сообщения
объединенного
института
ядерных
исследований
дубна

E2-92-19

S. Manoff*

# DEVIATION EQUATIONS OF SYNGE AND SCHILD IN SPACES WITH AFFINE CONNECTION AND METRIC, AND EQUATIONS FOR GRAVITATIONAL WAVES DETECTORS

*Bulgarian Academy of Sciences, Institute
for Nuclear Research and Nuclear Energy,
Boul. Trakia 72, 1784 Sofia - Bulgaria

1992

Манов С. E2-92-19

Уравнения девиации Синга - Шильда в пространствах с аффинной связностью и метрикой и уравнения для гравитационных волновых детекторов

Уравнение девиации Синга - Шильда рассмотрено в пространствах с аффинной связностью и метрикой. Условие $\mathcal{L}_{\xi}u = 0$, из которого следует это уравнение, является достаточным, но не необходимым условием. С помощью неизотропного векторного поля и ортогональной к нему метрики получено проекционное уравнение Синга - Шильда для другого векторного поля, ортогонального к первому. При заданном неизотропном (а точнее - времениподобном), автопараллельном и нормированном векторном поле это уравнение может принимать в специальных случаях форму осцилляторного уравнения.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Сообщение Объединенного института ядерных исследований. Дубна 1992

Manoff S. E2-92-19

Deviation Equations of Synge and Schild in Spaces with Affine Connection and Metric, and Equations for Gravitational Waves Detectors

Deviation equation of Synge and Schild has been investigated in spaces with affine connection and metric. It is shown that the condition $\mathcal{L}_{\xi}u = 0$ for obtaining this equation is only a sufficient (but not necessary) condition. By means of a non-isotropic vector field u ($g(u,u) = e \neq 0$) and the orthogonal to it metric $h_u$ a projected deviation equation of Synge and Schild has been obtained for the orthogonal to u vector field $\xi_{\perp}$ and its square $L^2 = g(\xi_{\perp},\xi_{\perp})$. For a given non-isotropic, autoparallel and normalized vector field u this equation could have in some special cases the form of an oscillator equation.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Communication of the Joint Institute for Nuclear Research. Dubna 1992

INTRODUCTION

1. In general relativity, as a basis for the theoretical scheme for gravitational waves detectors proposed by Weber[1] in 1958-1959 and discussed by many authors[2-5], the deviation geodesic equation (proposed by Levi-Civita in 1925[6]) in the form

$$\frac{D^2\xi^i}{ds^2} = R^i{}_{jkl}u^j u^k \xi^l \ , \qquad u^i{}_{;j}u^j = 0 \ , \tag{1'}$$

or in the indexfree form

$$\nabla_u \nabla_u \xi = [R(u,\xi)]u \ , \qquad a = \nabla_u u = 0 \ , \tag{1}$$

has been used. Its generalization for non-geodesic trajectories ($a \neq 0$) (which has been proposed by Synge and Schild[7] in 1956) in the form

$$\frac{D^2\xi^i}{ds^2} = R^i{}_{jkl}u^j u^k \xi^l + a^i{}_{;j}\xi^j \ , \qquad a^i = u^i{}_{;j}u^j \ , \tag{2'}$$

or in the indexfree form

$$\nabla_u \nabla_u \xi = [R(u,\xi)]u + \nabla_\xi a \ , \tag{2}$$

has also been used by Weber in a special form for construction of gravitational waves detectors of the type of massive cylinders reacting to periodical gravitational processes. The application of these equations in experiments for detecting gravitational waves turned the attention of many authors to considerations and proposals for new deviation equations. Two types of prerequisites for obtaining such equations are usually used:

a) Physical interpretation of deviation equations as equations for the relative acceleration between particles, moving on trajectories in (pseudo)Riemannian spaces without torsion ($V_n$-spaces), considered as models of space-time in general theory of relativity, or in relativistic continuum media mechanics[8-11].

b) Mathematical models for obtaining deviation equations by means of (covariant) differential operators, acting on vector fields in spaces with affine connection and metric ($L_n$-spaces) (special case:

(pseudo)Riemannian spaces with torsion ($U_n$-spaces) or without torsion).

In both types of methods problems arise, connected with the physical interpretation of the quantities defined in the equations as well as with finding exact solutions of the proposed equations. At the same time there are many tangential- and cross-points between these methods /12-15/

2. From mathematical point of view many of the proposed by different authors deviation equations can be obtained from the s.c. generalized deviation identity (generalized deviation equation) in $L_n$-spaces /13,14/

$$\nabla_u \nabla_u \xi = - [R(\xi,u)]u + \nabla_a \xi + T(\xi,a) - \nabla_u [T(\xi,u)] + [\pounds\Gamma(\xi,u)]u, \tag{3}$$

or in index form

$$(\xi^i{}_{/j} u^j)_{/k} u^k = R^i{}_{klj} u^k u^l \xi^j + \xi^i{}_{/j} a^j + T_{kl}{}^i \xi^k a^l - (T_{kl}{}^i \xi^k u^l)_{/j} u^j + + \pounds_\xi \Gamma^i_{kl} . u^k u^l , \tag{3'}$$

where

$a = \nabla_u u = u^i{}_{/j} u^j E_i = a^i E_i$ , $u, \xi \in T(M)$ ,

$R(\xi,u)$ is the curvature operator

$$R(\xi,u) = \nabla_\xi \nabla_u - \nabla_u \nabla_\xi - \nabla_{\pounds_\xi u} = [\nabla_\xi, \nabla_u] - \nabla_{[\xi,u]} , \tag{4}$$

