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SOLITON EQUATIONS AND SELF-DUAL GAUGE FIELDS

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1. We consider a gauge field A_μ ($\mu,\nu,\ldots = 1,\ldots,4$) of an arbitrary Lie group G in the space $\mathbb{R}^{2,2}$ of signature (−++−). It is known that for A_μ one may define the self-duality equations, solutions of which satisfy the Yang-Mills (YM) equations in $\mathbb{R}^{2,2}$ owing to the Bianchi identity. Many integrable equations in (1+1) dimension may be obtained by reduction of the self-dual Yang-Mills (SDYM) equations in (2+2) dimension. These reductions were discussed by Ward [1], Mason and Sparling [2], Ablowitz, Chakravarty and Clarkson [3] and many others.

It is known, in particular, that the SDYM equations in $\mathbb{R}^{2,2}$ may be reduced to the equations in $\mathbb{R}^{2,1}$ which are equivalent to the equations of modified principal chiral model (see [4,5]) and to the following equations in $\mathbb{R}^{1,1}$: the equations of principal chiral model, Toda model; the Ernst, Liouville, sine-Gordon, Bullough-Dodd, NLS and KdV equations; the equations of relativistic string moving in the space $\mathbb{R}^{1,q-1}$ (see, e.g., [1,2,6,7]). If A_μ depend on one coordinate (null or non-null) then the SDYM equations admit reductions to the Euler-Arnold equations for n-dimensional rigid body, the generalized Kovalevskaya top and modified Nahm equations (see [1,3,8,9]).

The reduction of the SDYM equations in d=4 to the soliton equations in d=2 dimension is interesting from different points of view. Firstly, the SDYM equations in d=4 play a part of a universal integrable system from which many others could be obtained by reduction. Secondly, this reduction permits one to make correspondence between the inverse scattering method and the twistor method (see [1,2,4,5]). Thirdly, taking any solution of the soliton equations in d=2 (see, e.g., [10–13]) one can obtain the solution of the SDYM equations in $\mathbb{R}^{2,2}$ or the solution of the modified chiral model equations in $\mathbb{R}^{2,1}$.

In this paper we shall show that the SDYM equations in the space $\mathbb{R}^{2,2}$ may be reduced to the equations mentioned in abstract as well as to the Burgers equation, modified KdV (MKdV) equation and to the generalized MKdV equations introduced by Athorne and Fordy [14].

2. In $\mathbb{R}^{2,2}$ we introduce null coordinates $t = 2^{-1/2}(x_1+x_2)$, $u = 2^{-1/2}(x_2-x_1)$, $y = 2^{-1/2}(x_3+x_4)$, $z = 2^{-1/2}(x_3-x_4)$ and set $A_t = 2^{-1/2}(A_1+A_2)$, $A_u = 2^{-1/2}(A_2-A_1)$, $A_y = 2^{-1/2}(A_3+A_4)$, $A_z = 2^{-1/2}(A_3-A_4)$. A gauge field A_μ takes values in the Lie algebra $\mathfrak{g}$ of the Lie group G. The SDYM equations

$$F_{14} = F_{23}\ ,\quad F_{24} = -F_{31}\ ,\quad F_{34} = -F_{12}$$

for $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu+[A_\mu,A_\nu]$ in null coordinates can be rewritten in the following way:

$$F_{tz} = 0\ ,\quad F_{yu} = 0\ ,\quad F_{tu}+ F_{zy} = 0\ . \tag{1}$$

Let us suppose that the gauge field depends only on t and z. In this case Eqs.(1) take the form

$$\partial_t A_z-\partial_z A_t+[A_t,A_z] = 0,\quad [A_y,A_u] = 0,$$
$$\partial_t A_u + \partial_z A_y + [A_t,A_u]+[A_z,A_y] = 0\ . \tag{2}$$

We set

$$U_0 := A_z\ ,\quad V_0 := A_t\ ,\quad U_1 := A_u\ ,\quad V_1 := -A_y\ . \tag{3}$$

