сообщения
объединенного
института
ядерных
исследований
дубна

E2-92-20

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ON THE FIRST INTEGRALS OF GEODESICS

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1992

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О первых интегралах уравнений геодезических

Рассмотрены квадратичные по импульсам первые интегралы уравнений геодезических в римановых пространствах, допускающих различные виды геометрических симметрий. Исследованы некоторые возможности их применения и физической интерпретации.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Сообщение Объединенного института ядерных исследований. Дубна 1992

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On the First Integrals of Geodesics

First integrals of the geodesics in the Riemannian spaces admitting various geometric symmetries are considered. Some possibilities of their physical interpretation and application are investigated.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Communication of the Joint Institute for Nuclear Research. Dubna 1992

It is known that the existence of certain geometric symmetries in curved spaces leads to conservation laws expressible in the form of first integrals along geodesics in these spaces (i.e., along trajectories of the test particles). In [1] quadratic first integrals (QFI) of geodesics are obtained.

In this report the method of the covariant 3+1-decomposition of the 4-space is used to establish the physical meaning of the above mentioned first integrals as functions of the reference frame's (RF) characteristics and to give some examples of their possible applications.

Henceforth we shall use natural units ($c = 1$). Greek indices run from 0 to 3 and latin ones run from 1 to 3.

In order to describe a reference frame (RF) it is convenient to use the congruence of time-like world lines with the field of correspondent tangent vector $\tau^{\mu} = dx^{\mu}/d\tau$ where $d\tau$ is the proper time along the world lines of the RF congruence (see, for example, [2]). Let us write some formulae of projection formalism which we shall use in what follows. We have the following decomposition for metric tensor

$$g_{\mu\nu} = \tau_{\mu}\tau_{\nu} - h_{\mu\nu}, \qquad /1/$$

where $h_{\mu\nu}$ is the metric tensor of the physical 3-space. Further, we have

$$dx^{\mu} = \tau^{\mu}d\tau + dl^{\mu}, \qquad /2/$$

where $dl^2 = dl_{\rho}dl^{\rho} = h_{\rho\sigma}dl^{\rho}dl^{\sigma}$ and $d\tau = \tau_{\mu}dx^{\mu}$ is proper time along a world line of the RF congruence.

The "time" and the "space" components of an arbitrary vector are defined as

/3/ $$A^{(\tau)} = A_\mu \tau^\mu .$$

/4/ $$a_\rho = h_{\lambda\rho} A^\lambda .$$

The covariant derivative of the vector τ^ρ may be determined by a set of geometrical objects which characterise the RF, namely, the vorticity tensor $A_{\alpha\beta}$, the expansion of Born tensor $D_{\alpha\beta}$ and the acceleration F_α in the following way

/5/ $$\tau_{\mu;\nu} = A_{\mu\nu} - D_{\mu\nu} + \tau_\mu F_\nu ,$$

where

$$G_\mu = \tau_{\mu;\alpha}\tau^\alpha ,$$

$$D_{\mu\nu} = \frac{1}{2} h^\alpha{}_\mu h^\beta{}_\nu (\tau_{\alpha;\beta} + \tau_{\beta;\alpha}),$$

$$A_{\mu\nu} = \frac{1}{2} h^\alpha{}_\mu h^\beta{}_\nu (\tau_{\alpha;\beta} - \tau_{\beta;\alpha}).$$

Let $U^\mu = dx^\mu/ds$ be the tangent vector of the geodesic, i.e., 4-velocity vector of a free particle in gravitational field; s is affine parameter along geodesic. In correspondence with /2/ the decomposition of the vector "U" in the system "τ" is

/6/ $$U^\mu = \gamma(\tau^\mu + v^\mu) .$$

We denote $v^2 = -v_\rho v^\rho$ and $\gamma = \frac{d\tau}{ds} = (U^\mu \tau_\mu) = \sqrt{1 - v^2}$. Then $v^\mu = dl^\mu/ds$ is the evident covariant generalization of the usual notion of the relative 3-velocity.

Let us consider one example of symmetry, namely, the affine collineation (AC) defined as a point transformation

/7/ $$\bar{x}^\mu = x^\mu + \xi(x)\delta\varepsilon ,$$

where $\delta\varepsilon$ is infinitesimal and for the vectors ξ^μ the following conditions are satisfied

/8/ $$\mathcal{L}_\xi \Gamma^\mu_{\nu\rho} = \xi^\mu_{;\rho\nu} + \xi^\lambda R^\mu_{\rho\lambda\nu} = 0 ,$$

or equivalenttly

/9/ $$(\xi_{\mu;\nu} + \xi_{\nu;\mu})_{;\rho} = 0 ,$$

where $\mathcal{L}_\xi$ indicates the Lie-derivative with respect to the vector ξ. This implies that

/10/ $$I_{AC} = (\xi_{\mu;\nu} + \xi_{\nu;\mu})U^\mu U^\nu = \text{Const}$$

is a quadratic first integral of the geodesics.

