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PARAGRASSMANN ANALYSIS AND QUANTUM GROUPS

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1 Introduction

Paragrassmann algebras (PGA) are interesting for several reasons. First, they are relevant to conformal field theories [1],[2]. Second, studies of the topological field theories show the necessity of unusual statistics [3] and, in particular, of the Green-Volkov parastatistics which was earlier discussed mainly in the context of the standard field theory [4]. There are also some hints (e.g., Ref.[5]) that PGA have a connection to quantum groups. Finally, it looks aesthetically appealing to find a generalization of the Grassmann analysis [6] that proved to be so successful in describing supersymmetry.

Recently, some applications of PGA have been discussed in literature. In Ref.[7], a parasupersymmetric generalization of quantum mechanics had been proposed. Ref.[8] has attempted at a more systematic consideration of the algebraic aspects of PGA based on the Green ansatz [4] and introduced, in that frame, a sort of paragrassmann generalization of the conformal algebra. Applications to the relativistic theory of the first-quantized spinning particles have been discussed in Ref.[9]. Further references can be found in [2],[5],[7],[8].

The aim of this paper is to construct a consistent generalization of the Grassmann algebra (GA) to a paragrassmann one preserving, as much as possible, those features of GA that were useful in physics applications. The crucial point of our approach is defining a generalized derivative in paragrassmann variables. This is shown to relate PGA, in a natural way, to q-deformed algebras and quantum groups with q being a root of unity. In this paper, we mainly concentrate on the algebraic aspects leaving the applications to future publications. It should be stressed that we do not use the Green ansatz although natural matrix realizations of the algebraic constructions are given.

Section 2 treats the algebra generated by one paragrassmann variable θ, $\theta^{p+1}=0$, and automorphisms of this algebra. In Section 3, a notion of generalized differentiation is introduced and discussed. It uses special automorphisms preserving the natural grading and naturally introduces into action the roots of unity, q $(q^{p+1}=1)$. The generalized differentiation coincides with the Grassmann one for $p=1$, and with the standard differentiation when $p\to\infty$. For intermediate cases $1<p<\infty$, the structure of the algebra depends on the arithmetic nature of its order $p+1$. This is briefly discussed in Section 4 where the simplest PGA with many variables θ_i are defined (PGA with N variables will be denoted as $\Gamma_p(N)$). They satisfy the nilpotency condition $\theta^{p+1}=0$ where θ is any linear combination of θ_i, and appear to be naturally related to the non-commutative spaces satisfying the commutation relations $\theta_i\theta_j=q\theta_j\theta_i$, $i<j$. These and other relations presented in this paper demonstrate a deep connection between PGA and quantum groups with deformation parameters q being roots of unity. Two of the most obvious are presented in the Sections 4 and 5.

2 Paragrassmann Algebra with One Variable

We start by defining the paragrassmann algebra $\Gamma_p(1)$ (or simply Γ), generated by one nilpotent variable θ ($\theta^{p+1}=0$, p is some positive integer). Any element of the algebra, $a\in\Gamma$, is a polynomial in θ of the degree p,

$$a=a_0+a_1\theta+\ldots+a_p\theta^p\ , \tag{1}$$

where a_i are real or complex numbers or, more generally, elements of some commutative ring (say, a ring of complex functions) [10]. It is useful to have a matrix realization of this algebra. One may regard a_i as coordinates of the vector a in the basis $(1,\ \theta,\ \ldots,\ \theta^p)$. Defining the operator of multiplication by θ,

$$\theta(a)=a_0\theta+\ldots+a_{p-1}\theta^p, \tag{2}$$

we see that it can be represented by the triangular $(p+1)\times(p+1)$-matrix acting on the coordinates of the vector a:

$$(\theta)_{mn}=\delta_{m,n+1}\ ,\ (\theta^k)_{mn}=\delta_{m,n+k}\ , \tag{3}$$

$m,n=0,1,\ldots,p$. We may now treat elements of the algebra as matrices. In view of Eq.(3), any element $a\in\Gamma$ can be represented by the matrix

$$(a)_{mn}=\begin{cases} a_{m-n} & \text{if } m\geq n, \\ 0 & \text{if } m<n. \end{cases} \tag{4}$$

This matrix representation of the algebra is obviously an isomorphism.

