92-136

объединенный
институт
ядерных
исследований
дубна

E2-92-136

S.V.Shabanov*

THE PROPER FIELD OF CHARGES AND GAUGE-INVARIANT VARIABLES IN ELECTRODYNAMICS

Submitted to "Journal of Physics A: Math. and Gen."

*E-mail: shabanov@theor.jinrc.dubna.su

1992

1 Introduction

Modern elementary particle physics is based on gauge theories [2]. Equations of motion of these theories are invariant with respect to gauge transformations containing arbitrary functions of coordinates and time. Owing to the principle of the gauge invariance, all physical quantities (observables) must be gauge-invariant [3]. For instance, electromagnetic potentials are defined up to the gradient of an arbitrary function, but electromagnetic field strengths being observable are gauge-invariant. In a quantum gauge theory, all physical dynamical variables (operators) must also be gauge-invariant [3].

A charged field $\psi(x)$ in electrodynamics cannot play the role of a physical variable because it changes its phase under gauge transformations,

$$\psi(x) \to \exp\left(ie\omega(x)\right)\psi(x) \tag{1.1}$$

with $\omega = \omega(x) = \omega(\mathbf{x}, t)$ being an arbitrary function, e is a coupling constant. Dirac proposed [1] to use the gauge-invariant field

$$\Psi = \exp\left(-ie\Delta^{-1}\partial_i A_i\right)\right)\psi \equiv \exp(ie\chi)\psi \tag{1.2}$$

as a physical dynamical variable describing charged particles. Here A_i, $i = 1,2,3$, is the vector potential; $A_i \to A_i + \partial_i\omega$ under gauge transformations; Δ means the Laplace operator. In QED the field $\hat{\Psi}$ describes creation and annihilation of charges together with their proper electric field (the Coulomb one) being non-dynamical [1]. This proper field gives rise to the static interaction of charges that obeys the Coulomb law.

Generalizing Dirac's idea one could expect that determination of all gauge-invariant operators is a key for finding a proper field of charges in any gauge theory and, hence, the law of a static interaction of the charges, which is important for understanding the confinement mechanism in non-Abelian gauge theories.

In the present paper we verify this idea with an example of electrodynamics. We find out that except the choice (1.2) there exists an infinite number of ways to determine gauge-invariant variables in electrodynamics (the choice of χ in (1.2) is not the only one). The proper field of charges strongly depends on the way of separating physical variables. For example, it may be localized on a contour or on a surface [4],[5]. The electromagnetic energy of two opposite charges surrounded by the proper field of that kind does not coincide with the Coulomb one. For the previous examples, it is proportional to the distance between them or to the logarithm of the distance, respectively [4],[5]. So, the natural question arises: what is a connection between gauge-invariant variables and the static interaction of charges?

A subsequent analysis of the dynamics shows that all the proper fields turn out unstable except the Coulomb one. Their decay is accompanied by radiation of electromagnetic waves outgoing to spatial infinity and creation of the Coulomb field surrounding charges, i.e., the Coulomb interaction is restored in due course. Therefore, the choice (1.2) of physical dynamical variables suggested by Dirac stands out against all others. The example of electrodynamics teaches that the choice of gauge-invariant variables (and/or gauge-invariant states) does not guarantee yet finding the right law of the potential interaction of charges because states induced by these variables may turn out unstable.

In Sec.2 the choice of gauge-invariant variables in electrodynamics of point-like particles is discussed. The Hamiltonian dynamics in these variables is constructed.

In Sec.3 we investigate the dependence of the proper field of charges on the choice of gauge-invariant variables. We prove that the Coulomb field is the only stable proper field of a charge. Other proper field configurations break down with radiating electromagnetic waves and creating the Coulomb proper field of charges.

In Sec.4 we analyze in detail the decay of the classical electromagnetic string when in an initial state the proper electric field of two opposite charges is localized on a contour connecting them.

Secs. 5 and 6 are devoted to QED. We calculate the proper fields in states induced by different gauge-invariant operators describing dynamics of charges and investigate their evolution. We demonstrate that the proper field of charges in QED behaves like in the classical case, i.e., it breaks down with photon emission and creation of the Coulomb field that is stable. In particular, the electromagnetic field energy of the electromagnetic string state

$$\hat{\bar{\psi}}(\mathbf{x}) \exp\left(ie\int_{\mathbf{y}}^{\mathbf{x}} dz_i\, \hat{A}_i(\mathbf{z})\right)\hat{\psi}(\mathbf{y})\,|0\rangle\,, \tag{1.3}$$

where the integration is carried out over the contour connecting points $\mathbf{x}$ and $\mathbf{y}$, is proportional to the contour length in the limit of infinite masses of charges [4]. Due to this fact, the state (1.3) is usually identified with the string one. However, during the time evolution the photon radiation takes away a part of this energy. The remaining energy is equal to the Coulomb interaction energy. The final state contains two charges (1.2) with their Coulomb field and photons going to infinity, therefore, the electromagnetic field configuration in the state (1.3) turns out unstable. For this reason the state (1.3) cannot be treated as the string one.

In Sec.7 an incomplete elimination of unphysical degrees of freedom (when there remains a residual gauge symmetry in the theory) is considered.

2 Classical electrodynamics of point-like particles in gauge-invariant variables

Consider classical electrodynamics of point-like particles, as a simple example, to explain a connection between the choice of gauge-invariant variables and dynamical description. Let the Lagrangian read

$$L = \frac{1}{2}\int d^3x\,(\mathbf{E}^2 + \mathbf{B}^2) + L_{mat}\,, \tag{2.1}$$

$$L_{mat} = \sum_\alpha \left[\frac{1}{2}m_\alpha \dot{\mathbf{r}}_\alpha^2 + e_\alpha(\dot{\mathbf{r}}_\alpha, \mathbf{A}(\mathbf{r}_\alpha)) - e_\alpha A_0(\mathbf{r}_\alpha)\right] \tag{2.2}$$

where $\mathbf{E} = -\dot{\mathbf{A}} - \partial A_0$, $\mathbf{B} = \mathbf{curl}\,\mathbf{A}$ are electric and magnetic field strengths, respectively, $\mathbf{A} = (A_i)$, A_0 are vector and scalar potentials; the brackets in the second term in (2.2) denote the scalar product of vectors $\mathbf{r}_\alpha$; α enumerates particles, $\mathbf{r}_\alpha$, e_α are their position vectors and charges, respectively. Lagrangian (2.1) gets the total time derivative

$$L \to L + \frac{d}{dt}\sum_\alpha e_\alpha\omega(\mathbf{r}_\alpha, t) \tag{2.3}$$

under the gauge transformations

$$\mathbf{A} \to \mathbf{A} + \partial\omega\,, \quad A_0 \to A_0 - \dot{\omega} \tag{2.4}$$

with ω being an arbitrary function of coordinates and time. Therefore, equations of motion induced by the Lagrangian are invariant with respect to the gauge transformations.

The existence of the symmetry transformations containing one arbitrary function does not mean that the electrodynamic field has just one unphysical degree of freedom. Actually, it has two such degrees of freedom. It follows from the Hamiltonian formalism for the theory [1],[3]. Indeed, defining canonical momenta in the standard way $\pi = \delta L/\delta \dot{\mathbf{A}} = -\mathbf{E}$,

$$\mathbf{p}_\alpha = \frac{\partial L}{\partial \dot{\mathbf{r}}_\alpha} = m_\alpha \dot{\mathbf{r}}_\alpha + \mathbf{A}(\mathbf{r}_\alpha) , \tag{2.5}$$

$$\pi_0 = \frac{\delta L}{\delta \dot{A}_0} = 0 \tag{2.6}$$

one finds the Hamiltonian

$$H = \int d^3x \left(\frac{1}{2}\pi^2 + \mathbf{B}^2 + A_0 \sigma \right) + H_{mat} , \tag{2.7}$$

$$H_{mat} = \sum_\alpha \frac{1}{2m_\alpha} (\mathbf{p}_\alpha - e_\alpha \mathbf{A}(\mathbf{r}_\alpha))^2 , \tag{2.8}$$

$$\sigma = (\boldsymbol{\partial}, \boldsymbol{\pi}) + J_0 \tag{2.9}$$

where $J_0(\mathbf{x}) = \sum_\alpha e_\alpha \delta^3(\mathbf{x} - \mathbf{r}_\alpha)$ is the charge density. Equation (2.6) shows that the momentum conjugated to A_0 vanishes. It is the primary constraint in the theory [3] which must be satisfied at all time moments. This requirement leads to the secondary constraint (the Gauss law) [3]

$$\dot{\pi}_0 = \{\pi_0, H\} = -\sigma = 0 ; \tag{2.10}$$

here $\{,\}$ denotes the Poisson brackets. By definition

$$\{r_{\alpha i} , p_{\beta j}\} = \delta_{ij}\delta_{\alpha\beta} , \tag{2.11}$$

$$\{A_i(\mathbf{x}) , \pi_j(\mathbf{y})\} = \delta_{ij}\delta^3(\mathbf{x} - \mathbf{y}) ; \tag{2.12}$$

the Poisson brackets for other canonical variables are equal to zero. It is easy to be convinced that $\dot{\sigma} = \{\sigma, H\} \equiv 0$, i.e., there are no more constraints in the theory [3]. The existence of two constraints (2.6) and (2.10) in the theory means that solutions of the Hamiltonian equations of motion, satisfying also (2.6) and (2.10), contain two arbitrary functions of coordinates and time [3]. Thus, the electromagnetic field has only two physical degrees of freedom corresponding to two independent polarizations of electromagnetic waves.

