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EFFECTIVE POTENTIAL IN $g\varphi_2^4$ THEORY

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1 Introduction

The phenomenon of spontaneous symmetry breaking, or in other words, the vacuum structure arrangement is an important part of many quantum field constructions. The simplest example, where the vacuum exhibits a nontrivial structure, is the φ_2^4 theory. Many papers [1-8] are devoted to the investigation of vacuum phase structure in this model. But their results are in contraduction. We shortly treat some nonperturbative methods seemed to be basic among investigations on this subject. An original approximation [1] using a Hartree-type renormalization exhibits a first order phase transition in this theory. A similar result was obtained [2] within the Gaussian effective potential approach. The dimensionless critical coupling constant, for which the first order phase transition takes place is $G = 1.62$ in both papers. These conclusions contradict mathematical theorems [3,4] proving that a second order phase transition occurs in the φ_2^4 model. There are papers [5-8], where different variational methods have been used for solving this problem and a second order phase transition has been observed in the region $G \sim 1$. In the previous studies [9,10], we have shown that the critical coupling constant leading to a second order phase transition cannot exceed the value $G_o = 1.4392$ and may be found near $G_{crit} \sim 0.53$.

In this paper, we consider the scalar field theory with the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\varphi(\mathbf{x})\cdot[\,\partial^2 - m^2\,]\cdot\varphi(\mathbf{x}) - \frac{g}{4}\cdot N_m\{\varphi^4(\mathbf{x})\}\,, \qquad (1.1)$$

where the normal product N_m of the fields $\varphi(\mathbf{x})$:

$$N_m\{\varphi^4(\mathbf{x})\} = \varphi^4(\mathbf{x}) - 6\cdot\varphi^2(\mathbf{x})\cdot D_m(0) + 3\cdot D_m^2(0)\,, \qquad (1.2)$$

$$D_m(\mathbf{x}) = \int \frac{d\mathbf{k}}{(2\pi)^2} \frac{\exp\{i\mathbf{k}\mathbf{x}\}}{m^2 + \mathbf{k}^2}.$$

removes all the divergences in this superrenormalizable theory. Here $\mathbf{x} = (x_1, x_2) \in \Omega$ and Ω is a finite volume in $\mathbf{R}^2$. Both the mass parameter m and coupling constant g are positive. At small g the Lagrangian (1.1) describes a system invariant with respect to the transformation $\varphi \leftrightarrow -\varphi$. The question is whether this symmetry remains for increasing g.

We will investigate this problem within the method of effective potential utilizing the techniques introduced in [11,12]. We demonstrate that the first order phase transition exhibited in [1,2] does not occur due to nontriviality of the obtained Gaussian effective potential. We shown also a possible occurrence of a second order phase transition and gave an estimation for the corresponding critical coupling constant.

2 Effective Potential

In this section, we will investigate the effective potential defined [13] as

$$\begin{aligned} V(\varphi_o) &= -\lim_{\Omega\to\infty} \frac{1}{\Omega} \ln I_\Omega(\varphi_o), \\ I_\Omega(\varphi_o) &= C_m \int \delta\varphi \cdot \delta\{\varphi_o - \frac{1}{\Omega}\int_\Omega d^2\mathbf{x}\varphi(\mathbf{x})\} \exp \int_\Omega d^2\mathbf{x}\cdot \mathcal{L}[\varphi(\mathbf{x})], \quad (2.1) \\ C_m &= \sqrt{det\{-\partial^2 + m^2\}}, \\ I_\Omega(\varphi_o, g=0) &= 1. \end{aligned}$$

We work in the Euclidean space. The functional integral in (2.1) is well defined at small g. The absolute minimum of the effective potential $V(\varphi_o)$ at the point $\varphi_o = \varphi_*$ determines the true ground state (vacuum) energy in the theory. For $g \ll 1$ the minimum is disposed at the origin: $\varphi_* = 0$. As g increases, there may occur new nontrivial minima $\varphi_* = \pm a$, which means the appearance of a phase transition in this system.