A very important construction related to the algebra Γ is its group of automorphisms consisting of the linear maps $a\to g(a)$ that preserve the multiplication i.e.,

$$g(\alpha a+\beta b)=\alpha g(a)+\beta g(b)\ , \tag{5}$$

$$g(ab)=g(a)g(b), \tag{6}$$

where α, β are numbers. It is clear that any automorphism is defined by p parameters γ_m , $m=0\ldots p-1$:

$$g(\theta)=\sum_{m=0}^{p-1}\gamma_m\theta^{m+1}, \tag{7}$$

or, in the infinitesimal form

$$\delta_\epsilon\theta=\sum_{m=0}^{p-1}\epsilon_m\theta^{m+1}. \tag{8}$$

Omitting the obvious summation symbols we have

$$\delta_\epsilon a\equiv\delta_\epsilon a_k\cdot\theta^k=a_k\cdot\delta_\epsilon\theta^k=a_k\cdot k\epsilon_m\theta^{k+m},$$

$$\delta_\epsilon a_k \equiv \epsilon_m (G^m)_{kl} a_l,$$

whereof the matrix elements of the Lie algebra generators G^m are

$$(G^m)_{kl} = l\delta_{k-l,m}\ , \tag{9}$$

and the commutation relations of them are

$$[G^m, G^n] = (n-m)G^{m+n}, \tag{10}$$

where $G^{m+n} = 0$, if $m+n \geq p$. Being the generators of the automorphism group, G^m define differentiations of the algebra Γ, the classical ones satisfying the Leibniz rule. However, it is impossible to treat any of them as a differentiation with respect to θ. In fact,

$$G^m(\theta^n) = \begin{cases} n\theta^{m+n} & \text{if } n+m \leq p, \\ 0 & \text{if } n+m > p, \end{cases}$$

but we would rather expect a differentiation $\partial \equiv \partial/\partial\theta$ to act as

$$\partial(1) = 0,\ \partial(\theta) = 1,\ \partial(\theta^n) \propto \theta^{n-1}\ ,\ n > 1, \tag{11}$$

It is easy to see that the condition $\partial(\theta) = 1$ together with the standard Leibniz rule, $\partial(ab) = \partial(a)\cdot b + a\cdot\partial(b)$, completely define the action of ∂ on any $a \in \Gamma$, but this immediately leads to a contradiction

$$0 \equiv \partial(\theta^{p+1}) = (\text{via Leibniz rule})\ = (p+1)\theta^p$$

This is a manifestation of the general fact about nilpotent algebras known even for the Grassmann case: once the normalization conditions of the type (11) are established, the Leibniz rule is to be deformed.

3 Generalized Differentiation

To introduce a useful definition of ∂ we suggest a generalized Leibniz rule (g-Leibniz rule)

$$\partial(ab) = \partial(a)\cdot b + g(a)\cdot\partial(b)\ , \tag{12}$$

where g is some automorphism of the algebra Γ_p. For the Grassmann case ($p = 1$) we have $g(a) = (-1)^{(a)}a$ where (a) is the Grassmann parity of the element a. The automorphism g and, hence, the derivative ∂ are completely fixed by the normalization conditions $\partial(\theta) = 1$ and $\partial(\theta^2) \propto \theta$. These, by (12) and (7), give

$$\gamma_m = 0 \text{ for } m > 0\ ,$$

$$\partial(1) = 0\ ,\ \partial(\theta^n) = (1 + \gamma_0 + \ldots + \gamma_0^{n-1})\theta^{n-1}$$

and from $\partial(\theta^{p+1}) \equiv 0$ we get

$$1 + \gamma_0 + \ldots + \gamma_0^p \equiv \frac{1-\gamma_0^{p+1}}{1-\gamma_0} = 0 \tag{13}$$

so that γ_0 is fixed to be a root of unity. For the moment, we choose γ_0 to be the *prime* root i.e.:

$$\gamma_0 = q \equiv e^{2\pi i/(p+1)} = (-1)^{2/(p+1)}. \tag{14}$$

By introducing the notation

$$(n)_q \equiv 1 + q + \ldots + q^{n-1} = \frac{1-q^n}{1-q}\ , \tag{15}$$

the action of ∂ can be performed as

$$\partial(\theta^n) = (n)_q\theta^{n-1}, \tag{16}$$

and so the matrix elements of ∂ in the basis $\{\theta^m\}, m = 0, \ldots, p$ are

