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EXPLICIT REALIZATIONS OF STATIC AND NONSTATIC SOLENOIDS AND CONDITIONS FOR THEIR EXISTENCE

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1. INTRODUCTION

Under the solenoids one usually understands the specific configurations of charge ρ and current $\vec{j}$ densities which generate electromagnetic field confined to the finite regions of space. It is the aim of present consideration to find general ρ and $\vec{j}$ distributions which meet these conditions.

2. NONSTATIC SOLENOIDS

Let charge and current densities be periodical functions of time: $\rho \cdot \exp(-i\omega t)$, $\vec{j} \exp(-i\omega t)$. In what follows we omit the factor $\exp(-i\omega t)$. From the continuity equation it follows that $i\omega\rho = \operatorname{div}\vec{j}$. We write out the general expansion of magnetic vector potential (VP) $\vec{A}$ and the scalar electric potential Φ (it is valid outside the region, where $\rho, \vec{j} \neq 0$)

$$\vec{A} = \frac{4\pi ik}{c} \sum_{lm\tau} \vec{A}_l^m(\tau) \cdot a_l^m(\tau), \tag{2.1}$$

$$\Phi = 4\pi ik \sum h_l Y_l^m \cdot q_l^m, \quad \operatorname{div}\vec{A} + \frac{1}{c}\frac{\partial\Phi}{\partial t} = 0. \tag{2.2}$$

The vector spherical harmonics (VSH) $\vec{A}_l^m(\tau)$ are the vector solutions of the Helmholtz equation. Index τ refers to electric (E), magnetic (M) and longitudinal (L) multipoles. The VSH with different l, m, τ are orthogonal on the sphere of an arbitrary radius. In the differential form they are given by [1]

$$\vec{A}_l^m(L) = \frac{1}{k}\vec{\nabla} h_l Y_l^m,$$

$$\vec{A}_l^m(M) = \frac{1}{\sqrt{l(l+1)}} h_l \vec{L} Y_l^m,$$

$$\vec{A}_l^m(E) = -\frac{i}{k}\frac{1}{\sqrt{l(l+1)}} \operatorname{rot}(\vec{L} h_l Y_l^m). \tag{2.3}$$

Here $\vec{L}$ is the operator of the orbital angular momentum $\vec{L} = -i\vec{r} x \vec{\nabla}$, $h_l \equiv h_l(kr) = H^{(1)}_{l+\frac{1}{2}}(kr) \cdot (\pi/2kr)^{1/2}$, $Y_l^m \equiv Y_l^m(\theta, \varphi)$; further $q_l^m = \int g_l Y_l^{m*} \cdot \rho dV$, $a_l^m(\tau) = \int \vec{B}_l^{m*}(\tau) \cdot \vec{j} dV$. The vector functions $B_l^m(\tau)$ are ob-

tained from Eq. (2.2) by substituting $g_l(x) = J_{l+1/2} \cdot (\pi/2x)^{1/2}$ instead of h_l. In the expansions (2.1), (2.2) it is suggested that ρ and $\vec{j}$ occupy the finite portion of space (S) which includes origin, while the VP is evaluated at the point $\vec{r}$ lying outside S. It may happen that S does not contain origin (e.g., for the toroidal solenoid $(\rho - d)^2 + z^2 = R^2$). In that case for $\vec{r}$ lying between origin and S the role of g_l and h_l should be interchanged in Eqs. (2.1), (2.2). Now we require the disappearance of $\vec{H} = \operatorname{rot}\vec{A}$ outside S. Taking into account that $\operatorname{rot}\vec{A}_l^m(L) = 0$, $\operatorname{rot}\vec{A}_l^m(M) = ik\vec{A}_l^m(E)$, $\operatorname{rot}\vec{A}_l^m(E) = -ik\vec{A}_l^m(M)$ we obtain outside S

$$\vec{H} = \frac{1}{c} 4\pi k^2 \sum_{lm} \left[\vec{A}_l^m(M)\, a_l^m(E) - \vec{A}_l^m(E)\, a_l^m(M)\right].$$