$\pounds\Gamma(\xi,u)$ is the deviation operator /13/

$$\pounds\Gamma(\xi,u) = \pounds_\xi \nabla_u - \nabla_u \pounds_\xi - \nabla_{\pounds_\xi u} = [\pounds_\xi, \nabla_u] - \nabla_{[\xi,u]}, \tag{5}$$

$\pounds_\xi u$ is the Lie derivative of the vector field u along the vector field $\xi$,

$$\pounds_\xi u = \nabla_\xi u - \nabla_u \xi - T(\xi,u) = [\xi,u] , \tag{6}$$

$\nabla_u \xi$ is the covariant derivative of the vector field $\xi$ along the vector field u

$$\nabla_u \xi = \xi^i{}_{/j} u^j E_i = (E_j \xi^i + \Gamma^i_{kj} \xi^k) u^j E_i . \tag{7}$$

3. In general relativity (GR) notions such as shear (shear velocity) $\sigma$, rotation (rotation velocity) $\omega$ and expansion (expansion velocity) $\theta$ are used. These notions /16-18/ can be defined in $L_n$-spaces in analogous way, as in $V_n$- and $U_n$-spaces by means of representation of the covariant derivative of a vector field along (another) non-isotropic vector field u ($g(u,u) = e \neq 0$) in the form

$$\nabla_u \xi = \frac{1}{e}.g(u,\nabla_u \xi).u + \bar{g}[h_u(\frac{1}{e}.a - \pounds_\xi u)] + + \bar{g}[\sigma(\xi)] + \bar{g}[\omega(\xi)] + \frac{1}{n-1}.\theta.\bar{g}[h_u(\xi)] , \tag{8}$$

$$d = \sigma + \omega + \frac{1}{n-1}.\theta.h_u , \tag{9}$$

or in index form

$$\xi^i{}_{/j} u^j = \frac{1}{e}.g_{kl} u^k \xi^l{}_{/m} u^m . u^i + g^{ij}[h_{jk}(\frac{1}{e}.a^k - \pounds_\xi u^k) + + (\sigma_{jk} + \omega_{jk} + \frac{1}{n-1}.\theta.h_{jk})\xi^k] ,$$

$$d_{ij} = \sigma_{ij} + \omega_{ij} + \frac{1}{n-1}.\theta.h_{ij} .$$

Here

$h_u = g - \frac{1}{e}.g(u) \otimes g(u) = h_{ij} E^i . E^j$ , $g = g_{kl} E^k . E^l$,

$\sigma$ is the shear velocity tensor (shear):

$$\sigma = \sigma_{ij} E^i . E^j = \frac{1}{2}\{h_u(\nabla_u \bar{g} - \pounds_u \bar{g}) h_u - \frac{1}{n-1}(h_u[\nabla_u \bar{g} - \pounds_u \bar{g}])h_u\} , \tag{10}$$

$\omega$ is the rotation velocity tensor (rotation):

$$\omega = \omega_{ij} E^i \wedge E^j = h_u(k_a) h_u, \quad k_a = s - q, \quad s = \frac{1}{2}(u^k{}_{/n} g^{nl} - u^l{}_{/n} g^{nk}) E_k \wedge E_l ,$$

$$q = \frac{1}{2}(T_{mn}{}^k g^{ml} - T_{mn}{}^l g^{mk}) u^n . E_k . E_l , \tag{11}$$

$\theta$ is the expansion velocity invariant (expansion):

$$\theta = \frac{1}{2}.h_u[\nabla_u \bar{g} - \pounds_u \bar{g}] = \frac{1}{2}.h_{kl}(g^{kl}{}_{/m} u^m - \pounds_u g^{kl}) . \tag{12}$$

In this way the notions of shear, rotation and expansion are generalized for $L_n$-spaces. In analogous way (after some more complicated computations) for the second covariant derivative notions such as shear acceleration, rotation acceleration and expansion acceleration can be introduced in $V_n$-, $U_n$- and $L_n$-spaces. These notions can be also connected with the generalized deviation identity which can be written in the form /13/

$$\nabla_u \nabla_u \xi = \frac{1}{e}.g(u,\nabla_u \nabla_u \xi).u + \bar{g}[h_u(\nabla_u \nabla_u \xi)] , \tag{13}$$

$$\bar{g}[h_u(\nabla_u \nabla_u \xi)] = \bar{g}(h_u)[\frac{1}{e}.\nabla_u a - \nabla_{\pounds_\xi u} u - \nabla_u(\pounds_\xi u) + T(\pounds_\xi u, u)] + + \bar{g}[{}_sD(\xi) + W(\xi) + \frac{1}{n-1}.U.h_u(\xi)] , \tag{14}$$

