_A\,g^4; \tag{11}$$

$$\begin{aligned}\beta_{\eta_1} = {} & \Big[\,(R_{31}+\tfrac12\,R_{t1}+T_R\,C_R+2C_R^2)\,\eta_1 + \tfrac14\,T_R\,C_A\,g^2 - \tfrac12\,R_{f1}\,(\eta_1+g^2) \\ & - \tfrac14\,C_R\,C_A\,(5\eta_1-3g^2) + \tfrac18\,C_A^2\,(\eta_1-3g^2)\,\Big]\,(\eta_1^2-g^4) \\ + & \Big[\,T_R\,(\eta_1^2+\eta_1g^2+g^4) + C_R\,(3\eta_1^2+4\eta_1g^2+3g^4) \\ & - \tfrac14\,C_A\,(5\eta_1^2+8\eta_1g^2+7g^4)\,\Big]\,R_{21}\,(\eta_1-g^2) \\ + & \Big[\,(6R_{21}+10\,C_R+3T_R-\tfrac32\,C_A)\,\eta_1^2 + (2n+11)\,\eta_1\eta_0 \\ & + 2C_R\,\eta_1g^2 - (2R_{21}+\tfrac12\,C_A)\,g^4\,\Big]\,\eta_0\,,\end{aligned} \tag{12}$$

$$\begin{aligned}\beta_{\eta_0} = {} & \Big[\,(R_{30}+\tfrac12\,R_{t0})\,\eta_1 - \tfrac12\,R_{f0}\,(\eta_1+g^2)\,\Big]\,(\eta_1^2-g^4) \\ + & \Big[\,T_R\,(\eta_1^2+\eta_1g^2+g^4) + C_R\,(3\eta_1^2+4\eta_1g^2+3g^4) \\ & - \tfrac14\,C_A\,(5\eta_1^2+8\eta_1g^2+7g^4)\,\Big]\,R_{20}\,(\eta_1-g^2) \\ + & \Big[\,7C_R\,\eta_1\eta_0 + 3(n+2)\,\eta_0^2 + C_R\,\eta_0g^2 \\ & + 2R_{20}\,(3\eta_1^2-g^4)+(2C_R+2T_R-C_A)\,C_R\,(\eta_1^2-g^4)\,\Big]\,\eta_0\,,\end{aligned} \tag{13}$$

$$\beta_{g^2} = 0, \tag{14}$$

$$\gamma_m = (C_R+T_R-\tfrac12\,C_A)\,C_R\,(\eta_1^2-g^4) + 2C_R\,\eta_1\eta_0 + (n+1)\,\eta_0^2\,. \tag{15}$$

A remarkable fact about these cumbersome expressions is that at the $N=2$ supersymmetric point ($\eta_1=g^2$, $\eta_0=0$) all the gauge-invariant RG functions (12)-(15) turn into zero. Note also that if, for a complicated gauge-group representation, there are more than two independent elements in the basis of the matter self-interaction terms, then we shall have more functions β_{η_j}, and the additional charges will contribute to each of them. However, as all η_j are equal to zero, except for η_1, the structure of the η_1 contributions to every β_{η_j} just repeats β_{η_0}, eq.(13), at $\eta_0=0$, with the subscript 0 of the reduction coefficients replaced with j. At $\eta_1=g^2$, all these contributions are nullified; hence, the conclusion about the UV finiteness of the $N=2$ supersymmetric Chern–Simons theory with matter is true at the two-loop level irrespective of the group and representation.

The vanishing of the mass renormalization (15) in the $N=2$ case needs a comment. The non-abelian action (1) contains no mass parameters whatsoever. But one can introduce [9] an abelian Chern–Simons $N=2$ superfield U in addition, with its own abelian charge e and a Fayet–Iliopoulos term κU in the action. The charge renormalization in this linear term can only be due to a U-field renormalization. The latter, however, determines also the UV divergencies in e, and thus, must be absent, which was verified by direct two-loop $N=2$ supergraph calculations in ref.[6]. Therefore, $\beta_\kappa=0$. On the other hand, coming down to $N=1$ superfields, we gain a mass $m=e\kappa$ for the matter superfields from the Fayet–Iliopoulos term. Since the RG β functions in the MS scheme are gauge independent, we still have both β_κ and $\beta_e=0$. The anomalous dimension of the mass must thus equal zero in the $N=2$ supersymmetric model at any value of the abelian charge e. In perturbation theory we can formally take a limit $e\to 0$, $\kappa\to\infty$, m fixed, which leaves us with the massive pure non-abelian model (2) without any mass renormalization. The mass starts to run (15) as soon as we break the $N=2$ relation between the couplings.

The Chern–Simons charge, in accordance with the expectations based on topological arguments [1, 5, 8], never requires an infinite UV renormalization, as eq.(14) shows us. This remains true for all particular cases which are considered below.

2.2 The abelian case, $U(1)\otimes SU(n)_{\rm global}$

From the general formulas (9)–(15) we can easily reproduce the abelian results [6] for n matter superfields with the global $SU(n)$ symmetry. For this purpose, we should substitute the following values of the group parameters

$$C_R=1,\qquad T_R=n,\qquad C_A=0,\qquad R_{2j}=R_{3j}=1,\qquad R_{fj}=0,\qquad R_{tj}=2n,$$

and set either $\eta_0=\eta$ and $\eta_1=0$, or $\eta_0=0$ and $\eta_1=\eta$. The two interactions (4) become indistinguishable; hence, the correct β_η ought to be a sum of β_{η_1} and β_{η_0}, and the results would depend on $\eta_1+\eta_0$ alone, if only the value $\mathrm{tr}(T^a)=0$, essentially used in computing the mixed $\eta_1\eta_0$-like terms of eqs.(9)–(15), were not to be replaced with n for the n-component abelian model.

The resulting expressions

$$\begin{aligned}
\gamma_\Phi &= \tfrac{1}{4}\,[\,(n+1)\,\eta^2-(2n+3)\,g^4\,],\\
\gamma_\Gamma &= 0,\\
\beta_\eta &= (\eta-g^2)[\,3(n+2)\,\eta^2+(3n+7)\,\eta\,g^2+(n+3)\,g^4\,],\\
\beta_{g^2} &= 0,\\
\gamma_m &= (n+1)\,(\eta^2-g^4)
\end{aligned}$$

exactly coincide with those obtained in ref.[6].