In the spaces which admit AC it is convenient to choose as a RF the congruence "τ" satisfiyng /9/. It follows immediately that $\tau^\rho{}_{;\rho} = D$.

Using /5/ and /6/ we obtain

/11/ $$I_{AC} = \gamma^2 (F_\nu v^\nu - D_{\mu\nu} v^\mu v^\nu).$$

In (2) the 3+1-decomposition of the geodesic's equation is given. Its "time part" is

/12/ $$\frac{dm}{d\tau} = p^\mu F_\mu - D_{\alpha\beta} p^\alpha v^\beta ,$$

where $p^\mu = mv$ are covariant 3-momentum components, $m = m_0 \frac{\tau_\alpha dx^\alpha}{ds} = \frac{m_0}{\gamma}$ is the dynamical mass and m_0 is the rest mass of the considered test particle.

The comparison of /11/ and /12/ implies the simple expression

/13/ $$\frac{dm}{d\tau} = \frac{I_{AC}\, m}{\gamma^2} = \frac{\text{Const}.m}{\gamma^2} .$$

In the case of the projective collineation (PC) the conditions on the vectors "ξ" are

/14/ $$(\xi_{\mu;\nu} + \xi_{\mu;\nu})_{;\rho} = b_{\mu\nu;\rho} = 2g_{\mu\nu}\phi_{,\rho} + g_{\nu\rho}\phi_{,\mu} + g_{\mu\rho}\phi_{,\nu}$$

and the quadratic first integral has the form

/15/ $$I_{PC} = (b_{\mu\nu} - 4\phi g_{\mu\nu})U^\mu U^\nu = \text{Const}.$$

where $\phi = (n + 1)^{-1}\xi^\lambda{}_{;\lambda} = \frac{1}{5} D$ in our case.

Making use of the fact that the norm of U is constant along a geodesic we have the following expressions

/16/ $$I_{PC} = (I_{AC} - c_2 D),$$

/17/ $$\frac{dm}{d\tau} = \frac{m}{\gamma^2}(c_1 - c_2 D),$$

where c_1 and c_2 are constants.

Hence we can get from /17/ that in the PC-case the dependence between proper times along the geodesics and the world lines of the RF depends on the volume expansion D.

world lines of the RF depends on the volume expansion D.

Let us now make some remarks on the problem of test particles' trajectories modelling. This method is based on the fact that one can introduce spaces with various metrics and connections on the same manifold and on the local character of tensor and vector quantities([3]). An interesting case are the trajectories of charged particles in given space V_4. They may be modelled by means of geodesics in some other space $\bar{V}_4$ chosen appropriately on the same manifold (See [4]).

For instance, let the metric $g_{\alpha\beta}$ of V_4 and the electromagnetic tensor $\Phi_{\alpha\beta}$ admit motion in some vector field's direction. Then the Lie-derivative of $\bar{V}_4$ connection in the same direction is equal to zero, i.e. this space admits AC (see [4]). Evidently it is more convenient to solve equations of motion as geodesic's equations in $\bar{V}_4$ and to use the QFI existence in the same way as above. The connection between the parameter s of the integral curve of charged particle's equations of motion $x = x(s)$ and the affine parameter $\bar{s}$ along the geodesic in $\bar{V}_4$ can be obtained by comparing correspondent equations

$$/18/ \qquad \frac{DU^\alpha}{ds} = U^\alpha_{;\rho}U^\rho = q\Phi^\alpha{}_\rho U^\rho ,$$

$$/19/ \qquad \frac{D\bar{U}^\alpha}{d\bar{s}} = \bar{U}^\alpha_{;\rho}\bar{U}^\rho = 0 .$$

Right-hand side of /18/ is the Lorentz force, $\Phi^\alpha{}_\rho$ is electromagnetic tensor and $\bar{U}^\rho = \frac{dx^\rho}{d\bar{s}}$. By definition in both cases the trajectory is the same and $s = s(\bar{s})$. It follows

$$/20/ \qquad \frac{d^2x^\alpha}{ds^2} = \frac{1}{\bar{s}'^2}\left(\frac{d^2x^\alpha}{ds^2} - \bar{s}''\frac{dx^\alpha}{ds}\right)$$

and taking into account that vector U^α is time-like and $U^\alpha U_\alpha = 1$ we get

$$/21/ \qquad \frac{\bar{s}''}{\bar{s}'} = \Pi_{\lambda,\mu\rho}U^\lambda U^\mu U^\rho ,$$

where $\Pi_{\lambda,\mu\rho} = \bar{\Gamma}_{\lambda,\mu\rho} - \Gamma_{\lambda,\mu\rho}$ is a tensor called "deformation of connection", $\bar{s}'' = \frac{d^2\bar{s}}{ds^2}$, $\bar{s}' = \frac{d\bar{s}}{ds}$.

For a cloud of non-interacting charged particles it may easily be obtained

$$/22/ \qquad \Pi_{\lambda,\mu\rho} = q(U^\lambda\Phi_{\mu\rho} + U_\rho\Phi_{\mu\lambda})$$

which follows from the symmetry properties of the Christoffel symbols $\Gamma^\sigma_{\mu\nu}$ and the electromagnetic tensor components $\Phi_{\mu\rho}$ (see /4/). Obviously in this special case $\bar{s}'' = 0$, $\bar{s}' =$ Const.