The variable A_0 is pure unphysical since its canonical momentum vanishes. The second unphysical variable can be selected with the help of a canonical transformation after which σ becomes a new canonical momentum. Then a generalized coordinate conjugated to σ turns out unphysical like A_0.

There are an infinite number of such transformations [5]. We restrict ourselves to linear transformations. Consider the change of variables [5]

$$\mathbf{A} = \boldsymbol{\alpha} + \boldsymbol{\partial}\chi \tag{2.13}$$

where the field $\boldsymbol{\alpha}$ obeys the additional condition

$$(\mathbf{K}, \boldsymbol{\alpha})(\mathbf{x}) \equiv \int d^3y K_i(\mathbf{x}, \mathbf{y}) \alpha_i(\mathbf{y}) = 0. \tag{2.14}$$

with K_i being a linear operator. The inverse transformations read

$$\chi = (\mathbf{K}, \boldsymbol{\partial})^{-1}(\mathbf{K}, \mathbf{A}) , \tag{2.15}$$

$$\boldsymbol{\alpha} = \mathbf{A} - \boldsymbol{\partial}(\mathbf{K}, \boldsymbol{\partial})^{-1}(\mathbf{K}, \mathbf{A}) . \tag{2.16}$$

The variable χ is translated under gauge transformations (2.4), but $\boldsymbol{\alpha}$ remains untouched.

One can take operators ∂_i or δ_{i3} as examples of the operator K_i. In the first case $\boldsymbol{\alpha}$ is a transverse field [1]; in the second one it is an axial field, $\alpha_3 = 0$. One should emphasize that the condition (2.14) does not mean a gauge fixing because $\boldsymbol{\alpha}$ is gauge-invariant. The problem of existence of the inverse operator $(\mathbf{K}, \boldsymbol{\partial})^{-1}$ is considered in Appendix A. Below the formal symbol $(\mathbf{K}, \boldsymbol{\partial})^{-1}$ is used. It does not affect our subsequent analysis (see Appendix A).

One can choose the quantity $-\sigma$ as a new canonical momentum $\pi_\chi = -\sigma$ conjugated to χ since it follows from (2.11) and (2.12) that

$$\{\chi(\mathbf{x}) , \pi_\chi(\mathbf{y})\} = \delta^3(\mathbf{x} - \mathbf{y}). \tag{2.17}$$

Moreover, due to the gauge invariance of $\boldsymbol{\alpha}$, we obtain $0 = \{\boldsymbol{\alpha}, \sigma\} = -\{\boldsymbol{\alpha}, \pi_\chi\} = 0$. Here and below all singular relations like (2.12) and (2.17) are defined as equalities of generalized functions. For instance, (2.17) implies

$$\left\{ \int d^3x \, f\chi , \int d^3y \, g\pi_\chi \right\} = \int d^3x \, fg \tag{2.18}$$

for two arbitrary trial functions f and g.

To finish derivation of the canonical transformation corresponding to the coordinate transformation (2.15), (2.16), one has to determine new canonical momenta of particles and the field $\boldsymbol{\alpha}$. The quantities

$$\tilde{\mathbf{p}}_\beta = \mathbf{p}_\beta + e_\beta \boldsymbol{\partial}\chi(\mathbf{r}_\beta) \tag{2.19}$$

are the new gauge-invariant canonical momenta of charged particles because they have zero Poisson brackets with σ and satisfy canonical relations (2.11) if $\mathbf{p}_\beta \to \tilde{\mathbf{p}}_\beta$ there.

Let us introduce operators $\mathbf{e}^a$, $a = 1, 2$ which obey the following identities [5]

$$(\mathbf{K} , \mathbf{e}^{aT}) = 0 , \quad (\mathbf{e}^a , \mathbf{K}^T) = 0 ; \tag{2.20}$$

$$(\mathbf{e}^a , \mathbf{e}^{bT}) = \delta^{ab} \tag{2.21}$$

where the sign "T" means the transposition. The operator transposed to the given one is defined by the rule

$$\int d^3x f(Og) = \int d^3x (O^T f) g \tag{2.22}$$

for arbitrary functions f and g. For example, put $\mathbf{K} = \boldsymbol{\partial}$, then $\boldsymbol{\partial}^T = -\boldsymbol{\partial}$. The operators ∂_i, $\epsilon_{ijk}x_j\partial_k$ and $\epsilon_{ijk}\partial_j \cdot \epsilon_{kln}x_l\partial_n$ form a system of linearly independent operators of which one can build the orthogonal operators (2.20), (2.21).

Due to the condition (2.14), the field $\boldsymbol{\alpha}$ contains only two independent field variables,

$$\boldsymbol{\alpha} = \mathbf{e}^{aT}\alpha^a , \tag{2.23}$$

$$\alpha^a = (\mathbf{e}^a , \boldsymbol{\alpha}) = (\mathbf{e}^a, \mathbf{A}) - (\mathbf{e}^a, \boldsymbol{\partial})(\mathbf{K}, \boldsymbol{\partial})^{-1}(\mathbf{K}, \mathbf{A}) . \tag{2.24}$$

Let us decompose the vector $\boldsymbol{\pi}$ over the operator system (2.20), (2.21)

$$\boldsymbol{\pi} = \mathbf{e}^{aT}\pi^a + \mathbf{K}^T\phi . \tag{2.25}$$

Using the rule (2.18) and definition (2.24) we are convinced that the quantities $\pi^a = (\mathbf{e}^a, \boldsymbol{\pi})$ are momentum canonically conjugated to α^a,

$$\{\alpha^a(\mathbf{x}) , \pi^b(\mathbf{y})\} = \delta^{ab}\delta^3(\mathbf{x} - \mathbf{y}) , \quad \{\pi^a , \chi\} = 0 . \tag{2.26}$$

[1] This case is considered in [6] in connection with the Aharonov-Bohm effect interpretation.

The quantity ϕ can be found as a functional of π^a and π_χ by applying the operator $\boldsymbol{\partial}$ to both parts of equality (2.25) with a subsequent representation of its left-hand side via π_χ and J_0. The final expression of $\boldsymbol{\pi}$ via the new canonical momenta reads

$$\boldsymbol{\pi} = \left(\mathbf{e}^{aT} + \mathbf{K}^T(\mathbf{K},\boldsymbol{\partial})^{-1T}(\boldsymbol{\partial},\dot{\mathbf{e}}^{aT})\right)\pi^a + \mathbf{K}^T(\mathbf{K},\boldsymbol{\partial})^{-1T}(\pi_\chi + J_0)\,. \tag{2.27}$$

Substituting (2.27) and (2.19) into (2.7) and taking into account the constraints $\pi_0 = \pi_\chi = 0$ we obtain the physical Hamiltonian,

$$H^{ph} = \int d^3x\left(\frac{1}{2}\pi^a M^{ab}\pi^b + \frac{1}{2}\alpha^a L^{ab}\alpha^b - \pi^a\varphi^a J_0 + \frac{1}{2}J_0\Lambda J_0\right) + H^{ph}_{mat}\,, \tag{2.28}$$

$$H^{ph}_{mat} = \sum_\beta \frac{1}{2m_\beta}\left(\tilde{\mathbf{p}}_\beta + e_\beta\boldsymbol{\alpha}(\mathbf{r}_\beta)\right)^2 \tag{2.29}$$

where we introduce the linear operators

$$L^{ab} = -(\mathbf{e}^a,\Delta\mathbf{e}^{bT}) - (\mathbf{e}^a,\boldsymbol{\partial})(\mathbf{e}^b,\boldsymbol{\partial})^T\,, \tag{2.30}$$

$$M^{ab} = \delta^{ab} + \Gamma^{ab}\,, \tag{2.31}$$

$$\Gamma^{ab} = \varphi^a\Lambda^{-1}\varphi^{bT}\,, \tag{2.32}$$

$$\varphi^a = (\mathbf{e}^a,\boldsymbol{\partial})\Lambda\,, \tag{2.33}$$

$$\Lambda = (\mathbf{K},\boldsymbol{\partial})^{-1}(\mathbf{K},\mathbf{K}^T)(\mathbf{K},\boldsymbol{\partial})^{-1T}\,. \tag{2.34}$$

The Hamiltonian equations of motion generated by (2.28) define dynamics in gauge-invariant variables [5]. So, the physical Hamiltonian as well as Hamiltonian equations of motion depend on the choice of physical variables. However, to get a complete physical picture, one should connect these variables with observables [5].