2.3 The fundamental representation of SU(n)

For one matter superfield in the fundamental representation of the special unitary $SU(n)$ gauge group, only one independent quartic self-interaction term is possible. The key relation, which allows us to reduce all the higher terms (4)–(8) to the first one, is

$$T^a_{j'j}\,T^a_{k'k}=T_R\,(\delta_{j'k}\,\delta_{k'j}-n^{-1}\,\delta_{j'j}\,\delta_{k'k}).\tag{16}$$

Owing to eq.(16), both interaction terms (4) become proportional to each other. In fact, everything depends on $\eta_0+\eta_1\,T_R\,(n-1)/n$. We choose the structure of rank 1 as the basic one, to make the $N=2$ identification clearer. The group-dependent constants, which we substitute into eqs.(9)–(15), acquire the following values

$$T_R=\tfrac{1}{2}\,,\qquad C_R=\tfrac{1}{2}\,(n^2-1)/n,\qquad C_A=n,$$

$$R_{21}=\frac{n-1}{2n}\,,\qquad R_{31}=\left(\frac{n-1}{2n}\right)^2,\qquad R_{t1}=\frac{n^2-4}{4n}\,,\qquad R_{f1}=-\tfrac{1}{2}\,n.$$

Setting $\eta_1=\eta$, $\eta_0=0$, we get

$$\begin{aligned}
\gamma_\Phi &= \frac{n^2-1}{16\,n^2}\,[\,(n-1)\,\eta^2+(n^2-2n+3)\,g^4\,],\\
\gamma_\Gamma &= -\gamma_\Lambda = \frac{n^2-1}{32\,n}\,g^4,\\
\beta_\eta &= \frac{(n-1)(\eta-g^2)}{4n^2}\,\Big\{(n-1)[3(n+2)\eta^2+(4n+7)\eta g^2]+(n^2+n-3)g^4\Big\}, \qquad (17)\\
\gamma_m &= \frac{(n-1)^2(n+1)}{4n^2}\,(\eta^2-g^4).
\end{aligned}$$

We now look for the RG fixed points. Besides the $N=2$ supersymmetric point $\eta=g^2$, which is IR stable, there are two more RG fixed points where the function β_η (17) equals zero. The ratio η/g^2 for these points depends on n

$$(\eta/g^2)_{\text{fixed}} = \frac{-(4n+7) \pm \sqrt{(4n^3+4n^2+5n+23)/(n-1)}}{6(n+2)} . \qquad (18)$$

It always lies between $-\frac{1}{4}$ (n=2) and approximately -0.34352 (n=8), approaching $-\frac{1}{3}$ as $n \to \infty$, for the first, saddle point, and between $-\frac{17}{18}$ (n=4) and -1 ($n \to \infty$) for the second, IR-stable point. This is different from the abelian case, where only one (N=2 supersymmetric) fixed point exists.

2.4 The fundamental representation of Sp(n)

The symplectic group $Sp(n)$, as opposed to $SU(n)$, preserves, besides the δ symbol, one more covariant tensor, a skew-symmetric metric

$$f_{jk} = -f_{kj} , \qquad f_{jk} f_{kl} = -\delta_{jl} , \qquad f_{jk} T^a_{kl} = f_{lk} T^a_{kj} ,$$

which also enters the contraction relation for the generators

$$T^a_{j'j}\, T^a_{k'k} = \tfrac{1}{2}\, T_R\, (\delta_{j'k}\, \delta_{k'j} - f_{j'k'}\, f_{jk}). \qquad (19)$$

In principle, this might add a new element to the basis of independent matter interactions. However, with just one matter multiplet, after applying eq.(19), we get, in addition to $(\overline{\Phi}_j \Phi_j)^2$, only $|\Phi_j f_{jk} \Phi_k|^2$, which is zero by antisymmetry of f_{jk}. Thus, again we are left with one matter-interaction charge $\eta = \eta_1 + (2/T_R)\eta_0$. The group-specific parameters are

$$C_R/T_R = \tfrac{1}{2}\,(n+1), \qquad C_A/T_R = n+2,$$
$$R_{21} = \tfrac{1}{2}\, T_R\, , \qquad R_{31} = \tfrac{1}{4}\, T_R^2\, , \qquad R_{f1} = -(n+2)\, T_R^2\, , \qquad R_{t1} = 0.$$

We do not fix the normalization of the generators, so that the value of T_R stays arbitrary. The results look as follows:

$$\begin{aligned}
\gamma_\Phi &= \tfrac{1}{16}\, T_R^2\, (n+1)\, [\, \eta^2 + (n+1)\, g^4 \,], \\
\gamma_\Gamma &= -\gamma_\Lambda \;=\; \tfrac{1}{8}\, T_R^2\, (n+2)\, g^4, \\
\beta_\eta &= \tfrac{1}{4}\, T_R^2\, [\, 3(n+2)\, \eta + (n+1)\, g^2 \,]\, (\eta^2 - g^4), \qquad (20) \\
\gamma_m &= \tfrac{1}{4}\, T_R^2\, (\eta^2 - g^4).
\end{aligned}$$

There are three RG fixed points: the IR-stable points $\eta = \pm g^2$ and the saddle point $-\frac{1}{3}\,(n+1)/(n+2)$. It is interesting to note that at the second fixed point with $\eta = -g^2$ the mass renormalization also vanishes, as well as at the first, N=2 supersymmetric point.

2.5 The fundamental representation of SO(n)

Though the orthogonal group $SO(n)$ has no other covariant tensors than the δ symbol, its fundamental representation is real, so that the indices of Φ_j and $\overline{\Phi}_k$ can be mixed arbitrarily. The contraction relation for the generators

$$T^a_{j'j}\, T^a_{k'k} = \tfrac{1}{2}\, T_R\, (\delta_{j'k}\, \delta_{k'j} - \delta_{j'k'}\, \delta_{jk}). \qquad (21)$$

allows us to reduce any element of the general basis (4)–(8) to a linear combination of two terms: the zero-rank element and the mixed term $\overline{\Phi}_j \overline{\Phi}_j\; \Phi_k \Phi_k$. Thus, we need two charges, to ensure the closed renormalizability. Evidently, the two first elements (4) will fit as well. We obtain the full set of the reduction coefficients, and the Casimir-operator values

$$C_R/T_R = \tfrac{1}{2}\,(n-1), \qquad C_A/T_R = n-2,$$
$$R_{20} = \tfrac{1}{4}\, T_R^2\, (n-1), \qquad R_{30} = R_{f0} = -\tfrac{1}{8}\, T_R^3\, (n-1)(n-2), \qquad R_{t0} = R_{t1} = 0,$$
$$R_{21} = -\tfrac{1}{2}\, T_R\, (n-2), \qquad R_{31} = \tfrac{1}{4}\, T_R^2\, (n^2 - 3n + 3), \qquad R_{f1} = \tfrac{1}{4}\, T_R^2\, (n-2)(n-3).$$

In that case the general formulas (9)–(15) give us

$$\begin{aligned}
\gamma_\Phi &= \tfrac{1}{16}\,(n-1)\, T_R^2\, [\, 3\eta_1^2 + (n-9)\, g^4 \,] \;+\; \tfrac{1}{4}\, [\, (n-1)\, T_R\, \eta_1 \eta_0 \;+\; (n+1)\, \eta_0^2 \,], \\
\gamma_\Gamma &= -\gamma_\Lambda \;=\; \tfrac{1}{8}\,(n-2)\, T_R^2\, g^4, \\
\beta_{\eta_0} &= (n-1) \Big[\, \tfrac{1}{8}\, T_R^3\, (5\eta_1^2 + 6\eta_1 g^2 + 5g^4)\, (\eta_1 - g^2) \;+\; T_R^2\, (3\eta_1^2 - 2g^4)\, \eta_0 \\
&\qquad + \tfrac{1}{2}\, T_R\, (7\eta_1 + g^2)\, \eta_0^2 \,\Big] \;+\; 3(n+2)\, \eta_0^3 , \\
\beta_{\eta_1} &= T_R^2\, [\, \tfrac{5}{4}\, \eta_1^3 + \tfrac{1}{2}\,(n-2)\, \eta_1^2 g^2 + (\tfrac{3}{4} - n)\, \eta_1 g^4 + \tfrac{1}{2}\,(n-2)\, g^6 \,] \qquad (22) \\
&+ \;[\, (\tfrac{1}{2}\, n + 7)\, \eta_1^2 + (n-1)\, \eta_1 g^2 + \tfrac{1}{2}\,(n-2)\, g^4 \,]\, T_R\, \eta_0 \;+\; (2n+11)\, \eta_1 \eta_0^2 , \\
\gamma_m &= (n-1)\, [\, \tfrac{3}{4}\, T_R^2\, (\eta_1^2 - g^4) \;+\; T_R\, \eta_1 \eta_0 \,] \;+\; (n+1)\, \eta_0^2 .
\end{aligned}$$