or in index form

$$(\xi^i{}_{/j} u^j)_{/k} u^k = \frac{1}{e}.g_{kl} u^k (\xi^l{}_{/j} u^j)_{/m} u^m . u^i + g^{ij} h_{jk} (\xi^k{}_{/l} u^l)_{/m} u^m ,$$

$$g^{ij} h_{jk} (\xi^k{}_{/l} u^l)_{/m} u^m = g^{ij} h_{jk} [\frac{1}{e}.a^k{}_{/l} u^l - u^k{}_{/l} \pounds_\xi u^l - (\pounds_\xi u^k)_{/l} u^l + + T_{mn}{}^k \pounds_\xi u^m u^n] + g^{ij}({}_sD_{jk} + W_{jk} + \frac{1}{n-1}.U.h_{jk})\xi^k .$$

Here: ${}_sD = {}_{sF}D_o - {}_{sT}D_o + {}_sM$ is the shear acceleration tensor (shear acceleration) constructed by three terms: ${}_{sF}D_o$ is the curvature- and torsion-free shear acceleration, ${}_{sT}D_o$ is the shear acceleration, induced by torsion, ${}_sM$ is the shear acceleration, induced by curvature, $W = {}_FW_o - {}_TW_o + N$ is the rotation acceleration tensor (rotation acceleration) which has also three terms: ${}_FW_o$ is the curvature- and torsion-free rotation acceleration, ${}_TW_o$ is the rotation acceleration, induced by torsion, N is the rotation acceleration, induced by curvature, $U = {}_FU_o - {}_TU_o + I$ is the expansion acceleration invariant (expansion acceleration) with the three terms: ${}_F U_o$ is the curvature- and torsion-free expansion acceleration, ${}_T U_o$ is the expansion acceleration, induced by torsion, I is the expansion acceleration, induced by curvature (this term appears as a generalization of Raychaudhuri identity[18] for $L_n$-spaces).

By means of different representation of the generalized deviation identity possibilities can be considered for writing down theoretical schemes in gravitational theories (and particular in GR theory) for construction of gravitational waves detectors.

4. In the present paper the deviation equation of Synge and Schild is generalized for $L_n$-spaces and specialized for description of an orthogonal to the non-isotropic (timelike) vector field u variation of the second covariant derivative of vector field $\xi$ (which has been interpreted as a deviation vector). In sec. I. the generalized deviation equation of Synge and Schild and its projected form (projected deviation equation of Synge and Schild) is considered in $L_n$-spaces as well as an analogous deviation equation for the square of a non-isotropic (spacelike) vector is found and investigated. In sec. II. the projected deviation equation of Synge and Schild for the square of an autoparallel ($\nabla_u u = 0$) (non-isotropic) and normalized ($g(u,u) = e = \text{const.} \neq 0$) vector field u in $L_n$-, $U_n$- and $V_n$-spaces is considered and examples for the case of $V_n$-spaces are given which can lead to an equation in the form of an oscillator equation.

## I. DEVIATION EQUATION OF SYNGE AND SCHILD IN SPACES WITH AFFINE CONNECTION AND METRIC

1. The deviation equation of Synge and Schild in $L_n$-spaces can be obtained from the generalized deviation identity by means of the additional condition

$$\mathcal{L}_\xi u = 0 \text{ or } \mathcal{L}_\xi u^i = 0 \tag{15}$$

in the form

$$\nabla_u \nabla_u \xi = [R(u,\xi)]u + \nabla_\xi a - \nabla_u[T(\xi,u)] \ , \tag{16}$$

or in index form

$$(\xi^i{}_{/j} u^j)_{/k} u^k = R^i{}_{klj} u^k u^l \xi^j + a^i{}_{/j} \xi^j - (T_{kl}{}^i \xi^k u^l)_{/j} u^j \ .$$

At the same time the conditions

$$\nabla_u \xi = \nabla_\xi u - T(\xi,u) \text{ or } \xi^i{}_{/j} u^j = u^i{}_{/j} \xi^j - T_{kl}{}^i \xi^k u^l \tag{17}$$

and

$$[\mathcal{L}\Gamma(\xi,u)]u = \mathcal{L}_\xi a \text{ or } \mathcal{L}_\xi \Gamma^i_{jk}.u^j u^k = \mathcal{L}_\xi a^i \tag{18}$$

are fulfilled.

The way of getting the deviation equation of Synge and Schild gives the possibility for proving the following proposition:



<u>Proposition 1.</u> Every vector field $\xi$, which satisfies the equations $\mathcal{L}_\xi u = 0$ ($\mathcal{L}_\xi u^i = 0$) for an arbitrary vector field u is a solution of the deviation equation of Synge and Schild.

Proof: There are at least two ways[13,15] for proving this proposition: 1. The proof follows immediately from the generalized identity and the condition (15). 2. Expression (17) follows from (15) and after covariant differentiation along u with condition (15), the deviation equation (16) follows.