Let us do some transformations of the functional integral $I_\Omega(\varphi_o)$ in (2.1). First, we introduce [12] a transformation of the field variable

$$\varphi(\mathbf{x}) = \phi_o + \phi(\mathbf{x}) + b(\mathbf{x}), \quad (2.2)$$

where ϕ_o is a constant field and the new field variable $\phi(\mathbf{x})$ and an arbitrary function $b(\mathbf{x})$ satisfy the conditions

$$\int_\Omega d^2\mathbf{x}\,\phi(\mathbf{x}) = 0, \quad \int_\Omega d^2\mathbf{x}\, b(\mathbf{x}) = 0 \quad \text{and} \quad b^2(\mathbf{x}) = b^2. \quad (2.3)$$

We can substitute (2.2) into (2.1) and perform integration over $d\phi_o$ taking into account the functional differential $\delta\varphi = d\phi_o\delta\phi$. Then, we obtain

$$I_\Omega(\varphi_o) = C_m \int \delta\phi\cdot \exp\Big\{\int_\Omega d^2\mathbf{x}\cdot\mathcal{L}[\varphi_o + \phi(\mathbf{x}) + b(\mathbf{x})]\Big\}. \quad (2.4)$$

Second, we go over to the normal ordering in the new fields $\phi(\mathbf{x})$ with a new mass μ using the well-known [14] formula as

$$\begin{aligned} N_m\Big\{\exp\{\beta\varphi(\mathbf{x})\}\Big\} &= N_\mu\Big\{\exp\{\beta(\varphi_o + b(\mathbf{x}) + \phi(\mathbf{x})) + \frac{\beta^2}{2}\Delta(m,\mu)\}\Big\}, \\ \Delta(m,\mu) &= D_m(0) - D_\mu(0), \quad (2.5) \\ D_\mu(\mathbf{x}) &= \int \frac{d\mathbf{k}}{(2\pi)^2}\frac{\exp\{i\mathbf{k}\mathbf{x}\}}{\mu^2 + \mathbf{k}^2} - \frac{1}{\mu^2\Omega}. \end{aligned}$$

Then, substituting (2.5) into (2.4) we get

$$\begin{aligned} I_\Omega(\varphi_o) &= e^{-\Omega V_o(\varphi_o)} \int d\sigma_\mu \cdot \exp\Big\{\int_\Omega d^2\mathbf{x} \\ & N_\mu\Big\{\frac{1}{2}\cdot[\mu^2 - m^2 + 3g(\Delta - \varphi_o^2)]\cdot\phi^2(\mathbf{x}) \\ & - \frac{g}{4}\Big[\phi^4(\mathbf{x}) + 4(\varphi_o + b(\mathbf{x}))\phi^3(\mathbf{x}) + 12\varphi_o b(\mathbf{x})\phi^2(\mathbf{x})\Big]\Big\}\Big\}, \quad (2.6) \end{aligned}$$

$$\int d\sigma_\mu = C_\mu \int \delta\phi \cdot \exp\left\{-\frac{1}{2}\int_\Omega d^2\mathbf{x}\phi(\mathbf{x})(-\partial^2+\mu^2)\phi(\mathbf{x})\right\} = 1,$$

where a leading term $V_o(\varphi_o)$ of the effective potential is introduced:

$$V_o(\varphi_o) = -\frac{1}{2}\int\frac{d\mathbf{k}}{(2\pi)^2}\left[\ln\left(1+\frac{m^2-\mu^2}{\mu^2+\mathbf{k}^2}\right)-\frac{m^2-\mu^2}{\mu^2+\mathbf{k}^2}\right]$$
$$+\frac{m^2}{2}(\varphi_o^2+b^2)+\frac{g}{4}(\varphi_o^4-6\Delta(\varphi_o^2+b^2)+3\Delta^2+6\varphi_o^2b^2+b^4). \quad (2.7)$$

We require [12] that all the quadratic field configurations be concentrated in the Gaussian measure $d\sigma_\mu$ and linear terms should be absent. The requirements lead to the following constraint equations for the parameters b and μ:

$$\begin{cases} b(\mathbf{x})[m^2-3g(\Delta-\varphi_o^2)+gb^2] = 0, \\ \mu^2-m^2+3g(\Delta-\varphi_o^2-b^2) = 0. \end{cases} \quad (2.8)$$