The RG equations for the two charges with the presented β functions (22) have several fixed points which are summarized in table 1 for n=2, 3, and $n \to \infty$. (The value of T_R is set to $\frac{1}{2}$). The five first fixed points exist for all n. The limit points # 6 and 7 are traced down to n=6, while points # 8 and 9 exist only when $n \geq 49$. A peculiar fact is the appearance of a UV-stable — absolutely IR-unstable — fixed point # 7 ($n \geq 6$), which occurs neither in the abelian theory with global $SU(n)$ [6], nor in the non-abelian $SU(n)$ component-field model (see sect.3 below). The N=2 supersymmetric fixed point # 4 becomes a saddle point in the extended space of the $SO(n)$ matter couplings.

The two running charges exhibit a complicated RG flow. The phase portraits of the RG equations $\partial\eta_j/\partial \log \mu^2 = \beta_{\eta_j}$ for n=2 and $n \to \infty$ are shown in figs.1 and 2. The arrows correspond to the IR direction $\mu^2 \to 0$. In the $SO(2)$ case, there exists a fixed straight line η_1=0. All other RG trajectories are curved. In fig.2, the far-distant fixed point # 1 is not shown.

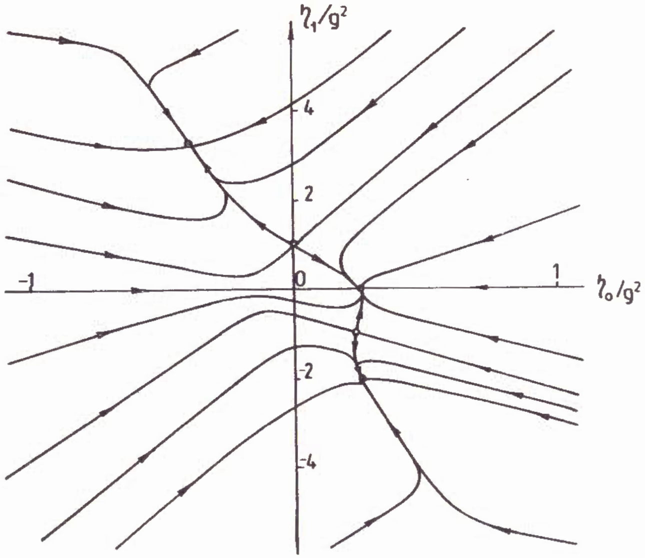

Figure 1: The renormalization-group flow of the matter couplings in the N=1 supersymmetric $SO(2)$ Chern–Simons field theory. Arrows indicate the infrared direction $\mu^2 \to 0$. Solid dots denote stable fixed points, while empty circles are saddle points.

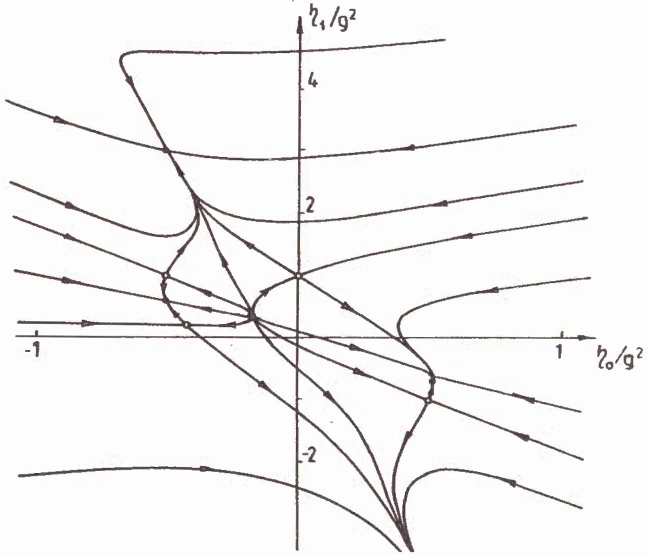

Figure 2: The renormalization-group flow of the matter couplings in the N=1 supersymmetric $SO(n)$ Chern–Simons field theory as $n \to \infty$. Arrows indicate the infrared direction $\mu^2 \to 0$. Solid dots denote stable fixed points, while empty circles are saddle points.

3 A nonsupersymmetric non-abelian SU(n) Chern–Simons field theory with matter

The component-field contents of the supersymmetric models considered in sect.2 can be obtained by rewriting the integral over the superspace as $\int d^3x\, d^2\theta = \int d^3x\, D^2$ at θ=0 and by defining component fields as various-order spinor derivatives of superfields. Everything is done in the same way as in the abelian case [6]. The Chern–Simons gauge superfield contains, besides the real vector field A^a_μ, two pure-gauge components, which can be set to zero by fixing the Wess–Zumino gauge. It also contains an auxiliary spinor field, which can be eliminated via its equation of motion. The complex matter superfield produces the physical charged spinor and scalar fields, ψ_j and ϕ_j, plus a non-propagating auxiliary scalar. After eliminating the auxiliary fields, we obtain the Yukawa-type (bispinor biscalar) interactions, and the scalar self-coupling of the sixth and fourth degrees. All the corresponding charges are functions of the parameters, present in the superfield action (2).

Our next step is a generalization of the model, where the correlations between the coupling constants, imposed by the N=1 supersymmetry, are put off, too. Again, having abandoned a symmetry, we may meet the necessity of introducing many new counterterms, to ensure closed renormalizability of the model. Since the number of different fields has grown up (scalars are not the same as spinors), we can construct more gauge-invariant structures which are possible as interaction terms. The complete set again depends on the gauge group and matter-field representation.

Not to make our calculations exceedingly cumbersome, we restrict our choice to the most popular group $SU(n)$ and its fundamental representation. Applying the contraction relation (16) to each pair of the gauge-group generators in the course of computing Feynman diagrams, we can always eliminate the generators from matter-field vertices. Thus, the "minimal" closed extension of the supersymmetric non-abelian model with matter fields in the fundamental representation of $SU(n)$ can be described by the following component-field lagrangian density:

$$\begin{aligned}\mathcal{L} \;=\; & \tfrac{1}{2}\,\epsilon_{\mu\nu\rho}A^a_\mu\partial_\nu A^a_\rho + \tfrac{1}{6}\,e\,f^{abc}\epsilon_{\mu\nu\rho}A^a_\mu A^b_\nu A^c_\rho \\ & + \;|\mathcal{D}_\mu\phi_j|^2 + i\,\overline{\psi}_j\hat{\mathcal{D}}\psi_j - m^2\phi^*_j\phi_j - M\,\overline{\psi}_j\psi_j + q\,(\phi^*_j\phi_j)^2 \qquad (23) \\ & + \;\alpha\,\overline{\psi}_j\psi_j\,\phi^*_k\phi_k + \beta\,\overline{\psi}_j\psi_k\,\phi^*_k\phi_j + \tfrac{1}{4}\,\gamma\,\left(\overline{\psi}_j\psi^*_k\,\phi_j\phi_k + \overline{\psi^*_j}\psi_k\,\phi^*_j\phi^*_k\right) - h\,(\phi^*_j\phi_j)^3,\end{aligned}$$