It is easy to see that Eqs.(2) may be written in the form of "a zero curvature condition"

$$\partial_t U(\lambda) - \partial_z V(\lambda) + [V(\lambda),U(\lambda)] = 0\ , \tag{4}$$

where U and V linearly depend on a spectral parameter λ:

$$U(\lambda)=U_0(z,t) + \lambda U_1(z,t),\ V(\lambda)=V_0(z,t)+\lambda V_1(z,t). \tag{5}$$

Proposition 1. *All the integrable equations in d=2, which have zero curvature representation (4),(5), may be obtained by reduction of the SDYM equations (1) if the components* A_t, A_u, A_y, A_z *of the gauge field are identified with* U_0, V_0, U_1, V_1 *by (3).*

Proof follows from formulae (1)–(5).

Note that **Proposition 1** actually follows from representation of the SDYM equations (1) as a zero curvature equation with spectral parameter λ (see [15]).

3. Example 0. *Burgers equation.* For $\mathfrak{g}=gl(1,\mathbb{R})$ let us choose

$$U_0 = u(z,t),\quad V_0 = u^2 + u_z\ ,\quad U_1 = V_1 = 0\ , \tag{6}$$

where u is a real function, $u_z \equiv \partial u/\partial z$. Substituting (6) into (4), (5) we obtain the Burgers equation:

$$u_t = 2\,u\,u_z + u_{zz}\quad . \tag{7}$$

Notice that this example is trivial from the point of view of the YM theory because (6) corresponds to the Abelian gauge field A_μ with $F_{\mu\nu}=0$.

Example 1. *N-wave model.* Let us consider $\mathcal{G}=gl(2n,\mathbb{C})$. Suppose that (see [10-12])

$$U_0 = [U_1, W] , \quad U_1 = \gamma\, diag(a_1,\dots,a_n) ,$$
$$V_0 = [V_1, W] , \quad V_1 = \gamma\, diag(b_1,\dots,b_n) , \tag{8}$$

where W is an arbitrary matrix with zero diagonal, γ, a_j, b_j ($j,\dots=1,\dots,n$) are constants and $a_1 > a_2 > \dots > a_n$. Substitution of (8) into (4), (5) leads to the equation

$$[U_1, \partial_t W] - [V_1, \partial_z W] + [[V_1,W], [U_1,W]] = 0 , \tag{9}$$

describing the evolution system with quadratic nonlinearity.

If one takes $\gamma=i$ and $W^+= -W$, i.e., W is an antihermitian matrix and $\mathcal{G}=u(n)$, then (9) will describe nonlinear interaction of $N=n(n-1)/2$ wave packets [10-12]. If one additionally requires $W \in so(n)$ and $\partial_t W=0$, then Eqs.(9) will coincide with the Euler-Arnold equations on the algebra $so(n)$ (see [16]).

Example 2. *One-dimensional gas dynamics equations.* For $\mathcal{G} = gl(2,\mathbb{R})$ we choose [17]:

$$U_0=[U_1,W], \quad V_0=[V_1,W]+Q, \quad U_1=-\begin{pmatrix} s_z & 0 \\ 0 & r_z \end{pmatrix}, \quad V_1=\begin{pmatrix} (u-\rho\alpha)s_z & 0 \\ 0 & (u+\rho\alpha)r_z \end{pmatrix},$$

$$W=\frac{\alpha'}{4\alpha^2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} \frac{1}{s_z}\{s_t+(u-\rho\alpha)s_z\}_z & 0 \\ 0 & \frac{1}{r_z}\{r_t+(u+\rho\alpha)r_z\}_z \end{pmatrix}+$$

$$+\frac{\alpha'}{4\alpha^2}\Big\{(r_t+(u+\rho\alpha)\,r_z)-(s_t+(u-\rho\alpha)\,s_z)\Big\}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \tag{10}$$

where r and s are real functions of (z,t), $\alpha=\alpha(\rho)$ is some function of ρ, $\alpha' \equiv d\alpha/d\rho$ and $\rho=\rho(r-s)$ is given by $\int_{\rho_0}^{\rho}\alpha(\xi)d\xi = r-s$. It was shown in [17] that after substitution of (10) into (5) Eqs.(4) are equivalent to the OGD equations of a one-dimensional plane isoentropic flow of gas.