3 Static fields in classical electrodynamics

Static electromagnetic fields are the simplest observables in electrodynamics since they determine a potential interaction of charges. Therefore, for describing the static fields, we consider an electromagnetic interaction of non-moving charges (static sources). With this purpose we take the limit $m_\alpha \to \infty$ in Hamiltonian (2.28). In this limit one can omit H^{ph}_{mat} in the whole Hamiltonian.

The interaction energy of charges is described by the term squared in J_0 in the Hamiltonian (2.28). For point-like sources, we obtain

$$\frac{1}{2}\int d^3x J_0\Lambda J_0 = \frac{1}{2}\,Q^2\Lambda(0) + \sum_{\alpha>\beta} e_\alpha e_\beta\Lambda(\mathbf{r}_\alpha - \mathbf{r}_\beta) \tag{3.1}$$

where $Q^2 = \sum_\alpha e_\alpha^2$ and $\Lambda(\mathbf{x}-\mathbf{y})$ is the operator Λ kernel viewed as the interaction potential. If, for instance, $\mathbf{K} = \boldsymbol{\partial}$, then $\Lambda = -\Delta^{-1}$. In this case the first term in (3.1) is the Coulomb self-energy of charges (it is infinite); the second term gives the energy of the Coulomb interaction of charges. However, taking $K_i = \delta_{i3}$ we do not obtain the Coulomb law. For this choice

$$(\mathbf{K},\boldsymbol{\partial})^{-1}f(\mathbf{x}) = \partial_3^{-1}f(\mathbf{x}) = \int_{-\infty}^{x_3} dx_3' f(\mathbf{x}_\perp, x_3')\,, \tag{3.2}$$

here $\mathbf{x}_\perp$ implies $x_{1,2}$. After simple calculations we derive the static energy (3.1) for two opposite charges being on the third coordinate axis,

$$\frac{1}{2}\int d^3x J_0\Lambda J_0 = -\delta^\perp(0)e^2 r \tag{3.3}$$

where r is a distance between the charges, i.e., the energy linearly rises with distance. The energy (3.1) can also be logarithmically rising with a distance between charges if we put $\mathbf{K} = \boldsymbol{\partial}_\perp$ being the gradient in the plane x_1Ox_2 [4],[5].

Assume for the moment that we do not know the Coulomb law as an experimental fact, and we have just Lagrangian (2.1) (in fact, this occurs in non-Abelian gauge theories). What could we say about the static interaction law then? Since all choices of gauge-invariant variables are a priori equivalent, this law looks quite arbitrary because Λ in (3.1) depends on the choice of $\mathbf{K}$. It is clear however that the interaction energy of static charges cannot depend on our arbitrariness. A solution of the paradox is to consider a total static energy of the system, being a value of the Hamiltonian (2.28) on static solutions of Hamiltonian equations of motion. It is always equal to the Coulomb energy and independent of choosing $\mathbf{K}$.

To prove this, consider the Hamiltonian equations of motion in the limit of static sources, $m_\alpha \to \infty$,

$$\dot{\pi}^a = \{\pi^a\,,\,H^{ph}\} = -L^{ab}\alpha^b\,, \tag{3.4}$$

$$\dot{\alpha}^a = \{\alpha^a\,,\,H^{ph}\} = M^{ab}\pi^b - \varphi^a J_0 \tag{3.5}$$

and require $\dot{\pi}^a = \dot{\alpha}^a = 0$, i.e., all fields are static. It follows from (3.4) that the second term in (2.28) gives no contribution to the total system energy (solutions of the equation $L^{ab}\alpha^b = 0$ are discussed in Appendix A). Equation (3.5) shows us that π^a has a static component induced by an external source J_0. The value of H^{ph} on static solutions of Eqs.(3.4), (3.5) reads

$$E_{st} = \frac{1}{2}\int d^3x J_0\left(\Lambda - \varphi^{aT}(M^{-1})^{ab}\varphi^b\right)J_0\,. \tag{3.6}$$

Thus, the relation (3.1) gives just a part of the interaction energy of static sources.

We rewrite the operator M^{-1} in the form of the series

$$M^{-1} = (1+\Gamma)^{-1} = \sum_{n=0}^{\infty}(-1)^n\Gamma^n\,. \tag{3.7}$$

Then the second term in the left-hand side of (3.6) is easily calculated

$$\begin{aligned}\varphi^T M^{-1}\varphi &= \sum_{n=0}^{\infty}(-1)^n\varphi^T\Gamma^n\varphi = \sum_{n=1}^{\infty}(-1)^{n+1}\left(\varphi^T\varphi\Lambda^{-1}\right)^n\Lambda = \\ &= -\sum_{n=1}^{\infty}(\Lambda\Delta+1)^n\Lambda = \Delta^{-1} + \Lambda\end{aligned} \tag{3.8}$$

where we use the relation

$$\varphi^T\varphi \equiv \varphi^{aT}\varphi^a = -\Lambda\Delta\Lambda - \Lambda \tag{3.9}$$

that follows from the equality

$$\mathbf{e}_i^{aT}\mathbf{e}_j^a = \delta_{ij} - K_i^T(\mathbf{K},\mathbf{K}^T)^{-1}K_j \tag{3.10}$$

defining a projector on K-transverse fields. Substituting (3.8) into (3.6) we obtain the total static energy

$$E_{st} = -\frac{1}{2}\int d^3x J_0 \Delta^{-1} J_0 \tag{3.11}$$

that coincides with the Coulomb energy of a source J_0.

However, if we put $\alpha^a = \pi^a = 0$ in (2.28) at an initial time moment (the zero initial conditions, $\alpha^a(t=0) = \pi^a(t=0) = 0$, in Eqs. (3.4),(3.5)), then the total system energy turns out equal to (3.1) during all the time evolution because of the energy conservation law. It gains the impression that a choice of the initial conditions (in fact, a way of preparing an initial state of the system) may influence the static interaction of charges. It is not the case indeed. As we demonstrate below, Eqs. (3.4), (3.5) have non-stationary solutions under such initial conditions, therefore, the total energy consists of two parts – the Coulomb energy (3.11) and electromagnetic radiation energy being taken away to infinity. In the limit of large time $t \to \infty$ the field energy localized in the neighbourhood of a source J_0 is equal to the Coulomb one.

The initial conditions $\alpha^a(t=0) = \pi^a(t=0) = 0$ pick out all admissible electric field configurations, in absence of magnetic fields, which do not contradict to the Gauss law (2.10), i.e., they might be viable in nature, at least in principle. Indeed, in this initial state magnetic fields vanish, $\mathbf{B}(\mathbf{x},0) = 0$, and the electric field has the form

$$\mathbf{E}(\mathbf{x},0) = -\boldsymbol{\pi}(\mathbf{x},0) = -\mathbf{K}^T(\mathbf{K},\boldsymbol{\partial})^{-1T} J_0(\mathbf{x}) \tag{3.12}$$

in accordance with (2.27). It satisfies the Gauss law. Note that the Gauss equation (2.10) fixes just the longitudinal part of the electric field; its transverse components remain arbitrary. We may choose them taking into consideration an initial state of electromagnetic fields. Thus, a choice of gauge-invariant variables in classical electrodynamics can be related with preparing of the initial state of an electrodynamic system. It turns out that the same occurs in QED (see Secs.5,6).

The initial state (3.12) is not stable if $\mathbf{K} \neq \boldsymbol{\partial}$. To understand this, note that electromagnetic radiation is described by transverse fields. The electric field (3.12) contains a transverse component when $\mathbf{K} \neq \boldsymbol{\partial}$. So, according to the Maxwell equations, it creates a magnetic field ($\mathbf{B} \neq 0$) that makes then the electric field changed etc., i.e., the electromagnetic radiation arises. The radiation fields exist by themselves, hence, their energy has no relations with the interaction energy of static sources (a consequence of the linearity of the Maxwell equations). Thus, one has to investigate the time evolution of the state (3.12) to separate the static interaction of the sources.