where the covariant derivative for the matter fields is of the form $\mathcal{D}_\mu\phi_j = \partial_\mu\phi_j - ieT^a_{jk}A^a_\mu\phi_k$. In the non-abelian Chern–Simons field lagrangian (23), only the triple vector-field vertex is present. The quartic interaction of superfields (2) involves the pure-gauge components which are absent in the Wess–Zumino gauge. We use the Majorana spinor basis, where the charge conjugate of a spinor ψ is obtained just by its complex conjugation ψ^*. The γ-type Yukawa vertices, permissible in three dimensions, lead to the fermion-number non-conservation. The relation of the parameters of eq.(23) in the supersymmetric special case to the superfield coupling constants in eq.(2) is

$$e = g/\sqrt{2}, \qquad \alpha = \left[(n-1)\eta + ng^2\right]/(4n), \qquad \beta = \left[(n-1)\eta - g^2\right]/(4n),$$
$$\gamma = (\eta - g^2)(n-1)/(2n), \qquad h = \eta^2(n-1)^2/(16n^2),$$
$$M = m, \qquad q = \eta m(n-1)/(2n). \tag{24}$$

3.1 Component-field calculations

The gauge-fixing and ghost terms for eq.(23) can be written as follows:

$$\mathcal{L}_G = (2\xi)^{-1}(\partial_\mu A^a_\mu)^2 - \overline{\lambda}^a \partial_\mu(\partial_\mu \lambda^a + e\ f^{abc} A^b_\mu \lambda^c). \tag{25}$$

In three dimensions the gauge parameter ξ for the component vector field in eq.(25) has the dimension of a mass, as opposed to the dimensionless supersymmetric-gauge parameter in eq.(3). Thus, every power of ξ lowers the divergency index of a diagram by one. Multiple ξ-term insertions in one propagator line seem then to cause infrared troubles. However, owing to the gauge Ward identity, all the corrections to the vector-field propagator are transverse, which prevents these longitudinal IR-divergent terms from appearing. The model is thus IR finite (off shell) at any choice of ξ. Although the RG β functions in the MS scheme are gauge invariant, and hence, independent of ξ, the appearance of an additional unphysical mass parameter in some Green functions is technically inconvenient. Like in the abelian case [6], we find it much easier to work in the Landau gauge $\xi \rightarrow 0$ with the transverse propagator

$$\langle A_\mu(-p)\ A_\nu(p)\rangle \ = \ \frac{\epsilon_{\mu\rho\nu}p_\rho}{p^2 + i0}\ .$$

In the MS scheme, the anomalous dimensions of the fields and the β functions for the dimensionless couplings (e, α, β, γ, and h) do not depend on the parameters of the dimension of a mass. The mass-type parameters themselves (M, q, and m^2) can be intermixed, one contributing to the β function of another, but these contributions are always polynomial. Thus, β_M and β_q will never depend on m^2.

The β functions are related to the anomalous dimensions of the vertex and propagator-type diagrams according to the following formulas

$$\beta_{e^2} = (\gamma_{\overline{\lambda}A\lambda} + 2\gamma_\lambda + \gamma_A)e^2, \qquad \beta_\alpha = (\gamma_\alpha + \gamma_\psi + \gamma_\phi)\alpha, \qquad \beta_\beta = (\gamma_\beta + \gamma_\psi + \gamma_\phi)\beta,$$
$$\beta_\gamma = (\gamma_\gamma + \gamma_\psi + \gamma_\phi)\gamma, \qquad \beta_h = (\gamma_h + 3\gamma_\phi)h,$$
$$\beta_M = (\gamma_{\overline{\psi}\psi} + \gamma_\psi)M, \qquad \beta_q = (\gamma_q + 2\gamma_\phi)q, \qquad \beta_{m^2} = (\gamma_{\phi^*\phi} + \gamma_\phi)m^2,$$

where γ_j are the anomalous dimensions of the corresponding vertex and propagator one-particle-irreducible Green functions.

The total number of the contributing component-field Feynman diagrams exceeded one hundred and fifty. The results are

$$\gamma_\phi = \tfrac{1}{12}\Big\{4(n\alpha^2 + 2\alpha\beta + n\beta^2) + (n+1)\gamma^2 + \left[(n^2-1)(3n^2-8n+10)/n^2\right] e^4\Big\}, \tag{26}$$

$$\gamma_\psi = \tfrac{1}{12}\Big\{4(n\alpha^2 + 2\alpha\beta + n\beta^2) + (n+1)\gamma^2 + \left[(n^2-1)(-2n+1)/n^2\right] e^4\Big\}, \tag{27}$$

$$\gamma_\lambda = \tfrac{1}{3}\, ne^4, \tag{28}$$

$$\gamma_A = -\tfrac{1}{3}\, ne^4, \tag{29}$$

$$\beta_{e^2} = 0, \tag{30}$$

$$\begin{aligned}\beta_\alpha = {} & (\tfrac{8}{3}n+2)\alpha^3 + \tfrac{16}{3}\alpha^2\beta + (\tfrac{8}{3}n+3)\alpha\beta^2 + (n+2)\beta^3 + (\tfrac{2}{3}n + \tfrac{17}{12})\alpha\gamma^2 \\ & + \tfrac{3}{4}(n+2)\beta\gamma^2 - \alpha\beta e^2 + \tfrac{1}{2}\left[(n^2-3)/n\right]\beta^2e^2 + \tfrac{1}{8}\left[(n^2-1)/n\right]\gamma^2e^2 \\ & - \tfrac{1}{12}\left[(40n^3 - 17n^2 - 40n + 62)/n^2\right]\alpha e^4 - \tfrac{1}{4}\left[(5n^3 + 6n^2 - 18n + 8)/n^2\right]\beta e^4 \\ & + \tfrac{1}{8}\left[(3n^4 - 4n^3 + 5n^2 - 8n + 16)/n^3\right] e^6,\end{aligned} \tag{31}$$

$$\begin{aligned}\beta_\beta = {} & (\tfrac{8}{3}n+6)\alpha^2\beta + (3n + \tfrac{16}{3})\alpha\beta^2 + (1 + \tfrac{2}{3}n)\beta^3 + \tfrac{3}{4}(n+2)\alpha\gamma^2 \\ & + (\tfrac{2}{3}n + \tfrac{17}{12})\beta\gamma^2 + n\alpha\beta e^2 + \beta^2e^2 + \tfrac{1}{4}\left[(n^2-1)/n\right]\gamma^2e^2 \\ & - \tfrac{5}{4}\left[(n^2-4)/n\right]\alpha e^4 - \tfrac{1}{12}\left[(22n^3 - 23n^2 - 64n + 62)/n^2\right]\beta e^4 \\ & - \tfrac{1}{2}\left[(n^2-4)(n-2)/n^2\right] e^6,\end{aligned} \tag{32}$$