$$(\partial)_{mn} = (m+1)_q\delta_{m+1,n}\ . \tag{17}$$

Since $(p+1)_q = 0$, the operator ∂ is nilpotent, $\partial^{p+1} = 0$. It is not hard to see that ∂ and θ satisfy the q-deformed commutation relation

$$[\partial, \theta]_q \equiv \partial\theta - q\theta\partial = 1\ . \tag{18}$$

The Grassmann case for $p = 1$ and the classical one in the limit $p \to \infty$ are evidently reproduced. The last equation is suggestive of a relation between PGA and much discussed q-deformed oscillators and quantum groups (see, e.g. Refs.[12] — [14], [17], [18]) with the deformation parameter q being a root of unity. We will return to this point at the end of the paper.

Consider now the algebra $\Pi_p(1)$ (or, simply Π) generated by both θ and ∂. Since Eq.(18) makes it possible to push all ∂'s to the right of θ's, the complete basis of Π might be given by $(p+1)^2$ monomials $\{\theta^m\partial^n\}$, $m, n = 0, \ldots, p$. (Their linear independence is quite evident in the matrix representation). Thus Π is isomorphic, as an associative algebra, to the general matrix algebra of the order $p+1$ with natural "along-diagonal" grading

$$deg(\theta^m\partial^n) = m - n\ . \tag{19}$$

Note that this grading makes it possible to rewrite the g-Leibniz rule (12) in a complete visual correspondence to the Grassmann case

$$\partial(ab) = (\partial a)b + (-1)^{\frac{2}{p+1}deg\ a}a(\partial b) \tag{20}$$

(one can interpret the quantity $(a) = \frac{2}{p+1} deg\ a$ as the paragrassmann parity of the element a)

Note also that since the automorphisms of Γ can be represented by $(p+1)$-matrices, they must have an expression in terms of θ and ∂. In particular, the operator g from Eq.(12) is expressed as

$$g = \partial\theta - \theta\partial = 1 + (q-1)\theta\partial. \tag{21}$$

Its matrix elements are

$$(g)_{mn} = q^m \delta_{mn} \tag{22}$$

In the mathematical literature (see, e.g. Ref.[11]), our generalized differentiation (12) is called g-differentiation. Mathematicians also consider a further generalization, called $(g, \bar{g})$-differentiation which satisfies the rule

$$\partial(ab) = \partial(a) \cdot \bar{g}(b) + g(a) \cdot \partial(b). \tag{23}$$

This generalization of the Leibniz rule is related to a special representation of the algebra Γ by 2×2-matrices with elements in Γ

$$a \longmapsto \begin{pmatrix} g(a) & \partial(a) \\ 0 & \bar{g}(a) \end{pmatrix} \equiv M(a). \tag{24}$$

If g and $\bar{g}$ are algebraic homomorphisms, i.e., satisfying Eq. (6), then Eq. (23) is equivalent to the homomorphism condition

$$M(ab) = M(a)M(b).$$

All this is obviously applicable to the g-Leibniz rule and to the standard one as well.

For physical applications, it seems more reasonable to use for g and $\bar{g}$ some automorphisms rather than just homomorphisms. Although we think that Eq.(12) looks more natural than Eq.(23), the latter can be used to define "real" differentiation, i.e., the one with real matrix elements. In fact, choosing for g and $\bar{g}$ the automorphisms defined by

$$g(\theta) = q^{1/2}\theta, \ \bar{g}(\theta) = q^{-1/2}\theta, \tag{25}$$

we find that

$$\partial(\theta^n) = [n]_{\sqrt{q}}\theta^{n-1}, \tag{26}$$

with the popular notation

$$[n]_{\sqrt{q}} \equiv \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}} = q^{(1-n)/2}(n)_q. \tag{27}$$

This is obviously a real number. The operators g and $\bar{g}$ have the matrix elements

$$(g)_{mn} = q^{m/2}\delta_{mn}, \ (\bar{g})_{mn} = q^{-m/2}\delta_{mn}, \tag{28}$$

and the following expression in terms of θ and ∂

$$g = \partial\theta - q^{-1/2}\theta\partial, \ \bar{g} = \partial\theta - q^{1/2}\theta\partial. \tag{29}$$

The first equation in (29) is an analog of Eq. (21) while the second one may be considered as an analog of (18). One can easily recognize in formulas (29) the definition of the quantum oscillator (see, e.g. [12], [17], [18]). We will exploit this variant of differentiation in the last Section of this paper.