For describing the decay of the state (3.12), one should find a spatial distribution of electromagnetic fields as a function of time. Equations of motion (3.4), (3.5) have the simplest form when $\mathbf{K} = \boldsymbol{\partial}$,

$$\partial_t \alpha^\perp = \pi^\perp, \qquad \partial_t \pi^\perp = \Delta \alpha^\perp \tag{3.13}$$

where $\alpha^\perp$ is defined by (2.23) with $\mathbf{e}^a$ obeying relations (2.20), (2.21) for $\mathbf{K} = \boldsymbol{\partial}$. The canonical momenta $\pi_i^\perp = P_{ij}^\perp \pi_j$, $P_{ij}^\perp = \delta_{ij} - \partial_i \Delta^{-1}\partial_j$, are the transverse components of $\boldsymbol{\pi}$, $(\boldsymbol{\partial}, \boldsymbol{\pi}^\perp) = (\boldsymbol{\partial}, \boldsymbol{\alpha}^\perp) = 0$. The initial electromagnetic field configuration (3.12), $\mathbf{B} = \mathbf{curl}\ \boldsymbol{\alpha}^\perp = 0$ induces the following initial conditions for Eqs. (3.13)

$$\alpha_i^\perp|_{t=0}(\mathbf{x}) = 0\,, \quad \pi_i^\perp|_{t=0}(\mathbf{x}) = P_{ij}^\perp K_j^T(\mathbf{K},\boldsymbol{\partial})^{-1T} J_0(\mathbf{x}) \tag{3.14}$$

The solution of Eqs. (3.13) read

$$\boldsymbol{\alpha}^\perp(\mathbf{x},t) = \partial_t D_t \boldsymbol{\alpha}^\perp(\mathbf{x},0) + D_t \boldsymbol{\pi}^\perp(\mathbf{x},0)\,, \quad \boldsymbol{\pi}^\perp(\mathbf{x},t) = \partial_t \boldsymbol{\alpha}^\perp(\mathbf{x},t) \tag{3.15}$$

where the operator D_t has the standard form

$$D_t f(\mathbf{x}) = \frac{1}{4\pi t}\int d^3x' \delta(t - |\mathbf{x}-\mathbf{x}'|) f(\mathbf{x}')\,, \quad t > 0\,. \tag{3.16}$$

Substituting (3.14) into (3.15) we find the following expressions for the electromagnetic fields [2]

$$\begin{aligned} \mathbf{E}(\mathbf{x},t) &= -\boldsymbol{\pi}^\perp(\mathbf{x},t) + \boldsymbol{\partial}\Delta^{-1}J_0(\mathbf{x}) = \\ &= \partial_t D_t \mathbf{K}^T(\mathbf{K},\boldsymbol{\partial})^{-1T} J_0(\mathbf{x}) + \boldsymbol{\partial}\int d^3x' \theta(t - |\mathbf{x}-\mathbf{x}'|)\frac{J_0(\mathbf{x}')}{4\pi|\mathbf{x}-\mathbf{x}'|}\,, \end{aligned} \tag{3.17}$$

$$\mathbf{B}(\mathbf{x},t) = \mathbf{curl}\ \boldsymbol{\alpha}^\perp(\mathbf{x},t) = \mathbf{curl}\ D_t \mathbf{K}^T(\mathbf{K},\boldsymbol{\partial})^{-1T} J_0(\mathbf{x})\,. \tag{3.18}$$

The first term entering into (3.17) and (3.18) describes the electromagnetic radiation outgoing from the effective source $\mathbf{K}^T(\mathbf{K},\boldsymbol{\partial})^{-1T} J_0$. The second term determines filling the space around a source J_0 by the Coulomb field.

Thus, the electromagnetic field configuration (3.12) breaks down creating radiation of electromagnetic waves. Each spatial point, where the right-hand side of (3.12) does not vanish, serves as a source of the radiation. In due course, the radiation fields go away to spatial infinity and a region around charges is filled by their Coulomb field. The radiation does not arise just at $\mathbf{K} = \boldsymbol{\partial}$ in (3.12) since in this case $\boldsymbol{\pi}^\perp(\mathbf{x}, t=0) \equiv 0$ (see (3.14), (3.15)). Thus, the only stable proper field of charges is the Coulomb one.

As an illustration of the general formulas (3.17), (3.18), let us elucidate what happens with a classical electromagnetic string, when the electric field in the initial state is localized on a contour connecting two opposite charges, during its evolution.

4 The decay of the classical electromagnetic string

Consider two opposite point-like charges e and $-e$ attached at points $\mathbf{y}_1$ and $\mathbf{y}_2$, respectively. Let a contour C pass through these points, and $\mathbf{x} = \mathbf{z}(s) \in C$ where s is a parameter describing a position point on the contour. Put

$$\mathbf{K} = \frac{d\mathbf{z}}{ds} \tag{4.1}$$

and take a plane so that the contour intersects it just once. Assume for simplicity that all planes, parallel to the chosen one, intersect the contour C also once. At each point of this plane, we attach the vector $d\mathbf{z}/ds$ taken at a common point of the plane and the contour. In so doing, we define the field of vectors $\mathbf{K}(\mathbf{x})$ parallel to tangent vectors of the contour C just in the whole space. At each spatial point $\mathbf{x}$ this field determines a contour passing through $\mathbf{x}$ and parallel to C. Thus, the vector field $\mathbf{K}(\mathbf{x})$ serves as the $\mathbf{K}$-operator in the supplementary condition (2.14) and

$$(\mathbf{K},\boldsymbol{\partial})^{-1}\mathbf{K} f(\mathbf{x}) = \int^{\mathbf{x}} d\mathbf{z} f(\mathbf{z}) \tag{4.2}$$

where the integral is taken over a contour passing through the point $\mathbf{x}$ and parallel to C in the sense pointed out above.

[2] They satisfy, of course, the Maxwell equations with the initial conditions (3.12) and $\mathbf{B} = 0$.

Substitution of $J_0(\mathbf{x}) = e\delta^3(\mathbf{x}-\mathbf{y}_1) - e\delta^3(\mathbf{x}-\mathbf{y}_2)$ and (4.2) into (3.17) and (3.18) leads to the following expressions for the electromagnetic field

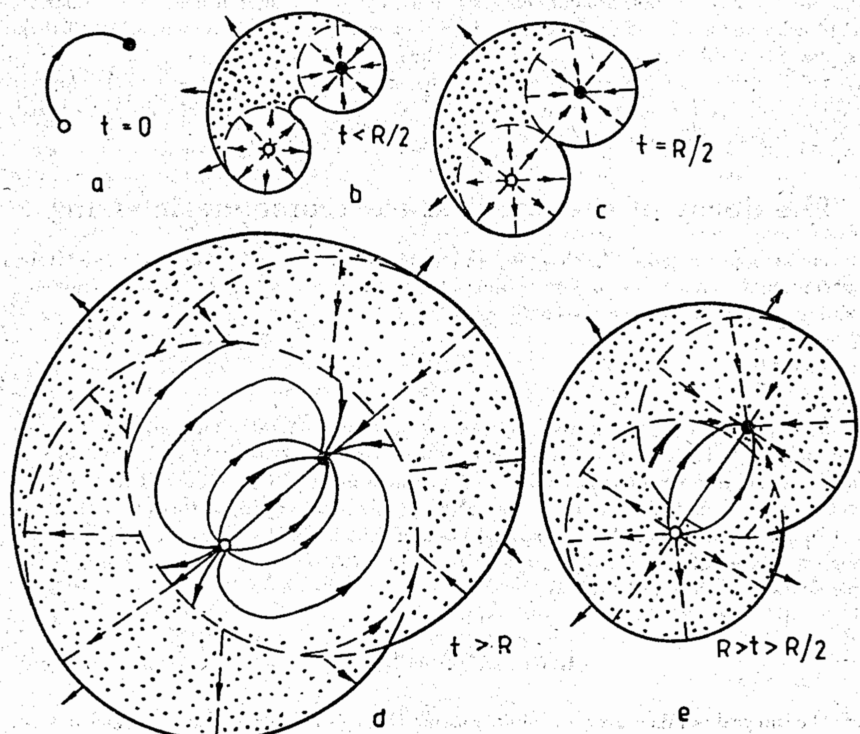
$$\mathbf{E}(\mathbf{x},t) = \partial_t \frac{e}{4\pi t}\int_{\mathbf{y}_2}^{\mathbf{y}_1} d\mathbf{z}\ \delta(t-|\mathbf{x}-\mathbf{z}|) + \theta(t-|\mathbf{x}-\mathbf{y}_1|)\partial\frac{e}{4\pi|\mathbf{x}-\mathbf{y}_1|} - \theta(t-|\mathbf{x}-\mathbf{y}_2|)\partial\frac{e}{4\pi|\mathbf{x}-\mathbf{y}_2|} + \mathbf{E}^s(\mathbf{x},t)\ , \quad (4.3)$$