$$\begin{aligned}\beta_\gamma = {} & \Big\{(\tfrac{8}{3}n+6)\alpha^2 + (6n + \tfrac{34}{3})\alpha\beta + (\tfrac{8}{3}n+6)\beta^2 + \tfrac{1}{6}(n+1)\gamma^2 \\ & + \left[(n-1)(n+2)/n\right]\alpha e^2 + \left[(n-1)(2n+1)/n\right]\beta e^2 \\ & - \tfrac{1}{6}\left[(n-1)(2n^2 - 2n + 5)/n^2\right] e^4\Big\}\gamma,\end{aligned} \tag{33}$$

$$\begin{aligned}\beta_h = {} & 12(3n+11)h^2 - \left[(n-1)(11n^2 + 13n - 16)/n^2\right] he^4 \\ & + \left[16n\alpha^2 + 32\alpha\beta + 4(n+3)\beta^2 + (n+4)\gamma^2\right] h \\ & - (n-1)\big\{2\alpha\beta^2 + \beta^3 + \tfrac{1}{2}\left[(n+2)/n\right]\alpha\gamma^2 + \tfrac{1}{4}\left[(3n+4)/n\right]\beta\gamma^2\big\}e^2 \\ & + \left[(n-1)/n^2\right]\left[n(n+1)\alpha^2 + (n^2+2n-2)\alpha\beta + (n-1)\beta^2 - \tfrac{1}{4}(n-1)\gamma^2\right]e^4 \\ & + \tfrac{1}{4}\left[(n-1)/n^3\right]\left[2n(n^2+2n-2)\alpha + (n^3 + 4n^2 - 8n + 4)\beta\right]e^6 \\ & + \tfrac{1}{16}\left[(n-1)/n^4\right](5n^4 + 4n^3 - 56n^2 + 76n - 28)e^8 \\ & - \left[(n+6)\alpha^2 + (3n+11)\alpha\beta + \tfrac{1}{2}(3n+11)\beta^2\right]\gamma^2 - \tfrac{1}{16}(n+3)\gamma^4 \\ & - \left[4n\alpha^4 + 16\alpha^3\beta + 4(n+5)\alpha^2\beta^2 + 4(n+3)\alpha\beta^3 + (n+3)\beta^4\right].\end{aligned} \tag{34}$$

We have computed the renormalization of the dimensional parameters in our model as well. Here are the results

$$\beta_M = \tfrac{1}{3}\Big\{4(n\alpha^2 + 2\alpha\beta + n\beta^2) + (n+1)\gamma^2 - \left[(n^2-1)(5n-4)/n^2\right] e^4\Big\}M, \tag{35}$$

$$\beta_q = \Big[\tfrac{1}{3} \Big\{ 20n\alpha^2 + 40\alpha\beta + 4(2n+3)\beta^2 + (2n+5)\gamma^2 \qquad (36)$$
$$-2\left[(n-1)(8n^2+7n-10)/n^2\right] e^4 \Big\} + 24(n+2)h \Big] q$$
$$- \Big\{ 8n\alpha^3 + 24\alpha^2\beta + 8(n+2)\alpha\beta^2 + 4(n+1)\beta^3 + 2(n+3)\alpha\gamma^2 + (3n+5)\beta\gamma^2$$
$$+2(n-1)\beta^2 e^2 + \tfrac{1}{2}\left[(n-1)(n+2)/n\right]\gamma^2 e^2 - 2\left[(n^2-1)/n\right]\alpha e^4$$
$$-\left[(n-1)(n^2+2n-2)/n^2\right]\beta e^4 - \tfrac{1}{2}\left[(n-1)(n^2+2n-2)/n^2\right] e^6 \Big\} M,$$

$$\beta_{m^2} = \tfrac{1}{3}\Big\{4(n\alpha^2+2\alpha\beta+n\beta^2) + (n+1)\gamma^2 - \left[(n^2-1)(5n-4)/n^2\right] e^4\Big\} m^2 \qquad (37)$$
$$- \Big\{4(n\alpha^2+2\alpha\beta+n\beta^2) + (n+1)\gamma^2 - \left[(n^2-1)/n\right] e^4\Big\} M^2 + 4(n+1) q^2.$$

Note that, in the leading two-loop approximation, the fermion mass happens to renormalize only through itself (35). In higher orders, a q contribution to β_M will also be present.

Eqs.(30)–(37) are compatible with the N=1 supersymmetry relations (24), substituting which, we get exactly formulas (17) from sect.2, except for the anomalous dimensions of the fields (27)–(29), which should not be the same in the supersymmetric and Wess–Zumino gauges.

3.2 Renormalization-group fixed points

The RG fixed points are determined by zeros of the β functions (30)–(34) for the dimensionless couplings of the model. The Chern–Simons charge e, with its identically zero β function (30) to all orders in perturbation theory [8], stays an external parameter while other charges become proportional to its powers at the fixed points. In a vicinity of a fixed point, the RG equations for the dimensional parameters, with the β functions presented in eqs.(35), (36), and (37), become linear differential equations (inhomogeneous for q and m^2) with asymptotically constant coefficients:

$$\mu^2 \frac{dM}{d\mu^2} = X_M M, \qquad \mu^2 \frac{dq}{d\mu^2} = X_q q + Y_q M,$$
$$\mu^2 \frac{dm^2}{d\mu^2} = 2(X_m m^2 + Y_m M^2 + Z_m q^2). \qquad (38)$$

Incidentally, $2X_m$ turned out to equal X_M. The asymptotics of the solutions is of the power form $(\mu^2)^X$, with $X{=}X_M$ for M, $X{=}\min(X_M, X_q)$ for q, and $X{=}\min(X_M, X_q, X_m)$ for m, in the IR ($\mu{\to}0$) limit. For the UV asymptotics, $\mu{\to}\infty$, one should replace *min* with *max*.

The fixed points are listed in tables 2, 3, and 4. The order of sorting is such that the parameters of the points with the same numbers smoothly depend on n. Points # 23 and 24 of table 3 flow together with # 21 and 22 as $n{\to}2$, so that points # 21 and 22 in table 2 are double. In table 4 we present the leading asymptotic terms only. Points # 25–28 appear as $n{\geq}96$. Points # 19 and 20 converge with # 7 and 8 to

double solutions as $n{\to}\infty$. Points # 15 and 16 meet with points # 17 and 18 between n=6 and n=7, then they diverge again.

Note that the non-abelian $SU(n)$ N=2 supersymmetric fixed point # 1 remains IR stable in the extended space of charges, as differs from the abelian case where it became a saddle point, stable only along the fixed line of the N=1 supersymmetry. Two more supersymmetric solutions, which correspond to eq.(18), are points # 21 and either # 16 for $n{\leq}6$, or # 18 for $n{\geq}7$, always with negative values of γ. They both turn out to be saddle points off the supersymmetry fixed line (24).

As explained in ref.[6] for the abelian case, in higher orders of perturbation theory the obtained above ratios of the charges at the fixed points will simply get corrections in e^4 (only even loops contributing), since we know that the gauge charge remains a fixed parameter, in which the perturbation-theory expansions are carried out. Thus, no new fixed points will appear, and the stability properties will not change.

4 The critical limit and final remarks

Among the possible applications of the RG analysis of our model is the description of critical phenomena with the methods of quantum field theory, by analogy with spin models and ferromagnets. Such an attempt in the case of the abelian Chern–Simons model, in the application to the high-T_c superconductivity, has been performed in refs.[6, 7].