<u>Corollary:</u> The condition (15) is a "first integral" for the deviation equation of Synge and Schild (for arbitrary vector field u).

<u>Remark:</u> Under "first integral" here one can define a quantity whose covariant derivative along an arbitrary vector field u leads to the deviation equation of a concrete type (here of Synge and Schild).

<u>Proposition 2.</u> The necessary and sufficient condition for the existence of the deviation equation of Synge and Schild is the condition (18): $\mathcal{L}_\xi a = [\mathcal{L}\Gamma(\xi,u)]u$ or $\mathcal{L}_\xi a^i = \mathcal{L}_\xi \Gamma^i_{kl}.u^k u^l$ .

Proof: a) Necessity: From the identity (3) and (16) it follows (18). b) Sufficiency: From (18) and the identity (3), the deviation equation (16) follows.

<u>Remark:</u> In finding out deviation equations different authors used only sufficient (or "first integrals") condition for these equations (like those of proposition 1.). They don't take into account that the obtained equations can fulfill also other sufficient condition than the considered one (s. for example[13] and[15]).

2. The second covariant derivative of a vector field $\xi$ along a non-isotropic vector field u can be written in two parts: the one is collinear to u, the other is orthogonal to the vector field u (s. (13,14)). The second term can be interpreted as a relative acceleration between two points, lying on a hypersurface orthogonal to the vector field u. Since the (infinitesimal) vector has also to lie on this hypersurface, then in this case $\xi$ has to obey the condition

$$g(\xi,u) = 0 \ , \tag{19}$$

or $\xi$ has to be in the form

$$\xi_\perp = \bar{g}(h_u(\xi)) \ , \qquad g(\xi_\perp,u) = 0 \ . \tag{20}$$

The deviation equation which is obtained for $h_u(\nabla_u\nabla_u\xi)$ under the conditions

$$\mathcal{L}_{\xi_\perp} u = 0, \quad g(u,\xi_\perp) = 0 \ , \quad \xi_\perp = \bar{g}(h_u(\xi)) \tag{21}$$

can be defined as a projected deviation equation of Synge and Schild. It follows from (14) that this equation can be written in the form

$$\bar{g}[h_u(\nabla_u\nabla_u\xi_\perp) = \bar{g}[A(\xi_\perp)] = \bar{g}[{}_sD(\xi_\perp)] + \bar{g}[W(\xi_\perp)] + \frac{1}{n-1}.U.\xi_\perp \ , \tag{22}$$

or in index form

$$g^{ij}h_{jk}(\xi^k_\perp{}_{/l}u^l)_{/m}u^m = g^{ij}A_{jk}\xi^k_\perp =$$
$$= g^{ij}({}_sD_{jk} + W_{jk})\xi^k_\perp + \frac{1}{n-1}.U.\xi^i_\perp \ ,$$
where
$$\xi^k_\perp = g^{kl}h_{lm}\xi^m \ , \quad h_u(\xi_\perp) = h_u(\bar{g})h_u(\xi) = h_u(\xi) \ ,$$
$$\bar{g}[h_u(\xi_\perp)] = \bar{g}[h_u(\xi)] = \xi_\perp \ .$$

The equation (22) can also be written in an equivalent form
$$h_u(\nabla_u\nabla_u\xi_\perp) = {}_sD(\xi_\perp) + W(\xi_\perp) + \frac{1}{n-1}.U.g(\xi_\perp) \ . \qquad (23)$$
Every vector field $\xi_\perp$ (for arbitrary non-isotropic vector field u) which fulfills the conditions (21), is a solution of equation (22) or (23). Therefore the solution of equation $\pounds_{\xi_\perp}u = 0$ (or $\pounds_u\xi_\perp = 0$) for a vector field $\xi_\perp(x^k)$ and given vector field $u(x^k)$ is also a solution of the deviation equation (22). It follows in this case that if the components of the vector field $\xi = \xi^i E_i = \xi^k\partial_k$ should be solutions of a homogeneous (or nonhomogeneous) oscillator equation, then an additional equation for the vector field u has to be proposed, which could lead to such properties of $\xi$.

3. A deviation equation under the same condition (21), used for getting the deviation equation (22), can also be written for the square of $\xi_\perp$, i.e. for $g(\xi_\perp,\xi_\perp) = L^2 \neq 0$. If the vector field u is considered as a timelike vector field which is orthogonal to $\xi_\perp$, then $\xi_\perp$ could be interpreted as a spacelike vector field which length is considered as the length of a material object or the length of the distance between two particles, lying on an orthogonal to u hypersurface.