Thus, we finally obtain the formula for the effective potential

$$\begin{aligned} V(\varphi_o) &= V_o(\varphi_o) + V_{sc}(\varphi_o), \\ V_{sc}(\varphi_o) &= -\lim_{\Omega\to\infty}\frac{1}{\Omega}\ln J_\Omega(\varphi_o), \end{aligned} \quad (2.9)$$

where

$$J_\Omega(\varphi_o) = \int d\sigma_\mu \cdot \exp\left\{-\frac{g}{4}\int_\Omega d^2\mathbf{x}N_\mu\left[\phi^4(\mathbf{x})+4(\varphi_o+b(\mathbf{x}))\phi^3(\mathbf{x})\right]\right\}. \quad (2.10)$$

Eqs (2.7)-(2.10) define self-consistently the effective potential at arbitrary coupling g.

We note that in the particular case of trivial $b(\mathbf{x})=0$ our leading term $V_o(\varphi_o)$ and constraint equation (2.8) reproduce the Hartree-type potential and the corresponding condition of its minimum on the parameter μ , obtained in [5] . This coincidence may be explained by our particular choices of linear field transformation (2.2) and Gaussian type of measure $d\sigma_\mu$ in (2.6). It is well known [1,2] that the potential (2.7) for trivial $b(\mathbf{x})=0$ within the limitation for μ by (2.8) corresponds to the sum of 'cactus-type' diagrams and indicates in favor of a first-order phase transition at

$$\left(\frac{g}{m^2}\right)_* = 10.211 \quad \text{or} \quad G_o = \left(\frac{g}{2\pi m^2}\right)_* = 1.6251 \quad (2.11)$$

3 Gaussian Approximation

Now we will investigate the Gaussian Part (2.7) of the effective potential whose parameters $b(\mathbf{x})$ and μ are limited by constraint Eqs(2.8).

It will be convenient to work in units of m dealing with numerical results. We define

$$\xi = (\mu/m)^2, \quad \Phi_o{}^2 = 4\pi\varphi_o^2 \quad \text{and} \quad B^2 = 4\pi A^2. \quad (3.1)$$

Then, (2.7) becomes

$$V_o(\Phi_o) = \frac{m^2}{8\pi}\Big\{\xi - 1 - ln\xi + \Phi_o{}^2 + B^2$$
$$+\frac{G}{4}\left[\Phi_o{}^4+B^4+3ln^2\xi+6(B^2\Phi_o{}^2-B^2ln\xi-\Phi_o{}^2ln\xi)\right]\Big\}. \quad (3.2)$$

We note that the potential (3.2) is invariant for $\Phi_o \leftrightarrow B$.

The parameters ξ and B in (3.2) are limited by the following equations:

$$\begin{cases} B^2(\xi-GB^2)=0, \\ 2\xi-2+3G(ln\xi-\Phi_o{}^2-B^2)=0. \end{cases} \quad (3.3)$$

Let us consider the constraint Eqs(3.3). A pair of "trivial" solutions:

$$B=0 \quad \text{and} \quad \xi = 1+\frac{3G}{2}(ln\xi-\Phi_o{}^2) \quad (3.4)$$

can be found for an arbitrary coupling constant G. Since $G > G_o = 1.4392$ an additional pair of "nontrivial" solutions

$$B=\frac{\xi}{G} \quad \text{and} \quad \xi = -2+\frac{3G}{2}(ln\xi-\Phi_o{}^2) \quad (3.5)$$

appears here too. So for $G < G_o$ the only solution to be substituted into (3.2) is the "trivial" one, but since $G > G_o$ there is an alternative:

one can choose either (3.4) or (3.5). We choose the pair obeying the lowest value of $V_o(\Phi_o)$ for certain fixed Φ_o.

All necessary calculations can be performed numerically. The obtained potential $V_o(\Phi_o)$ is plotted in Fig.1. Near the origin $\Phi_o = 0$ the potential $V_o(\Phi_o)$ is presented by the "nontrivial" branch (if $G > G_o$) $B \neq 0$ as it is situated lower than the "trivial" one. But for larger values of Φ_o the "trivial" solution $B = 0$ provides the lowest value of the potential. This picture leads to an interesting result. Let us consider the local minima of both branches. For $B = 0$ the minimum point $\Phi_o = A$ in Fig.1 is given by the equations

$$\left\{ \begin{aligned} B &= 0\,, \\ 2 - 3Gln\xi + G{\Phi_o}^2 &= 0\,. \end{aligned} \right. \tag{3.6}$$