The critical behavior in quantum field theory is usually associated with the RG fixed-point behavior, the fixed points being identified with the critical (tricritical) points. Power-like asymptotics of the Green functions in a vicinity of a fixed point is then directly related to the scaling behavior in critical phenomena [12].

The critical theory, by definition, is the theory at $T{=}T_c$, where, by analogy with the Ising model, the temperature is identified with the mass term in the lagrangian (for the moment, we concentrate on a model with a single mass parameter), so that $m^2_{\rm bare}{\sim}t{=}(T{-}T_c)/T_c$. This means that at the critical point the mass term in the lagrangian vanishes: $m{\sim}m_{\rm bare}{=}0$ (m is renormalized multiplicatively, and if $m_{\rm bare}$ equals zero, so does m.)

Studying the critical behavior, we are usually interested in two things. The first thing is the long-range (infrared) behavior of the correlation functions (or the propagators in the field theory) at the critical point, that is $p^2{\to}0$, $m^2{=}0$. Then, according to the RG equations, the propagator behaves like

$$G(p^2/\mu^2,\ 0,\ g_*) \to (p^2/\mu^2)^{\gamma_\phi(g_*)}, \qquad (39)$$

where g_* is the value of the coupling constant at the fixed point. This defines the critical exponent $\eta{=}2\gamma_\phi$ through the propagator anomalous dimension.

Here we encounter a problem of the gauge dependence of propagators in a general situation. However, despite this fact, it is the anomalous dimension of the scalar field that seems to be insensitive to a change, at least, from the supersymmetric Feynman gauge (3) to the component-field Landau gauge (25): after using the supersymmetry relations between the component-field coupling constants (24) of the $SU(n)$ model, we obtain from γ_ϕ, eq.(26), just the anomalous dimension of the matter superfield γ_Φ, see eqs.(17). The same coincidence is observed in the abelian case [6]. Thus, in these models, the scalar-field propagator may be of interest in itself. In any case, one can compute the renormalization of the two-point Green functions for the gauge-invariant composite operators if necessary.

The second thing we are interested in is the behavior of the correlation length, identified with the reciprocal of the physical mass $m_{\rm ph}$ (a pole of the propagator, for instance). Here, we do not have a momentum tending to zero. Rather we are interested in the approach to the critical theory, that is in the limit $m^2 \to 0$. In fact, we must retrace the dependence of $m_{\rm ph}$ on the bare mass (related to the temperature), which is equivalent to the dependence on the renormalized mass at some fixed value of the RG scale μ.

The physical mass, being an RG-invariant quantity, obeys the equation without an anomalous dimension

$$\left(\mu^2 \frac{\partial}{\partial\mu^2} + \beta_{m^2} \frac{\partial}{\partial m^2} + \beta_g \frac{\partial}{\partial g}\right) m_{\rm ph}(\mu^2, m^2, g) = 0, \qquad (40)$$

while the renormalized mass runs according to

$$\mu^2 \frac{{\rm d}m^2}{{\rm d}\mu^2} = \beta_{m^2}, \qquad \beta_{m^2} = 2m^2\gamma_m .$$

At the same time, on dimensional grounds we have

$$2\left(\mu^2 \frac{\partial}{\partial\mu^2} + m^2 \frac{\partial}{\partial m^2}\right) m_{\rm ph} = m_{\rm ph} . \qquad (41)$$

Combining eqs.(40) and (41) in order to eliminate the partial derivative with respect to μ, at a fixed point where $\beta_g=0$, we find that

$$\left[(1-2\gamma_m)\, 2m^2 \frac{\partial}{\partial m^2} - 1\right] m_{\rm ph} = 0. \qquad (42)$$

The solution to eq.(42) is of the power form

$$m_{\rm ph} \sim m^{1/(1-2\gamma_m)} \sim t^\nu,$$

which defines the critical exponent $\nu^{-1}=2(1-2\gamma_m)$. [One supplements powers of μ whenever they are needed to restore the right dimension.]

We now proceed to a situation which is more complicated because of the presence of additional dimensional variables, as in our component-field model of sect.3. All the dimensionless couplings always behave like g and present no problems to us: they simply approach some constants near fixed points.

Because q and M have the dimension of a mass, the corresponding dimensionless quantities $\overline{q}=\mu^{-1}q$ and $\overline{M}=\mu^{-1}M$, obeying the RG equations

$$\mu^2 \frac{\rm d}{{\rm d}\mu^2}\,\overline{q} = -\tfrac{1}{2}\,\overline{q} + \beta_q, \qquad \mu^2 \frac{\rm d}{{\rm d}\mu^2}\,\overline{M} = -\tfrac{1}{2}\,\overline{M} + \beta_M,$$

behave, to the leading order, like

$$\overline{q(p)}^{\,2} \sim q_0^2/p^2, \qquad \overline{M(p)}^{\,2} \sim M_0^2/p^2,$$

as $p^2 \to 0$, that is they go to infinity. This fact would definitely spoil our fixed-point analysis.

However, if we define the critical theory in such a way that not only $m=0$, but also any other dimensional variable — the scale — goes to zero, everything will be all right. Since these dimensional quantities are renormalized multiplicatively within the whole set, it is self-consistent to put them all to be zero. So, if the critical theory is $m, q, M \to 0$, then eq.(39) is not changed, while instead of eq.(40) we have

$$\left(\mu^2 \frac{\partial}{\partial\mu^2} + \beta_{m^2} \frac{\partial}{\partial m^2} + \beta_q \frac{\partial}{\partial q} + \beta_M \frac{\partial}{\partial M} + \beta_g \frac{\partial}{\partial g}\right) m_{\rm ph}(\mu^2, m^2, q, M, g) = 0. \qquad (43)$$

The running parameters q and M obey the RG equations

$$\mu^2 \frac{{\rm d}q}{{\rm d}\mu^2} = \beta_q = q\,\gamma_q, \qquad \mu^2 \frac{{\rm d}M}{{\rm d}\mu^2} = \beta_M = M\,\gamma_M$$

— we do not keep an eye on the mixing of the parameters for a while. On dimensional grounds we have, instead of eq.(41),

$$\left(2\mu^2 \frac{\partial}{\partial\mu^2} + 2m^2 \frac{\partial}{\partial m^2} + q \frac{\partial}{\partial q} + M \frac{\partial}{\partial M}\right) m_{\rm ph} = m_{\rm ph} . \qquad (44)$$

Combining eq.(44) with eq.(43), at a fixed point we obtain

$$\left[(1-2\gamma_m)\, 2m^2 \frac{\partial}{\partial m^2} + (1-2\gamma_q)\, q \frac{\partial}{\partial q} + (1-2\gamma_M)\, M \frac{\partial}{\partial M} - 1\right] m_{\rm ph} = 0, \qquad (45)$$

with the solution of the form

$$m_{\rm ph} \sim m^{1/(1-2\gamma_m)}\, f\left(\frac{m^{1/(1-2\gamma_m)}}{q^{1/(1-2\gamma_q)}}, \frac{m^{1/(1-2\gamma_m)}}{M^{1/(1-2\gamma_M)}}\right), \qquad (46)$$

where f is an arbitrary function, and the powers of μ are supplemented as needed.