In addition to the g-differentiation, one can also construct an inverse operation, or g-integration, $(\partial)^{-1} = \int_\theta$. To do that, one has to "regularize" θ and ∂ by introducing a formal parameter dependence to θ and $(n)_q$, e.g., $\theta_\epsilon = \theta + \epsilon^2$, $q_\epsilon = q^{1+\epsilon}$. Then, the following simple definition

$$\int_\theta \theta_\epsilon^n = \frac{\theta_\epsilon^{n+1}}{(n+1)_{q_\epsilon}},$$

makes sense and one can check that

$$\partial \int_\theta = 1$$

in the limit $\epsilon \to 0$. This definition satisfies the g-modified partial integration rule

$$\int_\theta (\partial a)b = ab - \int_\theta g(a)\partial b.$$

In the limit $p \to \infty$ this definition reproduces the usual indefinite integral. Our definition of the θ-integration has no relation to the standard Grassmann integration. A possible definition of the integration over θ that generalizes the Grassmann integration to the paragrassmann one has earlier been addressed in Ref.[15].

Up to now, we have been discussing the paragrassmann algebra and its satellites with coefficients being complex (or real) numbers. In some applications (e.g., in constructing parasupersymmetries) one has to deal with a_n (Eq.(1)) being taken from a wider commutative ring, for instance, the ring of the differentiable functions of a real or complex variable t i.e., $a_n = a_n(t)$. For such an algebra, it is possible to define a sort of "covariant derivative"

$$D = \partial_\theta + \frac{1}{(p)_q!}\theta^p\partial_t, \tag{30}$$

where $\partial_\theta \equiv \partial$ and the standard notation is used

$$(p)_q! = (p)_q(p-1)_q \ldots (1)_q. \tag{31}$$

This derivative obviously satisfies the g-Leibniz rule (12) and may be considered as a root of ∂_t since

$$D^{p+1}a(t;\theta) = \partial_t a(t;\theta). \tag{32}$$

Unlike ∂_θ, the derivative D possesses eigenfunctions, the q-exponentials

$$e_q(t;\theta) = e^t \sum_{n=0}^{p} \frac{\theta^n}{(n)_q!} ,$$

$$De_q(\lambda^{p+1}t;\lambda\theta) = \lambda e_q(\lambda^{p+1}t;\lambda\theta).$$

In the limit $p \to \infty$ we have $e_q(t;\theta) \to \exp(t+\theta)$.

4 Many Paragrassmann Variables

Our discussion of the paragrassmann algebras $\Gamma_p(1)$ and $\Pi_p(1)$ was completely general and did not rely on special matrix representations for θ and ∂. In fact, different representations could be classified if we relaxed our assumption for q to be the prime root of unity, $q_p = \exp(2\pi i/(p+1))$. Then, one would find that the structure of the algebra $\Pi_p(1)$ depended on the arithmetic properties of $(p+1)$. The simplest case is when $(p+1)$ is a prime integer. Then, the multiplicative group of roots of unity, $\mathbf{Z}_{p+1}$, has no subgroups; any root generates the whole group and may be used for defining ∂. If $p+1$ is a composite number having divisors p_i, the group of roots contains subgroups, $\mathbf{Z}_{p_i}$, generated by the roots $q_i = \exp(2\pi i/p_i)$. Correspondingly, the algebra $\Gamma_p(1)$ has the subalgebras generated by θ^{p_i} having the following property: if we define ∂, with q in Eq.(16) replaced by q_i, we will find that $\partial \equiv 0$ over all subalgebra generated by θ^{p_i}. It follows that we can choose q only of the *primitive* roots, i.e., those that generate the entire group $\mathbf{Z}_{p+1}$ not just a subgroup.

In summary, when $(p+1)$ is a prime number, any root is primitive (except unity) and, hence, there are p possibilities to define ∂. For a composite $(p+1)$, the number of possible differentiations is equal to $\phi(p+1)$ which is the number of positive integers smaller than $(p+1)$ and relatively prime to it. Such an ambiguity becomes crucial when we turn to the many-θ case giving rise to the existence of a series of nonequivalent paragrassmann algebras $\Gamma_p(N)$. Needless to say, it is a pure $p>1$ effect.

