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ENHANCEMENT OF THE PALUMBO RESULT CONCERNING THE UNBOUNDEDNESS OF THE SPINOR QED HAMILTONIAN FROM BELOW

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1. INTRODUCTION

Recently F.Palumbo [1] proved that the QED Hamiltonian, as well as the QCD Hamiltonian and the Hamiltonian (1.3) of the Yukawa model with $M^2 = 0$ are unbounded from below.

This result seems to me to be extremely important.

The standard approach to the QED is based on the use of the Hamiltonian (see, e.g., the text books by Dirac [2], Wentzel [3], Heitler [4], Schiff [5])

$$H = H_{oF} + H_{oph} + H^1 + H_c. \tag{1.1}$$

Here H_{oF} is the Hamiltonian of free Fermions (electrons and positrons), H_{oph} is the Hamiltonian of free transverse photons, the term H^1 describes the interaction between these three kinds of particles, H_c describes the Coulomb interaction between the electrons and positrons,

The authors of books [2—5] derived Hamiltonian (1.1) from the QED Lagrangian

$$= -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 - \psi^* \gamma_4 [\gamma_\alpha(\partial_\alpha - ieA_\alpha) + m]\psi$$

using the representation

$$\vec{A} = \mathrm{rot}\vec{a} + \mathrm{grad}\varphi \tag{1.2}$$

of the vector potential. Palumbo took a note of the fact that in a periodicity cube one has to introduce into the r.h.s. of eq. (1.2) one more additional spatially independent, zero momentum mode term $\vec{q}_0/\sqrt{V} = \int \vec{A}(x)d^3x/V$. Then one gets in the Hamiltonian (1.1) a new term H_P;

$$H_P = -\frac{1}{2}\left(\frac{\partial}{\partial \vec{q}_0}\right)^2 + \frac{e}{\sqrt{V}}\vec{q}_0 i \int \psi^* \gamma_4 \vec{\gamma} \psi(x) d^3x.$$

1.1. This (Palumbo) term makes the QED Hamiltonian unbounded from below. This result makes, probably, impossible to describe any physics by using the above approach to QED.

1.2. F.Palumbo succeeded to catch main physical feature of the phenomenon: that the (spinor) QED Hamiltonian, when written correctly, cannot describe any physics. However, the result by F.Palumbo can be considerably

enhanced. Palumbo's result for the Yukawa model is that the Hamiltonian (1.3) with $M^2 = 0$ is unbounded from below. It is possible to prove that there exists some positive value M_o^2 of the squared mass parameter such that the Hamiltonian (1.3) is unbounded from below if $M^2 < M_0^2$ (see Sec.2). This result is provoked by the (well-known) observation that the second order perturbation theory correction to the squared boson mass for Hamiltonian (1.3) is negative (and diverges).

1.2.1. Usually, one represents the quantity M^2 in eq.(1.3) as $M^2 = N^2 + \delta N^2$ and includes into the unperturbed Hamiltonian H_0 the part of the free Boson Hamiltonian, where N^2 is substituted for M^2. Then, one uses the perturbation theory and chooses the quantity δN^2 so as to eliminate the (divergent) perturbation theory corrections to the squared Boson mass N^2. Thus, one simply does not consider the values of the parameter M^2, $M^2 < M_0^2$, for which the squared Boson mass is negative and the Hamiltonian (1.3) is unbounded from below.

1.2.2. Let us note that scalar QED gives positive (and divergent) squared photon mass, and that one cannot prove the unboundedness from below of the corresponding Hamiltonian via the method of this work. Thus, combining spinor field with several charged scalar fields, one can construct the QED model, whose Hamiltonian is, maybe, bounded from below.