By means of the relations
$$\nabla_u\xi_\perp = {}_{rel}v + \bar{g}(\nabla_u h_u)(\xi) + (\nabla_u\bar{g})(h_u(\xi)) \ , \qquad (24)$$
$${}_{rel}v = \bar{g}[h_u(\nabla_u\xi)] = g^{ij}h_{jk}\xi^k{}_{/l}u^l E_i \ , \qquad (25)$$
$$\nabla_u\nabla_u\xi_\perp = {}_{rel}a + 2\bar{g}(\nabla_u h_u)(\nabla_u\xi) + \bar{g}(\nabla_u\nabla_u h_u)(\xi) + 2(\nabla_u\bar{g})g({}_{rel}v) +$$
$$+ 2(\nabla_u\bar{g})(\nabla_u h_u)(\xi) + (\nabla_u\nabla_u\bar{g})g(\xi_\perp) \ , \qquad (26)$$
$${}_{rel}a = \bar{g}[h_u(\nabla_u\nabla_u\xi)] = g^{ij}h_{jk}(\xi^k{}_{/l}u^l)_{/m}u^m E_i \ ,$$
$$(\nabla_u\bar{g})g({}_{rel}v) = g^{ij}{}_{/k}u^k g_{jl}\cdot{}_{rel}v^l E_i = (\nabla_u\bar{g})(g)({}_{rel}v) \ ,$$
the deviation equation for $L^2$ can be obtained in the form
$$u(uL^2) = 2[g(\xi_\perp,{}_{rel}a) + 2g(\xi_\perp,\bar{g}(\nabla_u h_u)(\nabla_u\xi)) + g(\xi_\perp,\bar{g}(\nabla_u\nabla_u h_u)(\xi)) +$$
$$+ 2g(\xi_\perp,(\nabla_u\bar{g})g({}_{rel}v)) + 2g(\xi_\perp,(\nabla_u\bar{g})(\nabla_u h_u)(\xi)) +$$
$$+ g(\xi_\perp,(\nabla_u\nabla_u\bar{g})g(\xi_\perp))] + 2[g({}_{rel}v,{}_{rel}v) + 2g({}_{rel}v,\bar{g}(\nabla_u h_u(\xi)) +$$
$$+ g(\bar{g}(\nabla_u h_u)(\xi),\bar{g}(\nabla_u h_u(\xi)) + 2g({}_{rel}v,(\nabla_u\bar{g})g(\xi_\perp)) +$$
$$+ 2g(\bar{g}(\nabla_u h_u)(\xi),(\nabla_u\bar{g})g(\xi_\perp)) + g((\nabla_u\bar{g})g(\xi_\perp),(\nabla_u\bar{g})g(\xi_\perp))] +$$



$$+ 4(\nabla_u g)(\xi_\perp,\nabla_u\xi_\perp) + (\nabla_u\nabla_u g)(\xi_\perp,\xi_\perp) \ . \qquad (27)$$

For $U_n$- and $V_n$-spaces ($\nabla_u g = 0$ for $\forall u \in T(M)$) this equation will have the form
$$u(uL^2) = 2[g(\xi_\perp,{}_{rel}a) + 2g(\xi_\perp,\bar{g}(\nabla_u h_u)(\nabla_u\xi)) +$$
$$+ g(\xi_\perp,\bar{g}(\nabla_u\nabla_u h_u)(\xi))] + 2[g({}_{rel}v,{}_{rel}v) +$$
$$+ 2g({}_{rel}v,\bar{g}(\nabla_u h_u)(\xi)) + g(\bar{g}(\nabla_u h_u)(\xi),\bar{g}(\nabla_u h_u)(\xi))] \ . \qquad (28)$$

If the additional condition
$$\nabla_u h_u = 0 \qquad (29)$$
is required, then the equation (28) for $L^2$ will have the form
$$u(uL^2) = 2[g(\xi_\perp,{}_{rel}a) + g({}_{rel}v,{}_{rel}v)], \qquad (30)$$
or in index form
$$((L^2)_{,i}u^i)_{,j}u^j = 2(g_{kl}\xi^k_\perp\cdot{}_{rel}a^l + g_{kl}\cdot{}_{rel}v^k\cdot{}_{rel}v^l) \ .$$

The next task is to consider the deviation equation for $L^2$ for autoparallel ($\nabla_u u = a = 0$), non-isotropic ($g(u,u) = e \neq 0$) and normalized ($e = \text{const.} \neq 0$) vector field u.

## II. PROJECTED DEVIATION EQUATION OF SYNGE AND SCHILD FOR $L^2$ IN THE CASE OF AUTOPARALLEL VECTOR FIELD u IN $U_n$-SPACES

1. If the condition for autoparallelism is given for the vector field u, i.e.
$$\nabla_u u = a = u^i{}_{/j}u^j E_i = a^i E_i = 0 \ , \qquad (31)$$
then by means of the expression for $\nabla_u h_u$ in $L_n$-spaces
$$\nabla_u h_u = \nabla_u g + \frac{1}{e}\Big\{\frac{1}{e}(ue).g(u)\otimes g(u) - [g(a)\otimes g(u) + g(u)\otimes g(a)] -$$
$$- [(\nabla_u g)(u)\otimes g(u) + g(u)\otimes(\nabla_u g)(u)]\Big\} \qquad (32)$$
the following propositions can be proved for the case of $U_n$-spaces:

<u>Proposition 3.</u> For a non-isotropic, normalized and autoparallel vector field u in $U_n$-space the condition
$$\nabla_u\xi_\perp = {}_{rel}v \qquad (33)$$
is fulfilled.