On the other hand, the minimum of the "nontrivial" branch $B \neq 0$ is fixed at the origin $\Phi_o = 0$ for any $G > G_o$ and (3.3) becomes

$$\left\{ \begin{aligned} \Phi_o &= 0\,, \\ 2 - 3Gln\xi + GB^2 &= 0\,. \end{aligned} \right. \tag{3.7}$$

Due to the invariance of the potential $V_o(\Phi_o, B)$ in (3.2) for $\Phi_o \leftrightarrow B$ our Eqs (3.6) and (3.7) are identical. In other words, the minima of the potential (3.2) corresponding to different solutions of (3.3) are equal. The vacuum with $< \Phi(\mathbf{x}) >= \Phi_o \neq 0$ is not lower than the initial at the origin $< \Phi(\mathbf{x}) >= \Phi_o = 0$. There is no reason for occurrence of a first order phase transition.

4 Non-Gaussian Corrections

In the previous section, we have derived the expression for the effective potential consisting of two parts. Considering of only the "leading" term $V_o(\varphi_o)$ says one nothing about the nature of a phase transition in this theory. To answer this question one should consider also the remaining part $V_{sc}(\varphi_o)$ of the effective potential, defined in (2.9). At weak coupling limit one can estimate it expanding the exponential in (2.10) in perturbative series. But explicit calculation of the non-Gaussian functional integral $J_\Omega(\varphi_o)$ in (2.10) at arbitrary values of the coupling constant g and φ_o is a complicated problem. However, we are able to estimate it for small values of φ_o at arbitrary g.

We rewrite (2.10) in the form correct for small φ_o:

$$\begin{aligned} J_\Omega(\varphi_o) = \int d\sigma_\mu \cdot \exp\Big\{ &-\frac{g}{4}\int_\Omega d^2\mathbf{x} N_\mu\Big[\phi^4(\mathbf{x}) + 4b(\mathbf{x})\phi^3(\mathbf{x})\Big] \\ &+ \frac{g^2\varphi_o^2}{2}\Big[\int_\Omega d^2\mathbf{x} N_\mu(\phi^3(\mathbf{x}) + 3b(\mathbf{x})\phi^2(\mathbf{x}))\Big]^2\Big\}. \end{aligned} \tag{4.1}$$

This representation can easily be obtained due to the validity of the following transformation in (2.10):

$$\exp(-\varphi_o W) = cosh(\varphi_o W) \simeq \exp\Big\{\frac{1}{2}\varphi_o^2 W^2 + O(\varphi_o^4)\Big\}$$

for infinitesimal φ_o and finite functional W .

Applying to (4.1) Jensen's inequality we get an upper bound

$$\begin{aligned} V_{sc}(\varphi_o) \le V_{sc}^+(\varphi_o) = -\frac{g^2\varphi_o^2}{2\Omega}\int_\Omega d^2\mathbf{x}\int_\Omega d^2\mathbf{y}\int d\sigma_\mu\Big\{ \\ N_\mu\phi^3(\mathbf{x})N_\mu\phi^3(\mathbf{y}) + 9b(\mathbf{x})b(\mathbf{y})N_\mu\phi^2(\mathbf{x})N_\mu\phi^2(\mathbf{y})\Big\}. \end{aligned} \tag{4.2}$$

It is easy to show that

$$\begin{aligned} \int d\sigma_\mu N_\mu\phi^3(\mathbf{x})N_\mu\phi^3(\mathbf{y}) &= 6D_\mu^3(\mathbf{x}-\mathbf{y})\,, \\ \int d\sigma_\mu N_\mu\phi^2(\mathbf{x})N_\mu\phi^2(\mathbf{y}) &= 2D_\mu^2(\mathbf{x}-\mathbf{y})\,. \end{aligned} \tag{4.3}$$

Then, we rewrite (4.2) in the form

$$V_{sc}^+(\Phi_o) = -\frac{m^2}{8\pi}\cdot\frac{3G^2{\Phi_o}^2}{2\xi}(Q + 3B^2)\,, \tag{4.4}$$

$$Q = \iiint_0^1 d\alpha\, d\beta\, d\gamma\, \frac{\delta(1-\alpha-\beta-\gamma)}{\alpha\beta + \alpha\gamma + \beta\gamma} = 2.3439\,.$$