Example 3. *Boussinesq equation.* Now let us consider $\mathcal{G} = gl(3,\mathbb{R})$. We put

$$U_0=\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ w & \frac{3}{2}u+\beta & 0 \end{pmatrix}, \quad V_0 = \begin{pmatrix} -u & 0 & -1 \\ w-u_z & \frac{1}{2}u+\beta & 0 \\ w_z-u_{zz} & w-\frac{1}{2}u_z & \frac{1}{2}u+\beta \end{pmatrix},$$

$$V_1 = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \quad U_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \tag{11}$$

where u and w are real functions of z,t; β=const.

Substituting (11) into (5) and (4) we obtain the following system of equations:

$$w_t = w_{zz} - u_{zzz} - \frac{3}{4}(u^2)_z - \beta u_z , \quad u_t = \frac{4}{3}w_z - u_{zz} . \tag{12}$$

Excluding w from (12) we obtain the Boussinesq equation

$$3u_{tt} + 4\beta\, u_{zz} + 3(u^2)_{zz} + u_{zzzz} = 0 \tag{13}$$

for the function u.

4. Let us return to the SDYM equations (1). We now suppose that the components A_t, A_u, A_y, A_z of the gauge field depend only on t and $x\equiv y+z= 2^{1/2}x_3$. Then $\partial_y A_\mu = \partial_z A_\mu = \partial_x A_\mu$ and Eqs.(1) take the form

$$\partial_t A_z - \partial_x A_t + [A_t, A_z] = 0, \quad \partial_x A_u - [A_u, A_y] = 0,$$

$$\partial_t A_u - \partial_x (A_z - A_y) + [A_t, A_u] + [A_z, A_y] = 0 . \tag{14}$$

We define:

$$U_0 := A_z , \; V_0 := A_t , \; U_1 := A_u , \; V_1 := A_z - A_y , \; V_2 \equiv U_1 . \tag{15}$$

It is not difficult to verify that now Eqs.(14) may be rewritten as a zero curvature equation

$$\partial_t U(\lambda) - \partial_x V(\lambda) + [V(\lambda), U(\lambda)] = 0 , \tag{16}$$

where V and U depend on a spectral parameter λ in the following way:

$$V(\lambda) = V_0(x,t) + \lambda V_1(x,t) + \lambda^2 V_2(x,t),$$

$$U(\lambda) = U_0(x,t) + \lambda U_1(x,t), \quad V_2 = U_1 . \tag{17}$$

Proposition 2. *All the integrable equations in* d=2 *having zero curvature representation (16),(17) may be obtained by reduction of the SDYM equations (1) after the identification of the components* A_t, A_u, A_y, A_z *of gauge field with* U_0, V_0, U_1, V_1 *by formulae (15).*

Proof follows from formulae (14)-(17).

Remark. The SDYM equations (1) for gauge fields depending only on t, u and x are equivalent to the equations of the modified chiral model in $\mathbb{R}^{2,1}$ (see [4,5]). That is why each solution of the model (16), (17) gives the solution of the modified chiral model equations in $\mathbb{R}^{2,1}$.