In /5/ the close connection between first integrals of deviation equations in the spaces admitting motions and QFI of geodesics in these spaces has been pointed out. Let us write the following identity

$$/23/ \qquad \frac{D^2\xi^\alpha}{ds^2} = \frac{DV^\alpha}{ds} = R^\alpha{}_{\kappa\lambda\sigma}U^\kappa U^\lambda\xi^\sigma + \xi^\alpha{}_{;\sigma}f^\sigma + U^\lambda U^\sigma \mathcal{L}_\xi \Gamma^\alpha_{\lambda\sigma} ,$$

where f^σ is a non-gravitational force acting on the considered particles. From /23/ various equations of deviation can be obtained through imposing certain conditions on the considered vectors or on the spaces' characteristics.

"First integrals" of /23/ may be written as follows

$$/24/ \qquad V_\alpha = U^\rho b_{\alpha\rho} - \xi_{\rho;\alpha}U^\rho ,$$

or

$$/25/ \qquad V^\alpha = U^\alpha{}_{;\rho}\xi^\rho - \mathcal{L}_\xi U^\alpha .$$

Identifying vectors "ξ" and "τ" and using /5/ and /6/ one can show that /24/ takes the form

$$/26/ \qquad V_\alpha = \gamma v^\rho(A_{\rho\alpha} + D_{\rho\alpha}) .$$

When the basic trajectory is geodesic and the space in consideration admits AC we have

$$/26/ \qquad \frac{D^2\xi^\alpha}{ds^2} = \frac{DV^\alpha}{ds} = R^\alpha{}_{\kappa\lambda\sigma}U^\kappa U^\lambda\xi^\sigma$$

/27/ $$V_\alpha U^\alpha = -\frac{1}{2}\, b_{\rho\sigma} U^\rho U^\sigma = -\frac{1}{2}\, I_{AC} \cdot$$

So in the case of the AC $\frac{D}{ds}\,(V_\alpha U^\alpha) = 0$. If we consider projective collineation then from /15/, /25/ and /28/ we obtain the following relations

/28/ $$U_\alpha \mathcal{L}_\xi U^\alpha = U_\alpha V^\alpha = D + \text{Const}.$$

When the expansion D is constant this case reduces to the AC. Another example of space's symmetry is special curvature collineation (SCC). A space admits SCC if the requirement

/29/ $$b_{\alpha\beta;\rho\sigma} = 0$$

holds. To this symmetry the following cubic first integral of geodesics corresponds

/30/ $$b_{\alpha\beta;\rho} U^\alpha U^\beta U^\rho = \text{Const}.$$

By use of /30/, /31/, /28/ and /25/ one can easily verify that in this case for $\gamma = \frac{d\tau}{ds}$ the following requirement is satisfied

/31/ $$\frac{D^2}{ds}(U_\sigma \mathcal{L}_\xi U^\sigma) = \frac{D^2}{ds^2}\,(U_\sigma \xi^\sigma) = \frac{D^2\gamma}{ds^2} = \text{Const}.$$

Finally, let us consider the case when the basic trajectory is non-geodesic and for acting force the following condition holds

/32/ $$\mathcal{L}_\xi U^\alpha = \gamma f^\alpha;\;\; \gamma = U_\rho \xi^\rho$$

/see [5]/. By means of /5/ and /6/ one can obtain for V^α

/33/ $$V^\alpha = \gamma v^\sigma (A^\alpha{}_\sigma + D^\alpha{}_\sigma) \cdot$$

On the other hand, if we put $\gamma_{\alpha\beta} = U_\alpha U_\beta - g_{\alpha\beta}$ and consider the RF corresponding to this decomposition with the proper time ds and characteristics $\tilde{A}_{\mu\nu}$ and $\tilde{D}_{\mu\nu}$, then obviously $\gamma = \frac{d\tau}{ds}$, $\tilde{A}_{\mu\nu} U^\mu = \tilde{D}_{\mu\nu} U^\nu = 0$ and we have

/34/ $$V^\alpha = \frac{v^\rho}{\gamma}(\tilde{A}^\alpha{}_\rho + \tilde{D}^\alpha{}_\rho) \cdot$$

In the relativistic theory it is often convenient to consider motions satisfying particular kinematic conditions, e.g. rigid or irrotational motions. It has been pointed out in [6] that this problem can be reduced to the analysis of a space-time carrying the pair of time-like congruences. In general case one can assume that we are given information on these congruences' characteristics while the relative velocity of motion v^α is unknown. Usually expressions which connect the above mentioned characteristics are partial differential equations with respect to unknown functions v^α. In this special case, however, the problem is simplified and comparing the expressions /33/ and /34/ one can determine the v^α algebraically. More detailed consideration of such a problems will be the subject of further report.

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Received by Publishing Department
on February 4, 1992.