Proof: From the conditions $\nabla_u g = 0$ ($\nabla_u\bar{g} = 0$) ($U_n$-space), $g(u,u) = e = \text{const.} \neq 0$ (i.e. $ue = u^i\partial_i e = 0$), $\nabla_u u = a = 0$ and (32) it follows $\nabla_u h_u = 0$. From the last expression and (24) the expression (33) follows.

<u>Proposition 4.</u> For a non-isotropic, normalized and autoparallel vector field u in $U_n$-space the condition
$$\nabla_u\nabla_u\xi_\perp = {}_{rel}a = \nabla_u({}_{rel}v) \qquad (34)$$
is fulfilled.

Proof: 1. From proposition 4. after covariant differentiation along the vector field u it follows

$$\nabla_u\nabla_u\xi_\perp = \nabla_u({}_{rel}v).$$

2. From the definition (26) for ${}_{rel}a$ and $\nabla_u h_u = 0$ (from the prerequisites for u) one obtains

$${}_{rel}a = \bar{g}[\nabla_u\nabla_u(h_u(\xi))] = \nabla_u\nabla_u[\bar{g}(h_u(\xi))] = \nabla_u\nabla_u\xi_\perp .$$

Proposition 5. For a non-isotropic, normalized and autoparallel vector field u in $U_n$-space the condition for $L^2 = g(\xi_\perp,\xi_\perp)$

$$u(uL^2) = 2[g(\xi_\perp,{}_{rel}a) + g({}_{rel}v,{}_{rel}v)] \qquad (35)$$

is fulfilled.

Proof: The condition (35) follows immediately from (28) and (29)(which is fulfilled in this case).

2. Using the representation for $\nabla_u\xi_\perp$ by means of (8) and for $\nabla_u\nabla_u\xi_\perp$ by means of (13) under the conditions of proposition 5, the expression (35) can be written in the form

$$u(uL^2) - \frac{2}{n-1}.U.L^2 = 2[{}_sD(\xi_\perp,\xi_\perp) + \bar{g}(d(\xi_\perp),d(\xi_\perp))], \qquad (36)$$

where

$${}_sD(\xi_\perp,\xi_\perp) = (\xi_\perp)({}_sD(\xi_\perp)) = {}_sD_{kl}\xi^k_\perp\xi^l_\perp , \qquad (37)$$

and the following relations are fulfilled

$$g({}_{rel}v,{}_{rel}v) = {}_{rel}v^2 = \bar{g}(d(\xi_\perp),d(\xi_\perp)) , \qquad (38)$$

$$g(\xi_\perp,{}_{rel}a) = g(\xi_\perp,\bar{g}[{}_sD(\xi_\perp)]) + \frac{1}{n-1}.U.L^2 , \qquad (39)$$

$$g(\xi_\perp,\bar{g}[W(\xi_\perp)]) = 0 . \qquad (40)$$

If one uses the explicit form of d from (9) and after introducing the following abbreviations

$$\lambda = -\frac{2}{n-1}\left\{\bar{g}[\sigma(\bar{g})\sigma] + \bar{g}[\omega(\bar{g})\omega] + \theta' + \frac{2}{n-1}.\theta^2\right\} , \qquad (41)$$

$${}_sD(\xi_\perp,\xi_\perp) = D^2, \quad \theta' = \dot{\theta} = u\theta = u^i\partial_i\theta = u^i\theta_{,i} = u^jE_j\theta , \qquad (42)$$

$$\sigma(\xi_\perp) = \delta , \quad \omega(\xi_\perp) = \eta , \qquad (43)$$

$$\frac{2\theta}{n-1}.\sigma(\xi_\perp,\xi_\perp) = \frac{2\theta}{n-1}.(\xi_\perp)(\sigma(\xi_\perp)) = \sigma^2 , \qquad (44)$$

$$(\delta + \eta)^2 = \delta^2 + 2\delta\eta + \eta^2 = g(\delta,\delta) + 2g(\delta,\eta) + g(\eta,\eta) , \qquad (45)$$

$$L^2 = y , \quad u = \frac{d}{ds} ,$$

then after some computations equation (36) can be obtained in the form

$$\frac{d^2y}{ds^2} + \lambda(s).y = f(s) , \qquad (46)$$

where $y = y(x^k(s)) = y(s)$, $x^k = x^k(s)$ ,

$$f(s) = 2[D^2 + (\delta + \eta)^2 + \sigma^2] , \qquad (47)$$

$$D^2 = D^2(x^k(s)) = D^2(s), \quad \delta = \delta(x^k(s)), \quad \sigma = \sigma(x^k(s)) . \qquad (48)$$



The explicit form of $\lambda(s)$ and $f(s)$ determined the explicit form of equation (46) and therefore its solutions as well.

It is worth to mention that the explicit form of $\lambda$ and f can be found after solving the equations for the vector fields u and $\xi$: $\nabla_u u = 0$, $\pounds_u\xi = 0$ under the additional conditions $g(u,u) = e =$ const. $\neq 0$, $g(u,\xi) = 0$.