Leaving these subtleties to some further paper we present here just the simplest inductive construction of $\Gamma_p(N)$. Starting with $N=2$, define

$$\theta_1 = g \otimes \theta \ , \ \theta_2 = \theta \otimes 1 , \tag{33}$$

where θ and g have been defined in previous section. It is easy to see that

$$\theta_1\theta_2 = q\theta_2\theta_1 \ , \ \theta_i^{p+1} = 0 . \tag{34}$$

The crucial fact is that the definition (33) allows for nilpotency of any linear combination of θ_1 and θ_2. In fact, as one can easily derive by induction,

$$(a_1\theta_1 + a_2\theta_2)^n = \sum_{k=0}^{n} \binom{n}{k}_q a_1^k a_2^{n-k} \theta_2^{n-k}\theta_1^k , \tag{35}$$

where

$$\binom{n}{k}_q = \frac{(n)_q!}{(k)_q!(n-k)_q!} \tag{36}$$

are q-deformed binomial coefficients, the polynomials in q (a.k.a. Gauss polynomials). Remembering now the definitions (31) and (15), we immediately prove that

$$(a_1\theta_1 + a_2\theta_2)^{p+1} = 0 , \tag{37}$$

as long as q is a primitive root of unity.

Suppose now that we have constructed the algebra $\Gamma_p(N)$ satisfying the relations

$$\theta_i\theta_j = q\theta_j\theta_i \ , \ i<j \ , \ i,j = 1\ldots N , \tag{38}$$

$$(\sum_{i=1}^{N} a_i\theta_i)^{p+1} = 0 . \tag{39}$$

Then, $N+1$ matrices ϑ_i satisfying (38) and (39) can be constructed in analogy to (33)

$$\vartheta_i = g \otimes \theta_i \ , \ i = 1\ldots N \ , \vartheta_{N+1} = \theta \otimes 1. \tag{40}$$

The proof of the identity (39) is performed in full analogy with the $N=2$ case. Thus, the induction ensures the existence of the algebras $\Gamma_p(N)$ satisfying the conditions (38) for all N. As has been noted above, it is a simplest construction of the paragrassmann algebra with many generators. The complete classification of all admissible forms of $\Gamma_p(N)$ is an interesting but a separate problem.

It is rather amusing that the consideration of paragrassmann algebras naturally leads to the objects introduced in the context of quantum groups. In fact, the generators of the algebra $\Gamma_p(N)$, determined by the relations of type (38) and (39), might be considered as coordinates of a certain nilpotent quantum hyperplane similar to those of Refs.[13], [14]. Such an object and, especially, its ∂-extensions (defined by its automorphisms) look rather interesting both from algebraic and from quantum-geometric [16] points of view. Here, we just briefly outline problems arising in this area.

Let us consider an algebra $\Gamma_p(N)$ with the commutation relations

$$\theta_i\theta_j = q^{\rho_{ij}}\theta_j\theta_i \ , \ i,j = 1\ldots N , \tag{41}$$

where q denotes the prime root of unity. The requirement for $q^{\rho_{ij}}$ to be a primitive root is equivalent to the requirement for ρ_{ij} to be invertible elements of the ring $\mathbf{Z}_{p+1}$. Then, let us define differentiations ∂_i satisfying the normalization conditions,

$$\partial_i(\theta_k) = \delta_{ik} \tag{42}$$

and the g-Leibniz rule

$$\partial_i(ab) = \partial_i(a)\cdot b + g_i(a)\cdot\partial_i(b) \tag{43}$$

where the action of the automorphisms g_i on θ_k is

$$g_i(\theta_k) = q^{\nu_{ik}}\theta_k \,. \tag{44}$$

These conditions determine the commutation relations in the operator form

$$\partial_i\theta_k = \delta_{ik} + q^{\nu_{ik}}\theta_k\partial_i \,. \tag{45}$$