$$\mathbf{E}^s(\mathbf{x},t) = -e\delta(t-|\mathbf{x}-\mathbf{y}_1|)\frac{\mathbf{x}-\mathbf{y}_1}{t^2} + e\delta(t-|\mathbf{x}-\mathbf{y}_2|)\frac{\mathbf{x}-\mathbf{y}_2}{t^2}\ , \quad (4.4)$$

$$\mathbf{B}(\mathbf{x},t) = \frac{e}{4\pi t}\ \mathbf{curl}\int_{\mathbf{y}_2}^{\mathbf{y}_1} d\mathbf{z}\ \delta(t-|\mathbf{x}-\mathbf{z}|) \quad (4.5)$$

where θ means the Heaviside function. When $t \to 0$, the magnetic field tends to zero, and the electric field is localized on the contour C between the points $\mathbf{y}_1$ and $\mathbf{y}_2$,

$$\mathbf{E}(\mathbf{x},t) \to \mathbf{E}^{st}(\mathbf{x}) = e\int_{\mathbf{y}_2}^{\mathbf{y}_1} d\mathbf{z}\delta^3(\mathbf{x}-\mathbf{z})\ , \qquad \mathbf{B}(\mathbf{x},t) \to 0 \quad (4.6)$$



since $D_{t=0} = 0$ and $\partial_t D_t(\mathbf{x}-\mathbf{x}')|_{t=0} = \delta^3(\mathbf{x}-\mathbf{x}')$. The electric field is directed along the contour C and has a constant absolute value on it. The electromagnetic energy of this

state is proportional to the contour length. For this reason, this initial state is called the electromagnetic string. It is impossible to make a state like that on practice. However, it might exist in the dual electrodynamics with magnetic monopoles, like the Abrikosov vortex state in superconductors of the second kind [7].

The radiation field can be constructed with the help of the Huygens principle (each point of the contour C serves as the radiation source). A distribution of the electromagnetic fields (4.3)–(4.6) at different time moments is represented in Fig.1. An external boundary of the wave front is picked out by a thick continuous line. Arrows attached to it show a direction of its propagation. Field lines of the creating Coulomb field are pictured with thin continuous lines. A region containing the Coulomb field of two charges is outlined by two dashed circles. A spatial region occupied by the radiation field outgoing to infinity is filled by dots.

When $t < |\mathbf{y}_1 - \mathbf{y}_2|/2 \equiv R/2$ (Fig.1b) (the light velocity is assumed to be equal to 1), the Coulomb field differs from zero just inside spheres of the radius t and with centers at $\mathbf{y}_1$ and $\mathbf{y}_2$. Inside each sphere, this field coincides with the Coulomb field of an isolated charge. It is indicated by dashed lines with arrows. (Note that the field lines of the total electric field (Coulomb plus radiation) are meaningful, for this reason we picture the Coulomb field with the dashed lines in regions occupied also by the radiation fields.) The spheres widen with the light velocity as well as the wave front. They touch each other at $t = R/2$ (Fig.2c). When $t > R/2$ (Fig.1d), an overlapping region appears inside which the Coulomb field is equal to the sum of two Coulomb fields. A region without dots in Figs.1d,e is free of the radiation field and contains just the Coulomb one. This region widens with the light velocity. Therefore at $t \gg R$ the radiation field is localized inside an almost spherical front with a thickness $\sim R$ and goes away to spatial infinity. A space surrounding the charges is filled by the static Coulomb field of two charges. A distribution of the electromagnetic fields for $t > R$ is represented in Fig.1e.

Let on a distance from the string, say, larger than the string size R there be an observer with tools registering electromagnetic fields. The charges are invisible for the tools at the initial time moment because there are no electromagnetic fields outside the string. After a time interval, the tools will register a flash of radiation during $\tau \sim R$. Then charges become visible due to their Coulomb field.

5 An explicit solution of constraints in QED

To construct a quantum theory, one should change all canonical variables entering into Hamiltonian (2.7) and constraints by the corresponding operators so that their commutation relations are defined in accordance with the rule $[\ ,\ ] = i\{\ ,\ \}$ $(\hbar = 1)$. Operators of constraints must annihilate physical states [3]

$$\hat{\sigma}|\phi_{ph}\rangle = 0\ . \quad (5.1)$$

Here, as above, we ignore the pure unphysical degree of freedom π_0, A_0 [3].

Let the charge density operator read

$$\hat{J}_0(\mathbf{x}) = \sum_\alpha e_\alpha \hat{\psi}^+_\alpha(\mathbf{x})\hat{\psi}_\alpha(\mathbf{x}) \quad (5.2)$$

where

$$\left[\hat{\psi}_\alpha(\mathbf{x})\ ,\ \hat{\psi}^+_\beta(\mathbf{y})\right]_\pm = \delta_{\alpha\beta}\delta^3(\mathbf{x}-\mathbf{y})\ ; \quad (5.3)$$

the commutator and anticommutator are taken for Bose and Fermi fields, respectively. Dynamics of fields ψ can be determined in the standard way by gauging their free dynamics.

However, a concrete form of H_{mat} is not essential for what follows because we intend to investigate just configurations of proper fields of static sources. Therefore at the very beginning we assume these fields to be infinitely heavy (static) and neglect H_{mat} in the total Hamiltonian. In this way, we arrive at QED with static sources where the coupling of charges and the electromagnetic field is just due to the constraint (5.1).

Operators $\hat{\psi}^+_\alpha$ and $\hat{\psi}_\alpha$ describe respectively creation and annihilation of a point-like source with a charge e_α since

$$\left[\hat{J}_0(\mathbf{x})\, ,\, \hat{\psi}^+_\alpha(\mathbf{y})\right] = e_\alpha \delta^3(\mathbf{x}-\mathbf{y})\hat{\psi}^+_\alpha(\mathbf{y})\, , \tag{5.4}$$

i.e., the state $\hat{\psi}^+_\alpha|\phi\rangle$ is an eigenstate of the charge density operator if $\hat{J}_0|\phi\rangle = 0$. The operator $\hat{g} = \exp(-i\int d^3x\omega\hat{\sigma})$ defines gauge transformations of all field operators in the theory

$$\hat{g}\hat{\mathbf{A}}\hat{g}^+ = \hat{\mathbf{A}} + \partial\omega\, , \tag{5.5}$$

$$\hat{g}\hat{\psi}_\alpha\hat{g}^+ = \exp(-ie_\alpha\omega)\hat{\psi}_\alpha\, . \tag{5.6}$$

So, the requirement (5.1) means the gauge invariance of all physical states $\hat{g}|\phi_{ph}\rangle = |\phi_{ph}\rangle$.

There are several methods to take into account Eq.(5.1) in the quantum theory. One can define a physical ground state satisfying (5.1). Then any physical state may be built by applying gauge-invariant combinations of electromagnetic and charged field operators to the vacuum state [1]. We will consider this method in the next section. Here we discuss another method when Eq.(5.1) is solved explicitly.

Let us introduce new operators $\hat{\chi}$ and $\hat{\alpha}$ that are connected with the operators $\hat{\mathbf{A}}$ by the relations (2.15), (2.16) and new operators of charged fields

$$\hat{\Psi}_\alpha = \exp(ie_\alpha\hat{\chi})\hat{\psi}_\alpha\, , \qquad \hat{\Psi}^+_\alpha = \exp(-ie_\alpha\hat{\chi})\hat{\psi}^+_\alpha\, . \tag{5.7}$$

Operators (5.7) and $\hat{\alpha}$ are explicitly gauge-invariant and commute with $\hat{\sigma}$. Obviously, the commutation relations (5.3) remain valid for the operators (5.7).

Consider now the coordinate representation for electromagnetic fields operators, $\hat{\pi} = -i\delta/\delta\mathbf{A}$, and the holomorphic representation for charged fields, $\hat{\psi}_\alpha = \delta/\delta\psi^*_\alpha$, $\hat{\psi}^+_\alpha = \psi^*_\alpha$ with ψ^*_α being a complex field (it is the Grassmann one for fermion degrees of freedom, then both multiplication on it and the variation derivative with respect to ψ^*_α are assumed to be left). Vectors of states are treated as functionals of independent functional variables $\mathbf{A}$ and ψ^*_α. In this case relations (5.7), (2.15) and (2.16) define a change of variables, generally speaking, on a superspace [8]. It is easy to be convinced that

$$\hat{\sigma}\phi_{ph} = i\,\frac{\delta}{\delta\chi}\,\phi_{ph} = 0\, . \tag{5.8}$$

Thus, physical states are independent of χ that is an unphysical variable like A_0. This allows us to write the Hamiltonian only via physical degrees of freedom. It has the form (2.28) with $\hat{\pi}^a = -i\delta/\delta\alpha^a$ being the momentum operator conjugated to $\hat{\alpha}^a$.