As $\vec{A}_l^m(\tau)$ are linear independent and orthogonal, so $a_l^m(E) = a_l^m(M) = 0$ for $\vec{H} = 0$. Thus, $\vec{A}$ should be of the form

$$\vec{A} = \frac{4\pi ik}{c} \sum_{lm} \vec{A}_l^m(L) \cdot a_l^m(L), \tag{2.4}$$

where $a_l^m(L)$ are given by: $a_l^m(L) = \frac{1}{k}\int \vec{\nabla}(g_l Y_l^{m*})\vec{j} dV = -icq_l^m$. It is easy to check that $\vec{E} = -\operatorname{grad}\varphi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t} = 0$ for the electromagnetis potentials (2.2) and (2.4). Due to the completeness of expansion (2.1) it follows that these potentials realize nonstatic solenoid. In ref. [2] there was found realization of the nonstatic solenoid which has contained arbitrary function f. Comparing potentials of [2] with present ones we make the identification: $\rho = \Delta f$. The point-like realization of non-static solenoid was proposed in ref. [3]. It is obtained when the particular choice of the function f is made: $f = D \cdot \delta^3(\vec{r})$, D is const. In ref. [4] the nonstatic solenoid was realized by means of cylindrical capacitor. Beautiful experiments with such a capacitor were described in ref. [5]. The reason for treating nonstatic solenoid is that they emit the waves of electromagnetic potentials which propagate with the velocity of light [2, 4]. As they do not carry the electromagnetic energy ($\vec{E} = \vec{H} = 0$ in them), these waves can be detected only at the quantum level. However, electromagnetic potentials emitted by the non-static solenoids suggested in refs. [3,4] can be removed by the single valued gauge transformation and, thus, they are not observable. The electromagnetic potentials (2.2) and (2.4) are removed by the following transformation $\vec{A} \to \vec{A} - \operatorname{grad}\chi$, $\Phi \to \Phi + \frac{1}{c}\frac{\partial\chi}{\partial t}$, $\chi = ik \cdot \exp(-i\omega t) \cdot \int G_k \cdot f \cdot dV^1$,

$G_k(\vec{r}, \vec{r}^{\,1}) = \dfrac{\exp(-ik|\vec{r} - \vec{r}^{\,1}|)}{|\vec{r} - \vec{r}^{\,1}|}$. The physical effects originating from electromagnetic potential waves have chance to be observed if χ is nonsinglevalued function. This in turn depends both on the choice of the function f and on the space region to which $\rho, \vec{j}$ are confined (it should be multiconnected one).

3. STATIC MAGNETIC SOLENOIDS

The vector potential corresponding to the static current density equals

$$\vec{A} = \frac{1}{c}\int G_0 \cdot \vec{j}(\vec{r}^{\,1})\, dV^1, \quad G_0 \equiv |\vec{r} - \vec{r}^{\,1}|^{-1}. \tag{3.1}$$

Instead of $\vec{j}$ the equivalent magnetization can be used [6,7]: $\vec{J} = c\cdot \mathrm{rot}\,\vec{M}$. It is confined entirely within the solenoid. Thus,

$$\vec{A} = \int G_0\, \mathrm{rot}\,\vec{M}(\vec{r}^{\,1})\, dV^1. \tag{3.2}$$

The magnetic induction is given by

$$\vec{B} = \mathrm{rot}\,\vec{A} = \int G_0\, \mathrm{rot}\,\mathrm{rot}\,\vec{M}(\vec{r}^{\,1}) dV^1 = \int G_0\,(\mathrm{grad}\,\mathrm{div}\,\vec{M} - \Delta\vec{M}) dV^1,$$

or integrating by parts

$$\vec{B} = \vec{H} + 4\pi\vec{M}, \tag{3.3}$$

where the magnetic field strength equals

$$\vec{H} = \mathrm{grad}\int G_0 \cdot \mathrm{div}\,\vec{M}(\vec{r}^{\,1}) dV^1. \tag{3.4}$$