Substituting the parameters ξ and B in either (3.4) or (3.5) into (4.4) one gets the behavior of $V_{sc}^{+}(\Phi_o)$ for small values $\Phi_o \sim 0$. Omitting details of calculations we write the results

$$V_{sc}^{+}(\Phi_o) = -\frac{m^2}{8\pi}\cdot\left\{-\frac{3Q}{2}G^2{\Phi_o}^2 + O({\Phi_o}^4)\right\} \quad \text{for} \quad G < G_* \tag{4.5}$$

and

$$V_{sc}^{+}(\Phi_o) = -\frac{m^2}{8\pi}\cdot\left\{-\left[\frac{3QG^2}{2\xi}+\frac{9G}{2}\right]{\Phi_o}^2+O({\Phi_o}^4)\right\} \quad \text{for} \quad G > G_*, \tag{4.6}$$

$$3Gln\xi - \xi - 2 = 0\,.$$

From (3.2) we get the following asymptotic behavior:

$$V_o(\Phi_o) \;=\; \frac{m^2}{8\pi}\left\{{\Phi_o}^2 + O({\Phi_o}^4)\right\} \tag{4.7}$$

as $\Phi_o \to 0$ at any G.

Finally, taking into account Eq(2.9) we obtain the following behaviors of an upper bound of the effective potential in the region of small $\Phi_o \sim 0$:

$$V^{+}[\Phi_o] = V_o[\Phi_o] + V_{sc}^{+}[\Phi_o] = \frac{m^2}{8\pi}\cdot\left[\alpha(G){\Phi_o}^2 + O({\Phi_o}^4)\right], \tag{4.8}$$

where

$$\alpha(G) = \begin{cases} \alpha_1(G) = 1 - 3QG^2/2\,, & G \le 1.6251\,, \\ \alpha_2(G) = 1 - 3QG^2/(2\xi) - 9G/2\,, & G > 1.6251\,, \end{cases} \tag{4.9}$$

$$3Gln\xi - \xi - 2 = 0\,.$$

One can easily check that the coefficient $\alpha_1(G)$ in (4.9) becomes negative as $G > G_{crit} = 0.5333$ and remains negative for increasing G. But $\alpha_2(G)$ is negative at arbitrary $G > 1.4392$. In the author's opinion, it can indicate a possible occurrence of a second order phase transition in the model under consideration.

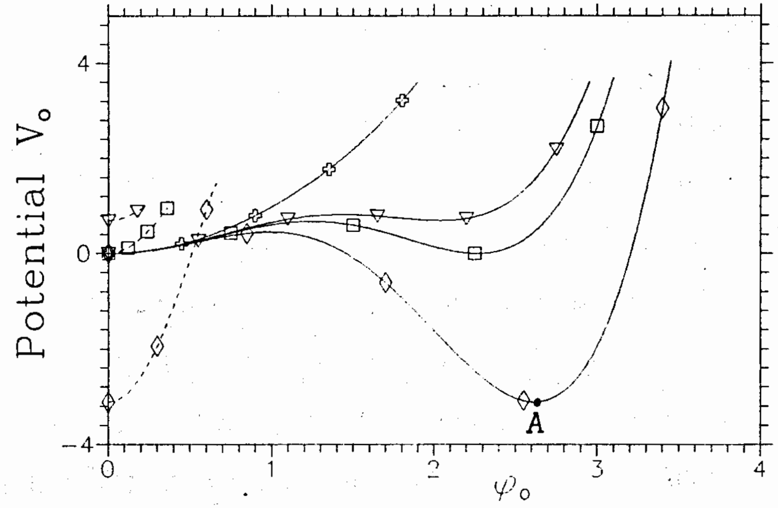

Fig.1. *The Gaussian part $V_o(\Phi_o)$ (in units of $m^2/8\pi$) of the effective potential as a function of Φ_o for different values of the coupling constant: crosses, $G = 0.5$; triangles, $G = 1.5$; squares, $G = 1.6251$ and rhombs, $G = 2.0$. The dashed lines represent the "nontrivial" branches. The "trivial" branches are denoted by the solid lines.*

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