In the static limit described above, we arrive at the quantum theory of two free fields α^a interacting with the quantum source $\hat{J}_0$ (the third term in (2.28)). Rewrite the free Hamiltonian of an electromagnetic field via the creation and annihilation operators

$$\hat{H}^{ph}_0 = \frac{1}{2}\int d^3x(\hat{\pi}^a M^{ab}\hat{\pi}^b + \hat{\alpha}^a L^{ab}\hat{\alpha}^b) = $$

$$= \int d^3x\ \hat{a}^{b+}\Omega^{bc}\hat{a}^c + E_0 = \sum_n \Omega_n \hat{a}^{b+}_n\hat{a}^b_n + E_0 \tag{5.9}$$

where E_0 is an energy of vacuum field fluctuations,

$$\hat{a}^c = \frac{1}{\sqrt{2}}\,(\Omega^{-1/2})^{cb}\left[(L^{1/2})^{ba}\hat{\alpha}^a - i(M^{1/2})^{ba}\hat{\pi}^a\right]\, , \tag{5.10}$$

$$\Omega = \frac{1}{2}\,(L^{1/2}M^{1/2} + M^{1/2}L^{1/2}) \tag{5.11}$$

so that

$$\left[\hat{a}^c(\mathbf{x})\, ,\, \hat{a}^{b+}(\mathbf{y})\right] = \delta^{cb}\delta^3(\mathbf{x}-\mathbf{y})\, . \tag{5.12}$$

Operators $\hat{a}^b_n$ are determined by decomposing operators $\hat{a}^b(\mathbf{x})$ over eigenstates of the operator Ω, Ω_n denote its eigenvalues and $\sum_n$ implies a spectral sum over the spectrum of Ω,

$$\left[\hat{a}^b_n\, ,\, \hat{a}^{c+}_{n'}\right] = \delta_{nn'}\delta^{bc} \tag{5.13}$$

with $\delta_{nn'}$ being the delta-function in the space of spectral parameters of Ω_n. For instance, putting $\mathbf{K} = \partial$ we obtain $\Omega = \sqrt{-\Delta}$, therefore, a decomposition over eigenfunctions of Ω is the Fourier transformation and the set $\{n\}$ means the momentum space. $\Omega_n = |\mathbf{k}|$, $\mathbf{k}$ is a photon momentum.

Thus, a basis of the Hilbert space may be realized as the Fock space with the vacuum state defined as follows

$$\hat{a}^b_n\ |0\rangle = 0\, , \qquad \hat{\Psi}_\alpha(\mathbf{x})\ |0\rangle = 0 \tag{5.14}$$

and with creation and annihilation operators satisfying (5.3), (5.13).

Eigenstates of the Hamiltonian (2.28) can be found by applying the operator

$$\hat{U} = \exp\left(-i\int d^3x\hat{\alpha}^a(M^{-1})^{ab}\varphi^b\hat{J}_0\right) \tag{5.15}$$

to the basis states because

$$\hat{U}^+\hat{H}^{ph}\hat{U} = \hat{H}^{ph}_0 - \frac{1}{2}\int d^3x\hat{J}_0\Delta^{-1}\hat{J}_0\, . \tag{5.16}$$

The states

$$|1,\cdots,m\rangle_{coul} = \hat{U}\hat{\Psi}^+_{\alpha_1}(\mathbf{y}_1)\cdots\hat{\Psi}^+_{\alpha_m}(\mathbf{y}_m)|0\rangle \equiv \hat{U}|1,\cdots,m\rangle \tag{5.17}$$

describe point-like charges with their Coulomb field. Indeed, the average value of the electric field operator $\hat{\mathbf{E}} = -\hat{\pi}$ (see (2.27)) over the state (5.17) is equal to the Coulomb field

$$\mathbf{E}(\mathbf{x}) = \frac{\langle 1,\cdots,m|\hat{U}^+\hat{\mathbf{E}}(\mathbf{x})\hat{U}|1,\cdots,m\rangle}{\langle 1,\cdots,m|1,\cdots,m\rangle} = -\partial\Delta^{-1}J_0(\mathbf{x}) \tag{5.18}$$

with

$$J_0(\mathbf{x}) = \sum_{i=1}^{m} e_{\alpha_i}\delta(\mathbf{x}-\mathbf{y}_i) \tag{5.19}$$

being an eigenvalue of $\hat{J}_0$ on its eigenvector $|1,\cdots,m\rangle$.

Equations (5.18) and (5.19) show that switching on the interaction of the fields $\hat{\alpha}^a$ with the quantum source (the quantity $H^{ph} - H^{ph}_0$ plays the role of the interaction Hamiltonian) "dresses" charges with the Coulomb field. The dressing process can be visually represented by average values of the Heisenberg operators of the fields $\hat{\mathbf{E}}(\mathbf{x},t)$ and $\hat{\mathbf{B}}(\mathbf{x},t)$ and the density $\mathcal{H}^{ph}$ of the total Hamiltonian $\hat{H}^{ph}$ over the state $|1,\cdots,m\rangle$ taken as the initial one. The average value of $\hat{\mathcal{H}}^{ph}$ characterizes changes of the energy distribution with time (the total energy is preserved and equal to (3.1) where J_0 is given by (5.19)). The average values of the field operators show the process of filling a space around charges with the Coulomb field.

Let us calculate them. The average of the operator $\hat{\mathbf{E}}(\mathbf{x})$ over the state $|1,\cdots,m\rangle$ is given by (3.12) since the vacuum expectation value $\langle\hat{\mathbf{E}}(\mathbf{x})\rangle_0$ vanishes. Obviously, the corresponding average of the magnetic field operator $\hat{\mathbf{B}}(\mathbf{x})$ is equal to zero. Introducing the transverse fields $\hat{\alpha}^{\perp}(\mathbf{x})$ and $\hat{\pi}^{\perp}(\mathbf{x})$ ($\hat{\mathbf{B}} = \mathbf{curl}\,\hat{\alpha}^{\perp}$, $\hat{\pi}_i^{\perp} = -P_{ij}^{\perp}\hat{E}_j$, see Sec.3) we are convinced that Eqs.(3.14) give their averages over the state $|1,\cdots,m\rangle$ and their Heisenberg operators $\hat{\alpha}^{\perp}(\mathbf{x},t)$ and $\hat{\pi}^{\perp}(\mathbf{x},t)$ satisfy Eqs.(3.13). Therefore, the fields (3.17), (3.18) coincide with the average values of the Heisenberg electromagnetic field operators if J_0 has the form (5.19). Apparently,

$$\frac{\langle 1,\cdots,m|\hat{\mathcal{H}}^{ph}(\mathbf{x},t)|1,\cdots,m\rangle}{\langle 1,\cdots,m|1,\cdots,m\rangle} = \frac{1}{2}\,\mathbf{E}^2(\mathbf{x},t) + \frac{1}{2}\,\mathbf{B}^2(\mathbf{x},t) + \mathcal{E}_0 \tag{5.20}$$

where $\mathcal{E}_0$ is the energy density of vacuum fluctuations of the electromagnetic field.

Thus, the process of dressing charges with their proper Coulomb field in the quantum theory is similar to the one in the classical theory investigated in Sec.3. In particular, put $\mathbf{K}$ as (4.1) and consider the following state with two opposite charges

$$\hat{\Psi}_1^+(\mathbf{y}_1)\hat{\Psi}_2^+(\mathbf{y}_2)|0\rangle = \hat{U}^+|1,2\rangle_{coul} \tag{5.21}$$

where $e_1 + e_2 = 0$ and $\mathbf{y}_1,\ \mathbf{y}_2 \in C$. Then dressing charges with their Coulomb field is qualitatively pictured in Fig.1. The average values of the Heisenberg electromagnetic field operators are given by formulas (4.3)–(4.6).

The appearance of the photon radiation during the time evolution of the state (5.21) could be expected at the very beginning. It follows from the right-hand side of the equality (5.21) that the initial state contains the stationary state of two charges with their Coulomb field $|1,2\rangle_{coul}$ and a coherent photon state $\hat{U}_{12}^+|0\rangle$, where $\hat{U}_{12}^+$ is the operator (5.15) with $\hat{J}_0$ changed by its eigenvalue (5.19) at $m = 2$, $e_1 + e_2 = 0$, (operators $\hat{a}_n^{b+}$ included in $\hat{U}_{12}^+$ create photons, being applied to the vacuum state). It is known from the theory of electromagnetic field coherent states that average values of the Heisenberg field operators in these states obey to the classical Maxwell equations. Thus, the state $|1,\cdots,m\rangle$ is unstable because it contains the unstable photon coherent state $\hat{U}_{12}^+|0\rangle$.