1.3. The Hamiltonian of the cut-off Yukawa model is

$$H = H_{oF} + H_{oB} + H_Y^1, \tag{1.3}$$

where*

$$H_{oB} = \frac{1}{2}\sum_{|k|<l}\left(-\frac{\partial^2}{\partial q_k \partial q_{-k}} + (M^2 + k^2) q_k q_{-k}\right), \tag{1.4}$$

$$H_{oF} = \sum_{|p|<l} E(p)\,(a_p^* a_p + b_p^* b_p), \tag{1.5}$$

$$H_Y^1 = \frac{g}{\sqrt{V}} \sum_{|p|,|k|,|p+k|<l} :\psi^*(p+k)\,\gamma_4 \psi(p)\, q_k: \tag{1.6}$$

Here l is the cut-off parameter, $E(p) = (p^2 + m^2)^{1/2}$,

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* We start from the usual form of the free boson Hamiltonian $H_{oB} = \frac{1}{2}\int [\pi^2 + (\nabla\varphi)^2 + M^2\varphi^2] dx$, here $\varphi(x) = \frac{1}{\sqrt{V}}\sum q_k e^{ikx}$.

$$\psi(p) = u(p,+)a_p + u(p,-)b^*_{-p}, \tag{1.7}$$

spin indices are omitted, $u(p, \pm)$ are the solutions to the Dirac equation with the positive and negative energy and momentum p, a, a^* and b, b^* are usual Fermion annihilation and creation operators.

1.4. The consideration of Sec.2 is based on my method of constructing the solutions of the second-quantized Schroedinger equation [6]. The method develops the idea by F.Coester and R.Haag [7].

1.5. The quantity M_0^2 has the order of the magnitude $g^2\ln(l/m)$, $g^2 l$, and $g^2 l^2$ if the number of spatial dimensions is one, two and three.

2. INVESTIGATION OF THE HAMILTONIAN (1.3)

Here I shall prove that the Hamiltonian (1.3) is unbounded from below if the condition (2.12) is satisfied.

At first, I shall average the Hamiltonian (1.3) over the normalized Boson state

$$\Omega_B = \text{const} \prod_{k \neq 0} e^{-\omega q_k q_{-k}}, \quad \omega > 0.$$

This operation evidently gives

$$H_{oB} \overset{\Omega_B}{\to} {}_1H_{oB} = \frac{1}{2}\left(-\frac{\partial^2}{\partial q_0^2} + M^2 q_0^2\right) + \text{const}, \tag{2.1}$$

$$H_{oF} \overset{\Omega_B}{\to} H_{oF}, \tag{2.2}$$

$$H_Y^1 \overset{\Omega_B}{\to} {}_1H_Y^1 = \frac{g q_o}{\sqrt{V}} \sum_{|p|<l} \left[(a_p^* b_{-p}^* + b_{-p} a_p) A_p + (a_p^* a_p + b_p^* b_p) B_p\right], \tag{2.3}$$

here $-1 \le A_p, B_p \le 1$, see eqs. (1.6) and (1.7).

2.1. Then, I shall consider the function

$$\Omega_F = e^{\kappa} |O_F>, \quad \kappa = \sum C_p a_p^* b_{-p}^*, \tag{2.4}$$

here $|O_F>$ is the state of the Fermion bare vacuum: $a_p|O_F> = b_p|O_F> = 0$ for all values of p. Let us determine constants C_p from the "Schroedinger equation"

$$(H_{oF} + {}_1H_Y^1 - h)\ \Omega_F = 0. \tag{2.5}$$

Here h is the constant to be found.

Using the standard formula

$$e^{-\kappa} P e^{\kappa} = P + [P, \kappa] + \frac{1}{2}\left[[P, \kappa], \kappa\right] + \dots$$

one can rewrite eq.(2.5) as

$$\left\{ {}_1H_Y^1 - h + [H_{oF} + {}_1H_Y^1, \kappa] + \frac{1}{2}\left[[{}_1H_Y^1, \kappa]\kappa\right]\right\} |O_F> = 0 \tag{2.6}$$

(the operator H_{oF} contains operators of annihilation linearly, whereas ${}_1H_Y^1$ contains them bilinearly, thus the commutators we have written down are the only ones which do not disappear).