It can easily be shown that

$$\partial_i\partial_j = q^{\rho_{ij}}\partial_j\partial_i \,, \tag{46}$$

and, for $i \neq k$,

$$\partial_i\theta_k = q^{\nu_{ik}}\theta_k\partial_i$$

$$\nu_{ik} = \rho_{ki} = -\rho_{ik} \,, \tag{47}$$

while the diagonal ν_{ii} remains not specified. There were no problems so far. But adding the requirement that any linear combination of ∂_i must also be a differentiation satisfying (43) with certain $\tilde{g}$ immediately gives

$$g_i(a) = g_j(a) = \tilde{g}(a) \tag{48}$$

and, therefore,

$$\nu_{ik} = \nu_{jk} \,. \tag{49}$$

The conditions (47) and (49) are in general hard to be satisfied together. For $N = 2$ the solution exists

$$\nu_{11} = \nu_{21} = -\nu_{12} = -\nu_{22} = (\text{some invertible element of } \mathbf{Z}_{p+1}\,) \tag{50}$$

But for $N > 2$ the equation (38) ensures the existence of the algebra (41) with all $\rho_{ij} = 1$ for $i < j$, which is evidently inconsistent with (49). Unless $p = 1$, of course. This demonstrates the necessity of a more detailed consideration of the algebra $\Gamma_p(N)$ and its automorphisms.

It is possible to construct another interesting extension of $\Gamma_p(N)$ (where p is even number) with generators θ_i and ∂_i if we even further relax the g-Leibniz rule (43) to the form familiar from the theory of quantum groups [16]

$$\partial_i(ab) = \partial_i(a)\cdot b + g_i^j(a)\cdot\partial_j(b) \,.$$

This makes it possible to construct operators ∂_i by the inductive procedure similar to (40)

$$\tilde{\partial}_i = g\otimes\partial_i, \quad i = 1\ldots N, \quad \tilde{\partial}_{N+1} = \partial\otimes 1 \,, \tag{51}$$

where we have also slightly modified the definition of ∂ and g

$$\partial\theta - q^2\theta\partial = 1 \,, \tag{52}$$

$$\partial\theta - \theta\partial = g^2 \,. \tag{53}$$

From these equations and from definitions of θ_i and ∂_i ($i = 1,\ldots,N$) we obtain the following algebra

$$\begin{aligned}
\theta_i\theta_j &= q\theta_j\theta_i \quad i<j\,,\\
\partial_i\partial_j &= q^{-1}\partial_j\partial_i \quad i<j\,,\\
\partial_i\theta_j &= q\theta_j\partial_i \quad i\neq j\,,\\
\partial_i\theta_i &- q^2\theta_i\partial_i = 1 + (q^2-1)\sum_{k>j}\theta_k\partial_k \,.
\end{aligned} \tag{54}$$

These are the well known formulas for differential calculus on the quantum hyperplane [16]. These formulas may also be interpreted as the definition of the covariant q-oscillators [18] or, else, as the central extension of the quantum symplectic space relations for the quantum group $Sp_q(2N)$ (see L.D.Faddeev a.o. [13]). Note that nilpotency of the linear combinations $a_i\theta_i$ and $b_i\partial_i$ as well as nondegeneracy of ∂ (52) are guaranteed since both q and q^2 are primitive roots of unity (for p even integer only).

This example demonstrates a dramatic relation between paragrassmann algebras and quantum groups. Another example will be presented in the next section.

5 Discussion

In this paper, we have introduced the basic ideas of a rather general approach to constructing paragrassmann algebras with differentiations. One may ask a question: what are the relations an algebra must satisfy to be called paragrassmann? In fact, one of them is clear — it is the p-nilpotency of any linear combination of generators θ_i ($i = 1\ldots N$) or, equivalently,

$$\sum_{\sigma\in S_{p+1}}\theta_{\sigma(i_0)}\theta_{\sigma(i_1)}\ldots\theta_{\sigma(i_p)} = 0 \,, \tag{55}$$

where the sum is taken over all permutations of the indices. It is clear that the algebra with the only identity (55) would be very hard to handle. So, one must impose some additional restrictions. A variant of those, known as the Green ansatz (see Ref.[4]), consists in taking each paragrassmann generator θ_i to be a sum of p mutually commuting Grassmann numbers. In addition to Eq.(55), this gives the condition

$$[[\theta_{i_1},\theta_{i_2}],\theta_{i_3}] = 0 \tag{56}$$

Such an algebra admits a sort of analysis (see [4]) which unfortunately quickly becomes messy as p increases.

As has been shown above, using the conditions (41) instead of (56) (with certain restrictions on ρ_{ij} coming from (55)) gives a much simpler algebra possessing matrix representation, differentiations and, as we might suspect, many other useful properties analogous to its Grassmann ancestor. These are the algebras we should call paragrassmann. One can easily check that conditions (41) and (56) are not particular cases of each other, and so the algebras $\Gamma_p(N)$ of the present paper are different from those of Ref.[4].