5. Example 4. *Generalized NLS equations.* These equations

were introduced by Fordy and Kulish [18] (see also [19, 20]). Here we shall follow [18, 20]. Let G/H be the hermitian symmetric space, $\mathcal{G}$ and $\mathcal{H}$ be the Lie algebras of the Lie groups G and H. Then $\mathcal{G} = \mathcal{H} \oplus \mathcal{P}$ and $[\mathcal{H},\mathcal{H}]\subset\mathcal{H}$, $[\mathcal{H},\mathcal{P}]\subset\mathcal{P}$, $[\mathcal{P},\mathcal{P}]\subset\mathcal{H}$. A special feature of hermitian symmetric spaces is the existence of an element $A\in\mathcal{H}$ such that $\mathcal{H} = \{ B\in\mathcal{G}: [A,B]=0\}$. Matrix ad_A has only three distinct eigenvalues 0, $\pm a$ and $[A,\mathcal{H}]=0$, $[A,X^{\pm}]=\pm a X^{\pm}$ for all $X^{\pm}\in\mathcal{P}^{\pm}$, $\mathcal{P}=\mathcal{P}^{+}\oplus\mathcal{P}^{-}$.

Let $e_{\pm\alpha}$ be a basis of the space $\mathcal{P}^{\pm}$ and Λ be an arbitrary constant diagonal matrix. Suppose that [18, 20]

$$U_0 = - \sum_{\alpha} (q^{\alpha} e_{\alpha} + r^{-\alpha} e_{-\alpha}), \; U_1 = -A, \; V_1 = U_0 , \; V_2 = U_1 ,$$

$$V_0 = -\frac{1}{a} \sum_{\alpha} (q^{\alpha}_x e_{\alpha} - r^{-\alpha}_x e_{-\alpha}) + \frac{1}{a} \sum_{\alpha,\beta} q^{\alpha} r^{-\beta} [e_{\alpha}, e_{-\beta}] - \Lambda, \tag{18}$$

where $q^{\alpha} = q^{\alpha}(x,t)$ and $r^{-\alpha} = r^{-\alpha}(x,t)$ are functions of x, t. Substitution of (18) into (17), (16) gives the generalized NLS equations

$$a\, q^{\alpha}_t = q^{\alpha}_{xx} + \sum_{\beta,\gamma,\delta} R^{\alpha}_{\beta,\gamma,-\delta} q^{\beta} q^{\gamma} r^{-\delta} + \omega_{\alpha} q^{\alpha} ,$$

$$-a\, r^{-\alpha}_t = r^{-\alpha}_{xx} + \sum_{\beta,\gamma,\delta} R^{-\alpha}_{-\beta,-\gamma,\delta} r^{-\beta} r^{-\gamma} q^{\delta} + \omega_{\alpha} r^{-\alpha}, \tag{19}$$

where $R^{\alpha}_{\beta,\gamma,-\delta}$ are components of the curvature tensor of the symmetric space G/H, given by $[e_{\beta},[e_{\gamma},e_{-\delta}]]=R^{\alpha}_{\beta,\gamma,-\delta}\, e_{\alpha}$. The numbers ω_{α} are linear combinations of the eigenvalues of Λ.

We can set $a=i$, $r^{-\alpha} = (q^{\alpha})^*$ or $r^{-\alpha} = -(q^{\alpha})^*$, where $*$ is a complex conjugation [18]. Some solutions of the generalized NLS equations with this restriction may be found, e.g., in

[13]. If $q^{\alpha}_t = r^{-\alpha}_t = 0$ and $r^{-\alpha} = -q^{\alpha}$, then we obtain the integrable hamiltonian system

$$q^{\alpha}_{xx} = \sum_{\beta,\gamma,\delta} R^{\alpha}_{\beta,\gamma,-\delta}\, q^{\beta} q^{\gamma} q^{\delta} - \omega_{\alpha} q^{\alpha} , \qquad (20)$$

connected with the hermitian symmetric space G/H [20]. On the other hand, if in (14) we require the components of gauge field be independent of t, then Eqs.(14) will coincide with the modified Nahm equations considered in [8,9]. Therefore, Eqs.(20) are reduction of the modified Nahm equations.