Now, we look for a trajectory of q and M going to zero together with m. We choose an RG-invariant particular pattern

$$m^{1/(1-2\gamma_m)} \sim q^{1/(1-2\gamma_q)} \sim M^{1/(1-2\gamma_M)} \to 0, \qquad (47)$$

which corresponds to the tricritical behavior. This is the only case when the arbitrary function in eq.(46) turns into a constant and causes no difficulties, so that the model has a unique correlation length.

Some complication of the formulas is due to the mixing in the renormalization of the mass-type parameters, eq.(38). Then, eq.(47) reads

$$q \sim M^{\kappa} \to 0, \qquad \kappa = \frac{1-2\gamma_q}{1-2X_M}, \qquad \gamma_q = \min\,(X_q\,, X_M);$$

$$m \sim M^{\lambda} \to 0, \qquad \lambda = \frac{1-2\gamma_m}{1-2X_M}, \qquad \gamma_m = \min\,(X_m\,, X_q\,, X_M),$$

where *min* corresponds to the IR fixed-point limit $\mu \to 0$. In this way we return to the previous case of just one mass parameter, with $m_{\rm ph} \sim m^{1/(1-2\gamma_m)}$. We obtain the same critical exponent $\nu^{-1}=2(1-2\gamma_m)$, and the situation is described by a single correlation length.

Although at the moment it is not quite clear to us which physical phenomena can be associated with the models that we have examined — according to the universality principle, there may be many: it is the symmetry of the problem that matters in the first place — we can analyze the critical properties of the found fixed points.

Table 1: The RG fixed points for the N=1 supersymmetric non-abelian Chern–Simons theory with matter superfields in the fundamental representation of $SO(n)$

n	#	η_0/g^2	η_1/g^2	stability	γ_m/g^4
2	1	0.25	−2	IR stable	0.5
2	2	0.25	0	IR stable	0
2	3	0.23151	−0.97939	saddle point	0.039770
2	4	0	1	saddle point	0
2	5	−0.39112	3.2440	IR stable	1.6102
3	1	0.44288	−3.6924	IR stable	3.8869
3	2	0.31302	−0.17320	IR stable	−0.026040
3	3	0.29453	−0.90015	saddle point	0.010721
3	4	0	1	saddle point	0
3	5	−0.46417	3.5009	IR stable	3.4579
∞	1	3.0687	−22.505	IR stable	69.665 n
∞	2	0.51764	−0.68835	IR stable	−0.0088671 n
∞	3	0.5	−1	saddle point	−0.125
∞	4	0	1	saddle point	0
∞	5	−0.5	3	IR stable	n
∞	6	−0.5	1	saddle point	1
∞	7	−0.16667	0.33333	UV stable	−0.16667 n
∞	8	−0.5	0.6	IR stable	−0.02 n
∞	9	−0.41969	0.19365	saddle point	−0.044968 n

Table 2: The RG fixed points in the non-abelian Chern-Simons field theory with matter fields in the fundamental representation of $SU(2)$

#	α/e^2	β/e^2	$\lvert\gamma\rvert/e^2$	h/e^4	stability	X_M/e^4	X_q/e^4
1	0.75	0	0	0.0625	IR stable	0	7.5
2	0.75	0	0	−0.084559	saddle point	0	−6.6176
3	0.11785	0	0	0.048319	saddle point	−1.4630	−1.1762
4	0.11785	0	0	0.015679	saddle point	−1.4630	−4.3097
5	0.090626	−0.91586	0	0.092391	saddle point	0.53737	9.7012
6	0.090626	−0.91586	0	−0.096719	saddle point	0.53737	−8.4534
7	−0.82524	0.91586	0	0.092391	saddle point	0.53737	9.7012
8	−0.82524	0.91586	0	−0.096719	saddle point	0.53737	−8.4534
9	−0.86785	0	0	0.11665	IR stable	0.50842	15.240
10	−0.86785	0	0	−0.16861	saddle point	0.50842	−12.145
11	0.53484	−0.70421	1.4084	0.066861	IR stable	1.5645	9.7904
12	0.53484	−0.70421	1.4084	−0.093437	saddle point	1.5645	−5.5983
13	0.46287	−0.78383	0.98307	0.066470	saddle point	0.70865	7.0339
14	0.46287	−0.78383	0.98307	−0.065649	saddle point	0.70865	−5.6495
15	0.4375	−0.3125	0.625	0.032629	saddle point	−0.70312	−0.055147
16	0.4375	−0.3125	0.625	0.0039062	saddle point	−0.70312	−2.8125
17	0.12106	−0.0063413	0.012683	0.048309	saddle point	−1.4627	−1.1763
18	0.12106	−0.0063413	0.012683	0.015680	saddle point	−1.4627	−4.3087
19	−0.83716	0.94955	0.35285	0.092416	saddle point	0.77798	10.406
20	−0.83716	0.94955	0.35285	−0.10354	saddle point	0.77798	−8.4054
21	0.25	−0.5	1	0.0625	saddle point	0	4.5
22	0.25	−0.5	1	−0.040441	saddle point	0	−5.3824

Of the utmost interest are the stable fixed points — saddle points can only be reached when a special relation between the couplings, a fine tuning, is strongly imposed, which does not seem feasible unless a symmetry protects this relation. To be able to define both critical exponents ν and η, we need an IR-stable fixed point.

Reviewing our tables, we find several points that fit. These are points # 1, 9, 11, and 27 for the $SU(n)$ group, also # 21 on the subset (24) protected by the N=1 supersymmetry; two supersymmetric IR-stable fixed points with $\eta=\pm g^2$ for the $Sp(n)$ group; and points # 1, 2, 5, and 8 for the supersymmetric $SO(n)$ model.

The $SO(n)$ UV-stable fixed point # 7 is quite interesting from another point of view: it fulfills an old hope to find a quantum-field model with a finite asymptotic charge renormalization, which would be reliable already to the lowest orders of perturbation theory. In a way, this is similar to the asymptotic freedom, only the charges stop at finite UV-limit values rather than tend to zero.

Table 3: The RG fixed points in the non-abelian Chern–Simons field theory with matter fields in the fundamental representation of $SU(3)$

#	α/e^2	β/e^2	$\|\gamma\|/e^2$	h/e^4	stability	X_M/e^4	X_q/e^4
1	0.83333	0.16667	0	0.11111	IR stable	0	17.111
2	0.83333	0.16667	0	−0.15833	saddle point	0	−15.222
3	0.059713	0.69359	0	0.11118	saddle point	−1.2103	7.4412
4	0.059713	0.69359	0	−0.052556	saddle point	−1.2103	−12.207
5	0.036656	−1.0951	0	0.14555	saddle point	1.4364	19.054
6	0.036656	−1.0951	0	−0.14744	saddle point	1.4364	−16.106
7	−0.80436	0.74065	0	0.12006	saddle point	−0.065663	13.690
8	−0.80436	0.74065	0	−0.11192	saddle point	−0.065663	−14.147
9	−0.90989	−0.18390	0	0.16644	IR stable	0.63381	26.871
10	−0.90989	−0.18390	0	−0.24475	saddle point	0.63381	−22.471
11	0.58789	−0.81555	1.8932	0.10555	IR stable	4.2843	22.014
12	0.58789	−0.81555	1.8932	−0.16884	saddle point	4.2843	−10.913
13	0.39350	−0.83617	1.1718	0.10031	saddle point	1.1102	11.876
14	0.39350	−0.83617	1.1718	−0.084415	saddle point	1.1102	−10.292
15	0.39488	−0.27178	0.87690	0.055222	saddle point	−1.6010	−0.27609
16	0.39488	−0.27178	0.87690	0.011050	saddle point	−1.6010	−5.5767
17	0.20632	−0.10741	0.70966	0.072576	saddle point	−2.4304	−1.0463
18	0.20632	−0.10741	0.70966	0.018986	saddle point	−2.4304	−7.4771
19	−0.82408	0.80331	0.86980	0.11283	saddle point	1.2818	16.516
20	−0.82408	0.80331	0.86980	−0.13402	saddle point	1.2818	−13.105
21	0.18290	−0.48377	1.3009	0.10056	saddle point	−0.16890	8.2730
22	0.18290	−0.48377	1.3009	−0.055247	saddle point	−0.16890	−10.423
23	0.27352	−0.66828	1.1609	0.10016	saddle point	0.13602	9.0836
24	0.27352	−0.66828	1.1609	−0.061763	saddle point	0.13602	−10.348