From the form of equation (46) one can draw a conclusion that (46) could have a form of oscillator equation (homogeneous or non-homogeneous) under the condition

$$\lambda(s) = \lambda_o = \text{const.} \neq 0 ,$$

which is a very special case, requiring additional discussion.

In the case of $U_n$-space admitting non-isotropic, autoparallel and normalized vector field u with shear $\sigma = 0$ and rotation $\omega = 0$

$$\lambda = -\frac{2}{n-1}(\theta' + \frac{2}{n-1}.\theta^2) , \qquad (49)$$

$$D^2 = 0 , \quad \delta = 0, \quad \eta = 0 , \quad \sigma^2 = 0 , \qquad (50)$$

the equation (46) will have the form

$$y'' = \{[g(s)]^2 + g'(s)\}y , \quad y' = \frac{dy}{ds} , \quad y'' = \frac{d^2y}{ds^2} , \qquad (51)$$

$$g(s) = \frac{2}{n-1}.\theta , \quad g'(s) = \frac{2}{n-1}.\theta' .$$

One solution of equation as (51) has been found by Ielchin [19] in the form

$$y = \exp\int g(s)ds . \qquad (52)$$

In the case of $V_n$-space (n=4) under the conditions (21) and the conditions for u to be non-isotropic, normalized and geodesic vector field for $L^2 = g(\xi_\perp,\xi_\perp)$ the following deviation equation can be obtained:

$$u(uL^2) - \frac{2}{n-1}.I.L^2 = 2[{}_sM(\xi_\perp,\xi_\perp) + \bar{g}(d(\xi_\perp),d(\xi_\perp))], \qquad (53)$$

which follows from equation (36) under the conditions

$$U = I \quad ({}_FU_o = 0) , \quad {}_sD = {}_sM . \qquad (54)$$

Equation (53) (if $u = \frac{d}{ds}$ , $\nabla_u = \frac{D}{ds}$) can be written therefore in the form (46) as

$$\frac{d^2y}{ds^2} + \lambda(s).y = f(s) , \qquad (46)$$

where

$$\lambda(s) = -\frac{2}{n-1}(I + \frac{1}{n-1}.\theta^2) ,$$

$$I = \bar{g}[\sigma(\bar{g})\sigma] + \bar{g}[\omega(\bar{g})\omega] + \theta' + \frac{1}{n-1}.\theta^2 \qquad (55)$$

(Raychaudhuri identity [18]),

$$f(s) = 2[M^2 + (\delta + \eta)^2 + \sigma^2] , \quad M^2 = {}_sM(\xi_\perp,\xi_\perp) = {}_sM_{kl}\xi^k_\perp\xi^l_\perp .$$

For $V_n$-spaces with Ricci = 0 ($R_{ij} = 0$) equation (46) takes the form

$$y'' - \frac{2}{(n-1)^2}.\theta^2.y = f(s) , \quad y' = \frac{dy}{ds} . \tag{56}$$

If for such a type of spaces the conditions $\sigma = 0, \omega = 0$ are fulfilled, then

$$_sM = 0 , \quad \delta = 0 , \quad \eta = 0 , \quad f(s) = 0 . \tag{57}$$

Equation (56) will have the form

$$y'' = \frac{2}{(n-1)^2}.\theta^2.y , \tag{58}$$

which by means of the substitution /19/ $y' = y.v(s)$ can be transformed in a Riccati equation

$$v' + v^2 = \frac{2}{(n-1)^2}.\theta^2 . \tag{59}$$

If v(s) is one solution of this equation, then the solutions of (58) are solutions of a 1st order linear differential equation /19/

$$y' - v(s).y = C.\exp(-\int v.ds) , \quad C = \mathrm{const.}. \tag{60}$$

For the special case, when $\theta = \theta_o = \mathrm{const.} \neq 0$, equation (56) has the form

$$y'' + \lambda_o.y = 0 , \quad \lambda_o = - \frac{2}{(n-1)^2}.\theta_o^2 < 0 , \tag{61}$$

and solutions of a type /19/

$$y = L^2 = a.\mathrm{Cosh}(\frac{\sqrt{2}}{n-1}.\theta_o.s) + b.\mathrm{Sinh}(\frac{\sqrt{2}}{n-1}.\theta_o.s) , \tag{62}$$

$a, b = \mathrm{const.}.$

Therefore, a deviation equation with non-isotropic, normalized and geodesic (timelike) vector field u for the square of a non-isotropic (spacelike) vector field $\xi$ can be considered as an eventual candidate for the theoretical scheme of gravitational waves detectors because such equations of type (46) could have, under certain conditions in $U_n$- and $V_n$-spaces, the form of an oscillator equation.

## CONCLUSION

1. The deviation equation of Synge and Schild in $L_n$-spaces can be considered as a corollary of the equation $\pounds_\xi u = 0$ ($\pounds_u \xi = 0$) for a vector field $\xi$ and an arbitrary vector field u. The last equation appears only as a sufficient, but not necessary condition for the existence of the deviation equation of Synge and Schild, which, therefore, allows other "first integrals" as well.