In this way one can get the equation

$$2C_p(E(p) + \frac{g q_0}{\sqrt{V}} B_p) - 2\frac{g q_0 C_p^2 A_p}{\sqrt{V}} + \frac{g q_0}{\sqrt{V}} A_p = 0 \tag{2.7}$$

which determines a value of C_p. Here the volume V is large ($V \to \infty$); thus it is possible to take

$$C_p = -\frac{g q_0 A_p}{\sqrt{V}\, 2E(p)}. \tag{2.8}$$

2.1.1. Later on, eqs.(2.6) and (2.8) give

$$h = \sum_{|p|<l} \frac{g q_0 A_p C_p}{\sqrt{V}} = -\sum \frac{(g q_0 A_p)^2}{V\, 2E(p)}. \tag{2.9}$$

2.2. Now we have

$$\Omega_F^* \left(\frac{1}{2} M^2 q_0^2 + H_{oF} + {}_1H_Y^1\right) \Omega_F = \tag{2.10}$$

$$= \frac{1}{2}\left(M^2 - \frac{1}{V}\sum_{|p|<l} \frac{(gA_p)^2}{E(p)}\right) q_0^2 Q \equiv \frac{1}{2} Z^2 q_0^2 Q,$$

$$Q \equiv \Omega_F^* \Omega_F = \prod_{|p|<l} (1 + C_p^2) \tag{2.11}$$

(cf. [8]).

One feels here that if

$$Z^2 < 0, \tag{2.12}$$

see eq. (2.10), our Hamiltonian (1.3) is unbounded from below.

2.3. Of course, we have to calculate yet the quantity

$$t = \frac{1}{2}\int dq_0\, \Phi(q_0)\, \Omega_F^* \left(-\frac{1}{2}\frac{\partial^2}{\partial q_0^2}\right) \Omega_F \Phi(q_0) =$$

$$= \frac{1}{2}\int dq_0\, \Omega_F^* (\Phi' + \kappa'^* \Phi)(\Phi' + \kappa' \Phi)\, \Omega_F = \frac{1}{2}\int dq_0 Q\left[\left(\Phi' + D\Phi\right)^2 + G\Phi^2\right]. \tag{2.13}$$

Here the prime denotes the differentiation with respect to q_o

$$D = \sum_p \frac{C_p^2/q_0}{1 + C_p^2} = \frac{1}{2Q}\frac{\partial Q}{\partial q_0}, \qquad (2.14)$$

$$G = \sum \frac{C_p^2/q_0^2}{(1 + C_p^2)^2}. \qquad (2.15)$$

Equations (2.1—3), (2.10—11), (2.13—14) enable one to easily prove the statement of this Section

$$\left(\Phi(q_0) = f(q_0)/\sqrt{Q}\,,\ \ f(q_0) = e^{-(q_0 - a)^2},\ \ a \to \pm\infty\right)$$

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Note added in proof.

Taking into account of spin degrees of freedom insignificantly changes eq. (2.11): $Q \to Q.\,[1 + O(\frac{1}{V})]$.

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Усиление результата Палумбо о неограниченности снизу гамильтониана спинорной КЭД

Показано, что гамильтониан модели Юкавы с бозонным массовым членом $(M^2/2)\int \varphi^2(x)dx$ не ограничен снизу, если $M^2 < M_0^2$; здесь величина M_0^2 положительна и имеет порядок величины $g^2\ln(l/|m|)$, $g^2 l$ и $g^2 l^2$ в случае одной, двух и трех пространственных размерностей. Здесь g — константа связи, m — фермионный массовый параметр и l — параметр обрезания.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

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Zastavenko L.G. E2-92-175

Enhancement of the Palumbo Result Concerning the Unboundedness of the Spinor QED Hamiltonian from Below

We have proved that the Hamiltonian of the Yukawa model with the Boson mass term $(M^2/2)\int \varphi^2(x)dx$ is unbounded from below if $M^2 < M_0^2$, here M_0^2 is positive and has the order of magnitude $g^2\ln(l^2/m^2)$, $g^2 l$, and $g^2 l^2$ in one, two and three space dimensions; g is the coupling constant; m, the fermion mass parameter; l, the cut-off parameter. This result is correct also for the spinor QED. Let us introduce into the Hamiltonian of the spinor QED the photon mass term $(M^2/2)\int (\vec{A}^{Tr})^2 dx$. Then the Hamiltonian is unbounded from below if $M^2 < M_0^2$, the quantity M_0^2 is estimated as above.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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