The $N=2$ supersymmetric fixed point is exceptional. There are no UV divergencies in the gauge-invariant quantities at this point. For the $SU(n)$ group (# 1), it is always IR stable and gives the gaussian critical exponents. However, even the $N=1$ supersymmetry does not provide a stable path to this point (# 4 of table 1) in the two-parametric space of the $SO(n)$ charges.

The nonsupersymmetric $SU(n)$ fixed point # 11 possesses a nonzero value of the γ coupling, which introduces the interaction that does not conserve the fermion number. This may lead to the existence of non-singlet two-fermion vacuum condensates, similar to the ones in the abelian model, considered in ref.[7].

Thus, we see that the quantum non-abelian Chern–Simons theories with matter present quite an interesting field of investigation and exhibit nontrivial properties to not less an extent than their abelian counterparts.

Table 4: The RG fixed points in the non-abelian Chern–Simons field theory with matter fields in the fundamental representation of $SU(n)$ as $n\to\infty$

#	α/e^2	β/e^2	$\|\gamma\|/e^2$	h/e^4	stability	X_M/e^4	X_q/e^4
1	1	0.5	0	0.25	IR stable	0	$8\,n$
2	1	0.5	0	−0.41667	saddle point	0	$-8\,n$
3	0.53664	0.86289	0	0.34077	saddle point	$-0.28992n$	$6.7505n$
4	0.53664	0.86289	0	−0.24593	saddle point	$-0.28992n$	$-7.3303n$
5	$-0.15/n$	−1.5	0	0.25	saddle point	$1.3333n$	$6.6667n$
6	$-0.15/n$	−1.5	0	−0.19444	saddle point	$1.3333n$	$-4\,n$
7	−1	0.5	0	0.25	saddle point	−1	$8\,n$
8	−1	0.5	0	−0.41667	saddle point	−1	$-8\,n$
9	−1.1679	−0.42638	0	0.30414	IR stable	$0.39444n$	$11.544n$
10	−1.1679	−0.42638	0	−0.62502	saddle point	$0.39444n$	$-10.755n$
11	1	−1.5	4	0.25	IR stable	$8\,n$	$24\,n$
12	1	−1.5	4	−1.0833	saddle point	$8\,n$	$-8\,n$
13	0.53954	−1.6469	1.5682	0.17234	saddle point	$3.1577n$	$9.6158n$
14	0.53954	−1.6469	1.5682	−0.36584	saddle point	$3.1577n$	$-3.3005n$
15	0.52054	−0.28184	1.3907	0.12366	saddle point	$-0.55477n$	$0.94220n$
16	0.52054	−0.28184	1.3907	−0.0010850	saddle point	$-0.55477n$	$-2.0517n$
17	0.33333	−0.16667	1.3333	0.17593	saddle point	$-0.88889n$	$0.88889n$
18	0.33333	−0.16667	1.3333	0.027778	saddle point	$-0.88889n$	$-2.6667n$
19	−1	0.5	$2/\sqrt{n}$	0.25	saddle point	$4.625/n$	$8\,n$
20	−1	0.5	$2/\sqrt{n}$	−0.41667	saddle point	$4.625/n$	$-8\,n$
21	$0.75/n$	−0.5	2	0.25	saddle point	−1	$4\,n$
22	$0.75/n$	−0.5	2	−0.083333	saddle point	−1	$-4\,n$
23	0.049649	−0.79486	1.5852	0.25360	saddle point	$0.016649n$	$4.1294n$
24	0.049649	−0.79486	1.5852	−0.089138	saddle point	$0.016649n$	$-4.0961n$
25	2.6201	−8.2243	0	1.4348	saddle point	$97.673n$	$255.24n$
26	2.6201	−8.2243	0	−11.696	saddle point	$97.673n$	$-59.895n$
27	1	−3.5	0	0.25	IR stable	$16\,n$	$40\,n$
28	1	−3.5	0	−1.75	saddle point	$16\,n$	$-8\,n$

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References

[1] A.S.Schwarz, Lett.Math.Phys. 2 (1978) 247; Commun.Math.Phys. 67 (1979) 1;
M.F.Atiyah, New invariants of three- and four-dimensional manifolds, in *The mathematical heritage of Hermann Weyl*, Proc.Symp.Pure Math. 48, ed. R.Wells (Providence, 1988);
E.Witten, Commun.Math.Phys. 121 (1989) 351

[2] D.J.Gross, R.D.Pisarski and L.G.Yaffe, Rev.Mod.Phys. 53 (1981) 43;
A.Linde, Rep.Progr.Phys. 42 (1979) 389

[3] S.Givrin, in *The quantum Hall effect*, ed. E.Prange and S.M.Givrin (Springer-Verlag, Berlin, 1986) 380

[4] see *e.g.* A.Zee, A theory of superconductivity based on fractional quantum statistics, in *Proc. XXV Int. Conf. on High Energy Physics*, ed. K.K.Phua and Y.Yamaguchi (Singapore, 1991) 206

[5] L.Alvarez-Gaumé, J.M.F.Labastida and A.V.Ramallo, Nucl.Phys. B334 (1990) 103;
E.Guadagnini, M.Martellini and M.Mintchev, Nucl.Phys. B336 (1990) 581;
A.Blasi and R.Collina, Nucl.Phys. B345 (1990) 472

[6] L.V.Avdeev, G.V.Grigoryev and D.I.Kazakov, Renormalizations in abelian Chern–Simons field theories with matter, CERN preprint, TH.6091/91 (Geneva, 1991); to appear in Nucl.Phys.B

[7] D.I.Kazakov, Mod.Phys.Lett. A7 (1992) 13

[8] F.Delduc, O.Piguet, C.Lucchesi and S.Sorella, Nucl.Phys. B346 (1990) 313

[9] E.Ivanov, Phys.Lett. B268 (1991) 203

[10] W.Siegel, Phys.Lett. B84 (1979) 193; B94 (1980) 37;
L.V.Avdeev and A.A.Vladimirov, Nucl.Phys. B219 (1983) 262

[11] S.J.Gates, M.T.Grisaru, M.Roček and W.Siegel, Superspace (Benjamin & Cummings, Reading, MA, 1983)

[12] J.Zinn-Justin, Quantum field theory and critical phenomena (Clarendon Press, Oxford, 1989)

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