6. Example 5. *Matrix KdV equation.* Consider $\mathcal{G}=sl(2n,\mathbb{C})$. We set (cf. [21]):

$$U_0 = \begin{pmatrix} \mathbf{0}_n & -\mathbf{1}_n \\ u & \mathbf{0}_n \end{pmatrix}, \quad U_1 = \begin{pmatrix} \mathbf{0}_n & \mathbf{0}_n \\ -\mathbf{1}_n & \mathbf{0}_n \end{pmatrix}, \quad V_1 = \begin{pmatrix} \mathbf{0}_n & -\mathbf{1}_n \\ \frac{u}{2} & \mathbf{0}_n \end{pmatrix},$$

$$V_0 = \frac{1}{4} \begin{pmatrix} u_x & -2u \\ 2u^2 + u_{xx} & -u_x \end{pmatrix}, \quad V_2 = U_1 , \qquad (21)$$

where $\mathbf{0}_n$ and $\mathbf{1}_n$ are zero and unit n×n matrices, u is an arbitrary n×n matrix. If one substitutes (21) into (17), then Eq.(16) coincides with the matrix KdV equation:

$$4\, u_t - 3(u\, u_x + u_x u) - u_{xxx} = 0. \qquad (22)$$

Equation (22) was introduced by Wadati and Kamijo [22] and was considered by Calogero and Degasperis [23], Athorne and Fordy [14] and by other authors.

Example 6. *KdV and modified KdV equations.* In Eq.(22) let us take n=1. Then Eq.(22) coincides with the standard KdV equation

$$4\, u_t - 6\, u\, u_x - u_{xxx} = 0$$

for the real function u(x,t).

Now let n=2 in (22) and

$$u = \begin{pmatrix} w^2 & iw \\ iw & w^2 \end{pmatrix}, \qquad (23)$$

where w(x,t) is a real function. Then, matrix KdV equation (22) is reduced to the standard MKdV equation [22]:

$$4\, w_t - 6\, w^2 w_x - w_{xxx} = 0.$$

If n=m+k (k≥1), then matrix u can be chosen in the form

$$u = \begin{pmatrix} vv^T & iv_x \\ iv_x^T & v^T v \end{pmatrix}, \qquad (24)$$

where v is the m×k matrix and superscript T means matrix transpose. After substitution of (24) into (22), Eq.(22) is reduced to the generalized MKdV equation

$$4\, v_t - 3(v_x v^T v + vv^T v_x) - v_{xxx} = 0 , \qquad (25)$$

introduced by Athorne and Fordy [14].

7. To each solution of the equations mentioned in Examples 1-3 the identification (3) permits one to correspond a real solution of the YM equations in the space $\mathbb{R}^{2,2}$ and a complex solution of the YM equations in the space $\mathbb{R}^{3,1}$. To each solution of the equations mentioned in Examples 4-6 the identification (15) permits one to correspond a solution of the modified chiral model equations in $\mathbb{R}^{2,1}$.

Moreover, as it was shown by Ooguri and Vafa [24], the strings with a gauged N=2 supersymmetry on the world-sheet have as a target space the space $\mathbb{R}^{2,2}$ with two time-like dimensions. Additional time-like dimensions in the theories of the Kaluza-Klein type, in the theories of strings and membranes were also considered by Sakharov [25], Aref'eva and Volovich [26], Blencow and Duff [27], Popov [28] and other authors. In the low-energy limit the equations of motion of the bosonic sector of the open and heterotic N=2 strings coincide with the SDYM equations (1) for gauge fields in $\mathbb{R}^{2,2}$. Therefore the solutions of the equations from Examples 1-6 give vacuum configurations for the open and heterotic N=2 strings.

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Солитонные уравнения
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Показано, что уравнения автодуальности для калибровочных полей могут быть редуцированы к уравнениям Буссинеска, модели N-волн, одномерной газовой динамики /ОГД/, матричному уравнению Кортевега-де Вриса /КДВ/ и обобщенным нелинейным уравнениям Шредингера /НШ/.

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Soliton Equations and Self-Dual
Gauge Fields

The Boussinesq, N-wave, one-dimensional gas dynamics (OGD), matrix Korteweg-de Vries (KdV) and generalised nonlinear Schrödinger (NLS) equations are shown to be reductions of the self-duality equations for gauge fields.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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