The most curious is the connection between paragrassmann and q-deformed algebras. In fact, our interest to paragrassmann algebras was initiated by searching for the parafermionic extensions of the Virasoro algebra (which we are going to present in the next paper). So, coming into play of roots of unity, q-oscillators, etc. was somewhat surprising. To make this connection more apparent, we give here a representation of the q-deformed algebra $U_q(su(1,1))$ in terms of the paragrassmann variable θ and $(g,\bar{g})$-differentiation ∂ (the analogous construction for $U_q(su(1,1))$ from q-deformed oscillator was considered in Ref.[17]). This can be done by representing the homomorphisms g and $\bar{g}$ from (23) as operators inverse to each other (see Eq. (25))

$$g = q^{\tilde{N}} \; , \quad \bar{g} = q^{-\tilde{N}} \; . \tag{57}$$

Then, defining the generators N, E_+ and E_-

$$\begin{aligned} N &= \tilde{N} + 1/2 \; , \\ E_+ &= \frac{1}{(q^{1/2}+q^{-1/2})^{1/2}}\theta^2 \; , \\ E_- &= \frac{1}{(q^{1/2}+q^{-1/2})^{1/2}}\partial^2 \; , \end{aligned} \tag{58}$$

and using Eq. (29), it is not hard to check that generators (58) satisfy the well-known relations of the quantum algebra $U_q(su(1,1))$ in the Drinfeld-Jimbo form

$$[N, E_\pm] = \pm E_\pm$$

$$[E_+, E_-] = -[2N]_{\sqrt{q}} \equiv -\frac{q^N - q^{-N}}{q^{1/2} - q^{-1/2}}$$

There exists a matrix representation of θ and of $(g,\bar{g})$-differentiation ∂, in which $(E_+)^\dagger = E_-$ and $N^\dagger = N$ (or $\theta^\dagger = \partial$). This representation is related to the slightly changed basis for the algebra $\Gamma_p(1)$

$$\theta^k \to e^{i\phi_k}([k]_{\sqrt{q}}!)^{-1/2}\theta^k \; ,$$

where ϕ_k are arbitrary real phases. For each p we obtain different $(p+1)$-dimensional representations for the algebra $U_q(su(1,1))$ when q is a root of unity. It would be interesting to compare these "parafermionic representations" of quantum algebras with other known representations of the similar kind (see, e.g. [2]).

One might suppose that larger q-deformed algebras could be constructed by virtue of PGA with many θ's and ∂'s (see e.g. Ref. [18] in view of the existence

of the PGA (54)). Anyway, for further applications one has to develop a detailed theory of PGA with many variables. In particular, it would allow for a systematic formal treatment of parasupersymmetries.

As a final remark, we would like to mention a possible relation of PGA to the finite-dimensional quantum models introduced by H.Weyl in his famous book [19] and further studied by J.Schwinger (Ref.[20]). They considered quantum variables described by unitary finite matrices U_i satisfying the relations: $U_iU_j = qU_jU_i$ and $(U_i)^{p+1} = 1$. (Obviously, q must be a root of unity). They realized that the $p = 1$ case is relevant for describing the spin variables and treated the infinite-dimensional limit $p \to \infty$ as a limit in which usual commutative geometry is restored.

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Параграссманнов анализ и квантовые группы

С алгебраической точки зрения без использования анзаца Грина рассматриваются параграссманновы алгебры с одной и многими параграссманновыми переменными. Путем естественного обобщения грассманного дифференциального исчисления до параграссманного вводится дифференцирование по параграссманновой переменной, а также вводится ковариантная парасуперсимметричная производная. Устанавливаются глубокие связи между параграссманновыми алгебрами и квантовыми группами с параметрами деформации, являющимися корнями из единицы.

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Paragrassmann Analysis and Quantum Groups

Paragrassmann algebras with one and many paragrassmann variables are considered from the algebraic point of view without using the Green ansatz. A differential operator with respect to paragrassmann variable and a covariant para-super-derivative are introduced giving a natural generalization of the Grassmann calculus to a paragrassmann one. Deep relations between paragrassmann algebras and quantum groups with deformation parameters being root of unity are established.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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