We conclude that all gauge-invariant variables are equivalent from the mathematical point of view. However, their physical interpretations are quite different. Different configurations of a proper electric field of charges in initial states correspond to different choices of $\mathbf{K}$. In this sense, the choice of the physical operators describing creation and annihilation of charges together with their Coulomb field turns out to be distinguished. If $\mathbf{K} = \boldsymbol{\partial}$, $\hat{U} = 1$, hence, the initial state $|1,\cdots,m\rangle$ does not contain the unstable photon coherent state, i.e., no electromagnetic radiation appears during its time evolution.

6 The method of gauge-invariant operators

Constraints in a quantum gauge theory can be taken into account without their explicit solution, i.e., without reduction of the total Hilbert space to the physical subspace as has been done in Sec.5. One can work in the total Hilbert space using, however, gauge-invariant operators for describing physical excitation of fields. If $[\hat{\sigma},\hat{\Phi}] = 0$ where $\hat{\Phi} = \Phi(\hat{\mathbf{A}},\hat{\psi},\hat{\psi}^+)$, then an operator $\hat{\Phi}$ being applied to the physical vacuum state creates a physical state satisfying (5.1).

The simplest gauge-invariant operators in QED are well-known [1], [9]

$$\hat{I}_{coul}(\mathbf{x}) = \exp\left(-ie_\alpha\Delta^{-1}\partial\hat{\mathbf{A}}(x)\right)\hat{\psi}_\alpha^+(\mathbf{x})\,, \tag{6.1}$$

$$\hat{I}_{st}(\mathbf{y}_1,\mathbf{y}_2) = \hat{\psi}_{\alpha_1}^+(\mathbf{y}_1)\exp\left(-ie\int_{\mathbf{y}_2}^{\mathbf{y}_1}\hat{\mathbf{A}}(\mathbf{z})d\mathbf{z}\right)\hat{\psi}_{\alpha_2}^+(\mathbf{y}_2) \tag{6.2}$$

where $e_{\alpha_1} = -e_{\alpha_2} = e$ and the integration is carried out over a contour connecting the points $\mathbf{y}_1$ and $\mathbf{y}_2$. A more general form of a gauge-invariant operator reads

$$\hat{I}_m(\mathbf{K};\ \mathbf{y}_1,\cdots,\mathbf{y}_m) = \prod_{j=1}^{m}\exp\left(-ie_{\alpha_j}\hat{\chi}(\mathbf{y}_j)\right)\hat{\psi}_{\alpha_j}^+(\mathbf{y}_j) \tag{6.3}$$

where the operator $\hat{\chi}$ is connected with the vector potential operator by the relation (2.15). The operators (6.1) and (6.2) are particular cases of (6.3) at special choices of $\mathbf{K}$: $\hat{I}_{coul} = \hat{I}_1(\boldsymbol{\partial})$ and $\hat{I}_{st} = \hat{I}_2(d\mathbf{z}/ds)$ (see (4.1)).

A physical meaning of the operators (6.3) is that they describe the creation of charges together with an electric field. Note that the Gauss law (5.1) connects an electric charge density in a system with an electric field, hence, charges cannot appear without their proper electric field satisfying the Gauss law. The operator $\hat{\chi}$ in (6.3) describes this field. Indeed, let there exist a state $|\mathbf{E}\rangle$ in which the electric field operator $\hat{\mathbf{E}}(\mathbf{x})$ has a numerous value $\mathbf{E}(\mathbf{x})$: $\hat{\mathbf{E}}|\mathbf{E}\rangle = \mathbf{E}|\mathbf{E}\rangle$. Then it follows from the equality

$$\hat{\mathbf{E}}(\mathbf{x})\exp\left(-e_\alpha\hat{\chi}(\mathbf{y})\right) = \exp\left(-e_\alpha\hat{\chi}(\mathbf{y})\right)\left[\hat{\mathbf{E}}(\mathbf{x}) + e_\alpha\mathbf{K}(\mathbf{K},\boldsymbol{\partial})^{-1T}\delta^3(\mathbf{x}-\mathbf{y})\right], \tag{6.4}$$

where all operators in the second term of the right-hand side act just on the variable $\mathbf{x}$ [3], that

$$\hat{\mathbf{E}}(\mathbf{x})\hat{I}_m(\mathbf{K})|\mathbf{E}\rangle = \left[\mathbf{E}(\mathbf{x}) + \mathbf{K}^T(\mathbf{K},\boldsymbol{\partial})^{-1T}J_0(\mathbf{x})\right]\hat{I}_m(\mathbf{K})|\mathbf{E}\rangle \equiv \mathbf{E}'(\mathbf{x})\hat{I}_m(\mathbf{K})|\mathbf{E}\rangle\,; \tag{6.5}$$

the charge density has the form (5.19). In particular, for operators (6.1) (see [1]) and (6.2) we obtain, respectively,

$$\mathbf{E}'(\mathbf{x}) = \mathbf{E}(\mathbf{x}) + \boldsymbol{\partial}\,\frac{e_\alpha}{4\pi|\mathbf{x}-\mathbf{y}|}\,, \tag{6.6}$$

$$\mathbf{E}'(\mathbf{x}) = \mathbf{E}(\mathbf{x}) + e\int_{\mathbf{y}_2}^{\mathbf{y}_1} d\mathbf{z}\,\delta^3(\mathbf{x}-\mathbf{z})\,. \tag{6.7}$$

Equation (6.5) shows that the operator (6.3) increases an electric field at each space point by the value $\mathbf{E}' - \mathbf{E}$ simultaneously with creating m charges. For instance, the additional electric field created by $\hat{I}_{coul}$ is equal to the Coulomb field of a point-like charge e_α, and for the operator $\hat{I}_{st}$ it turns out to be the electromagnetic string field considered in Sec.4 (see (4.6)).

Thus, the requirement of the gauge invariance for all quantities describing dynamics of physical degrees of freedom automatically leads to that charged particles must be created

[3] Note that the rule (2.22) leads to the relation $\hat{O}_{\mathbf{x}}\delta^3(\mathbf{x}-\mathbf{y}) = \hat{O}_{\mathbf{y}}^T\delta^3(\mathbf{x}-\mathbf{y})$ where the lower indices denote a variable on which the operator acts.

just with the proper electric field satisfying the Gauss law. A configuration of this field depends on the choice of gauge-invariant variables.

However, all the proper field configurations different from the Coulomb one (i.e. when $\mathbf{K} \neq \boldsymbol{\partial}$) are unstable. To prove this, one should again calculate average values of the Heisenberg operators of fields $\hat{\mathbf{E}}(\mathbf{x},t)$, $\hat{\mathbf{B}}(\mathbf{x},t)$ and the Hamiltonian density $\mathcal{H}(\mathbf{x},t)$ over the state $\hat{I}_m(\mathbf{K})|0\rangle = |I_m\rangle$. All charges are assumed to be static and, hence, the time evolution is determined by the Hamiltonian

$$\hat{H} = \int d^3x \mathcal{H}(\mathbf{x}) = \frac{1}{2} \int d^3x \left(\hat{\mathbf{E}}^2(\mathbf{x}) + \hat{\mathbf{B}}^2(\mathbf{x})\right) . \tag{6.8}$$

Therefore the average values of the Heisenberg field operators satisfy the Maxwell equation without sources and with the initial conditions

$$\langle I_m|\hat{\mathbf{B}}(\mathbf{x})|I_m\rangle = 0, \quad \langle I_m|I_m\rangle^{-1}\langle I_m|\hat{\mathbf{E}}(\mathbf{x})|I_m\rangle = \mathbf{K}^T(\mathbf{K},\boldsymbol{\partial})^{-1T} J_0(\mathbf{x}) . \tag{6.9}$$

Obviously, we come again to the formulas (3.17), (3.18) and (5.18) with J_0 defined by (5.19).

In conclusion, we consider the time evolution of a closed electrodynamic string, the state obtained by applying the gauge-invariant operator

$$\hat{I}_C = \exp\left(ie \oint_C (\hat{\mathbf{A}}(\mathbf{z}), d\mathbf{z}) \right) \tag{6.10}$$

to the physical vacuum. The average value of the electric field operator in this state is a transverse field and is given by (4.6) where the integral should be taken over a closed contour C. Therefore, the whole energy of the closed string state turns into the electromagnetic radiation energy (only the first term in (4.3) differs from zero).