2. A deviation equation can be also considered for the square $L^2$ of a non-isotropic (spacelike) vector field $\xi$, which equation appears in fact as equation for an invariant, carrying information about the length of this vector field. In the case of non-isotropic, normalized



and autoparallel vector fields u in $U_n$- and $V_n$-spaces this equation can have the form of an oscillator equation under certain conditions. This fact can be explored when theoretical schemes for gravitational waves detectors are considered in a fixed gravitational theory in $U_n$- and $V_n$-spaces.

3. In cases, when the deviation equation of Synge and Schild is considered in (pseudo)Riemannian spaces without torsion and with Ricci tensor equal to zero, the only acceleration, induced by curvature, which is not vanishing, is the shear acceleration (induced by the curvature). This fact should be taken into account in the theoretical schemes for gravitational waves detectors on the basis of general relativity theory.

## REFERENCES

1. Weber J., General Relativity and Gravitational Waves (New York, 1961) Ch. 8 (Russian translation: Izd. Inostr. Lit., Moscow, 1962), pp. 157-164; Gravity Research Foundation Prize Esseays (New Boston, New Hampshir, 1958,1959); Phys. Rev. 117 (1960), 306-313
2. Zacharov V.D., Gravitational Waves in Einstein's Theory of Gravitation (in Russian)(Izd. Nauka, Moscow, 1972), pp. 175-176
3. Will C.M., In: General Relativity: An Einstein Centenary Survey. Ed. S.W. Hawking, W. Israel (Cambridge U.P., Cambridge, 1979), pp. 24-89; Theory and Experiment in Gravitational Physics (Cambridge U.P., London, 1981) Ch.10 (Russian translation: Energoatomizdat, Moscow, 1985) pp. 183-213
4. Bičak J., Rudenko B.N., Gravitational Waves in GRT and the Problem of Their Detection (in Russian)(Izd. Moskovskogo Univ., Moscow, 1987), pp. 145-146
5. Amaldi E., Pizzella G., In: Astrofisica e cosmologia gravitatione. Quanti e relatività (Giunti Barbèra, Firenze, 1979), pp. 281-458 (Russian translation: Mir, Moscow, 1982), pp. 241-396
6. Ciufolini I., Phys. Rev. D 34 (1986)4, 1014-1017
7. Synge J.L., Schild A., Tensor Calculus (Toronto,1952), Ch. 3
8. Dixon W.C., Nuovo Cim. 34(1964), 317; Phil. Trans. Royal Soc. London 277(1974)Nr. 1264, 59
9. Maugin G.A., Gen. Rel. and Grav., 4(1973), 241; 5(1974), 13
10. Manoff S., Exp. Technik d. Physik 24 (1976)5, 425-431
11. Ciufolini M., Demianski M., Phys. Rev. D34 (1986)4, 1018-1020
12. Hawking S.W., Ellis G., The Large Scale Structure of Space-Time (Cambridge U.P., Cambridge, 1973), §§ 4.1,4.4. (Russian translation: Mir, Moscow, 1977) pp. 91-99, 110-116

13. Manoff S., Gen. Rel. and Grav. 11(1979), 189-204; 6th Sov. Grav. Conf., UDN, Moscow 1984. Contr. Papers (UDN, Moscow, 1984), pp. 229-230; In: Gravitational Waves (in Russian)(JINR Dubna, P2-85-667, Dubna, 1985), pp. 157-168; 11th Intern. Conf. on Gen. Rel. and Grav., Stockholm 6-12 July 1986. Contr. Papers. (Stockholm, 1986); 12th Intern. Conf. on Gen. Rel and Grav., Colorado 1989. Contr. Papers. (Univ. Colorado, Colorado Boulder, 1989); Proc. of the Workshop on Development and Building a Generator and Detector for Gravitational Waves (JINR Dubna, D4-89-221, Dubna, 1989), pp. 42-54
14. Iliev B., Manoff S., 5th Sov. Grav. Conf., Contr. Papers (in Russian)(MGU, Moscow, 1981), p. 122; Comm. JINR Dubna P2-83-897 (1983), pp. 1-16
15. Swaminarayan N.S., Safko J.L., J. Math. Phys. 24(1983)4, pp. 883--885
16. Ehlers J., Abhandlungen d. Mainzer Akademie d. Wiss., Math.-Naturwiss. Kl., Nr. 11(1961)
17. Greenberg Ph.J., J. Math. Anal. Appl. 30(1970), 128
18. Kramer D., Stephani H., MacCallum M., Herlt E., Exact solutions of Einstein's field equations. (VEB Deutscher Verlag d. Wiss., Berlin, 1980), § 6.1., pp. 76-81
19. Kamke E., Differentialgleichungen. Lösungsmethoden und Lösungen. Bd. 1. Gewöhnliche Differentialgleichungen. (Akademische Verlagsgesellschaft Geest & Portig K.-G., Leipzig, 1961), pp. 398, 410, 411,413

Received by Publishing Department
on January 17, 1992.