Let $\mathrm{div}\,\vec{M} = 0$. Then $\vec{B}$ differs from zero in those space regions, where $\vec{M} \neq 0$ (i.e., inside the solenoid). We conclude: to construct the static magnetic solenoid of the arbitrary geometrical form it is enough to fill this form by the substance with curl-free magnetization. The simplest example is the closed uniformly magnetized filament of an arbitrary geometrical form with magnetization parallel to the tangential vector of this filament. The linear magnetized whisker and magnetized ring which were used in the experiments testing Aharonov — Bohm effect [8] are the particular realizations of such a magnetized filament. The usual toroidal solenoids [9] and their generalizations considered in ref. [10] correspond to the patricular choice of curl-free magnetization. If one does not want to use the magnetization formalism he may alternatively work with the equivalent current density $\vec{J} = c\cdot \mathrm{rot}\,\vec{M}$.

The conditions for the disappearance of $\vec{H}$ outside the spatial region S to which current $\vec{j}$ is confined were also obtained in interesting ref. [11] (although

in a slightly different context). If j_μ denote the spherical components of $\vec{j}\,(j_0 = j_z,\ j_{\pm1} = \mp(j_x \pm ij_y)/\sqrt{2})$, then mentioned above conditions are

$$R^{lm}_{-1} = \left(\frac{1}{2}\,\frac{l-m}{l+m+1}\right)^{1/2} \cdot R_0^{l,\,m+1}, \quad R^{lm}_{1} = \left(\frac{1}{2}\,\frac{l+m}{l-m+1}\right)^{1/2} \cdot R_0^{l,\,m-1}. \tag{3.5}$$

Here $R^{lm}_\mu = \int r^l Y_l^{m*} j_\mu\, dV$. These Eqs. may be considered as checking points for the verification of $\vec{H}$ disappearance outside S for the chosen current configuration. However these Eqs. are not constructive in the sense that they do not give the receipt for the concrete realization of current density. In addition the evaluation of integrals occurring in R^{lm}_μ is not a trivial task even for the simplest toroidal configurations [12]. On the other hand, the use of magnetization formalism described above makes the construction of magnetic solenoids almost trivial. This fact is extensively used by the experimentalists. The components of current density corresponding to the chosen magnetization ($\vec{j} = c\cdot \mathrm{rot}\,\vec{M}$, $\mathrm{div}\,\vec{M} = 0$) satisfy Eqs. (3.5) automatically.

The final remark concerns the self-screened current distributions. Let the current be of the form $\vec{j} = c\cdot \mathrm{rot}\cdot \mathrm{rot}\,\vec{t}$, $\mathrm{div}\,\vec{t} = 0$. Substituting this into Eq. (3.1) we obtain $\vec{A} = 4\pi\vec{t}(\vec{r})$. Thus, VP $\vec{A}$ differs from zero in those space regions where $\vec{t}(\vec{r}) \neq 0$. Such a representation of $\vec{j}$ is valid, e.g., for the toroidal momenta [13]. Physically this means that toroidal momenta uniformly distributed over any closed curve and directed along the tangential vectors generate zero vector potential outside this curve. For the circular chain of toroidal momenta this fact was recently admitted in ref. [2].

4. CONCLUSION

We briefly summarize the main results obtained:

1) There are found current and charge densities which generate non-static solenoids; there are formulated conditions under which the waves of electromagnetic potentials can be detected.

2) There are given prescriptions for the construction of static magnetic solenoids of the arbitrary geometrical form.

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Явные реализации статических и нестатических соленоидов и условия их существования

Найдены плотности зарядов и токов, при которых электромагнитные напряженности заключены в конечных областях пространства. Обсуждается вопрос о наблюдаемости волн электромагнитных потенциалов.

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Explicit Realizations of Static and Nonstatic Solenoids and Conditions for Their Existence

There are found the charge and current densities for which the electromagnetic field strengths are confined to the finite regions of space. The question on the observability of electromagnetic potential waves is discussed.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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