This feature might be expected at the very beginning since the operator (6.9) creates only a coherent state of transverse photons. Indeed, substituting $\hat{\mathbf{A}} = \hat{\boldsymbol{\alpha}}^\perp + \boldsymbol{\partial}\Delta^{-1}(\boldsymbol{\partial},\hat{\mathbf{A}})$ into (6.9) we find that $\hat{I}_C$ depends just on $\hat{\boldsymbol{\alpha}}^\perp$. On the contrary, the states $\hat{I}_m(\mathbf{K})|0\rangle$ contain a coherent field excitation corresponding to the longitudinal (Coulomb) part of $\hat{\mathbf{A}}$ except the coherent state of photons described by $\hat{\boldsymbol{\alpha}}^\perp$, which follows from the identity

$$\hat{\chi} = (\mathbf{K},\boldsymbol{\partial})^{-1}(\mathbf{K},\hat{\mathbf{A}}) = (\mathbf{K},\boldsymbol{\partial})^{-1}(\mathbf{K},\hat{\boldsymbol{\alpha}}^\perp) + \Delta^{-1}(\boldsymbol{\partial},\hat{\mathbf{A}}) . \tag{6.11}$$

After its substitution into (6.3), the operator $\hat{I}_m(\mathbf{K})$ splinters into a product of m operators $\hat{I}_{coul}$ and an operator depending on $\hat{\boldsymbol{\alpha}}^\perp$ and creating an unstable coherent excitation of photons. This coherent state is absent just in the case when $\mathbf{K} = \boldsymbol{\partial}$. For this reason, charged states with the Coulomb field are stable.

7 Conclusion

Thus, the requirement of the gauge-invariance for dynamical variables in electrodynamics leads to that charges can be created or annihilated just together with their proper electric field (a consequence of the Gauss law). A proper field configuration of a charge depends on the choice of gauge-invariant variables that may be related with a way of preparing an initial state of an electrodynamic system. All configurations of the proper field of charges different from the Coulomb one are unstable and break down with radiation of electromagnetic waves and creation of the stable Coulomb field.

We conclude that the knowledge of all gauge-invariant operators describing dynamics of matter in gauge theories is not yet enough to determine a correct potential of the static interaction of charges. One should investigate dynamical stability of states generated by these operators. For instance, in QED there is a gauge-invariant state $\hat{I}_{st}$ describing two opposite charges connected by one force line of the electric field strength (by the string). Applying it to the physical vacuum we get the state whose energy linearly rises with the contour length (confinement). However, this state is unstable. After its decay, the charges will interact in accordance with the Coulomb law. Therefore, there is no confinement in the continuous electrodynamics. Note that in the lattice QED [10] the string-like excitations are stable in the strong coupling limit [11] because of the specific peculiarities of the lattice formulation of gauge theories [11]. On the contrary, our result concerning the string decay in the continuous QED does not depend on the value of the coupling constant e.

8 Appendix A

Let the operator $(\mathbf{K},\boldsymbol{\partial})$ have zero modes, i.e., there are functions satisfying the conditions

$$(\mathbf{K},\boldsymbol{\partial})\omega_0 = 0 , \qquad \boldsymbol{\partial}\omega_0|_\infty = 0 . \tag{A.1}$$

The boundary condition follows from the requirement of vanishing $\mathbf{A}$ at spatial infinity. Then the inverse operator $(\mathbf{K},\boldsymbol{\partial})^{-1}$ in (2.15) and (2.16) does not exist. Instead of it, one can use the operator $(\mathbf{K},\boldsymbol{\partial})^{-1}_{reg}$ whose action is defined on the reduced functional space containing functions $\omega' = (1-P_0)\omega$ where P_0 is a projector on zero modes of the operator $(\mathbf{K},\boldsymbol{\partial})$ in a functional space of ω's, $\omega_0 = P_0\omega$; $P_0^T = P_0$. In this case the variables $\boldsymbol{\alpha}$ (and, as a consequence, α^a) constructed with the help of $(\mathbf{K},\boldsymbol{\partial})^{-1}_{reg}$ are not gauge-invariant.

$$\boldsymbol{\alpha} \rightarrow \boldsymbol{\alpha} + \boldsymbol{\partial}\omega_0 , \tag{A.2}$$

$$\chi \rightarrow \chi + \omega' . \tag{A.3}$$

The constraint $\sigma = 0$ is equivalent to two equalities

$$\sigma' = (1-P_0)\sigma = 0 , \quad \sigma_0 = P_0\sigma = 0 \tag{A.4}$$

because of the linear independence of functions ω' and ω_0 (i.e., $\int d^3x\omega'\omega_0 = 0$). Obviously, $\sigma' = -\pi_\chi$ in the formalism of Sec.2. After eliminating the unphysical degree of freedom χ, π_χ, we have a theory with residual gauge symmetry (A.2) and the constraint $\sigma_0 = 0$. Note that the operator L^{ab} has zero modes when there remains a residual gauge symmetry

$$L^{ab}\alpha^b = 0 , \quad \alpha^b = (\boldsymbol{\partial}, \mathbf{e}^b)^T\omega_0 . \tag{A.5}$$

To make the analysis of Sec.3 correct, generally speaking, one should carry out one more canonical transformation of α^a, π^a like (2.13)–(2.16) so that σ_0 could become a new canonical momentum. However, if

$$P_0 J_0 = 0 , \tag{A.6}$$

all formulas of Sec.3 are kept without any changes (except the trivial one $(\mathbf{K},\boldsymbol{\partial})^{-1} \rightarrow (\mathbf{K},\boldsymbol{\partial})^{-1}_{reg}$). Indeed, as the functional $P_0(\boldsymbol{\partial},\boldsymbol{\pi})$ is linear and homogeneous in π^a, the initial condition $\pi^a = 0$ for equations (3.4) and (3.5) does not contradict to the residual constraint $\sigma_0 = P_0\sigma = P_0(\boldsymbol{\partial},\boldsymbol{\pi}) = 0$ due to (A.6).

The electrodynamic string considered in Sec.4 obeys Eq.(A.6). If $\mathbf{K}$ has the form (4.1), then the function $P_0 f(\mathbf{x})$ is an average of a function $f(\mathbf{x})$ on a contour passing through the point x and parallel to the contour C. So, the condition (A.6) means that the average charge on any contour parallel to C vanishes. This is fulfilled for the electrodynamic string.

In the quantum theory, Eq.(5.1) turns into $\hat{\sigma}_0|\phi_{ph}\rangle = 0$ after the elimination of the variable χ with the help of changing variables (2.15), (2.16) and (5.8). Under the residual gauge transformations, the operators (5.7) acquire the phase

$$\hat{\Psi}^+_\alpha(\mathbf{x}) \rightarrow \exp(ie_\alpha\omega_0(\mathbf{x}))\hat{\Psi}^+_\alpha(\mathbf{x}) = \exp\left(ie_\alpha P_0\omega(\mathbf{x})\right)\hat{\Psi}^+_\alpha(\mathbf{x}) = = \exp\left(ie_\alpha\int d^3x' P_0\delta^3(\mathbf{x}-\mathbf{x}')\omega(\mathbf{x}')\right)\hat{\Psi}^+_\alpha(\mathbf{x}) . \quad (A.7)$$

where the operator P_0 acts on the variable x (see (2.22) and Footnote 3). Therefore, the state $|1,\cdots,m\rangle$ is physical if the charge density (5.19) satisfies the condition (A.6). For instance, if $\mathbf{K}$ is given by (4.1) , then a total charge on any contour parallel to C must vanish for physical states, which is valid for the state (5.21).

Note that the equation (A.1) has solutions even in the case $\mathbf{K} = \partial$, $\omega_0 = const$. The quantity $P_0 f$ is an average value of a function f in the whole space. Because of the equality $\pi = 0$ at spatial infinity, we conclude that the equation $\sigma_0 = 0$ means vanishing the total electric charge of a system $Q = \int d^3x J_0 = 0$ [12].

References

[1] Dirac, P A M 1958 *The Principles of Quantum Mechanics*, Sec.80 (Oxford, Clarendon Press).

[2] Yang, C N and Mills, R L 1954 *Phys.Rev.* **96** 191.

[3] Dirac, P M A 1965 *Lectures on Quantum Mechanics* (New-York, Acadimic).

[4] Prokhorov, L V and Shabanov, S V 1991 JINR preprints E2-91-195, E2-91-266 (Dubna, JINR).

[5] Dubovik, V M and Shabanov, S V 1992 in: *Essays on Formal Aspects of the Electromagnetic Theory* (ed. by A. Lakhtakia) (New Jersey, WSPC) to appear.

[6] Dubovik, V M and Shabanov, S V 1990 *J.Phys.A:Math.Gen.* **23** 3245; 1989 *Phys.Lett.A* **142** 211.

[7] Abrikosov, A A 1957 *Sov.Phys.JETP* **5** 1174.

[8] Shabanov, S V 1991 *J.Phys.A:Math.Gen.* **24** 1199.

[9] Mandelstam, S 1962 *Ann.Phys.(NY)* **19** 1.

[10] Wilson, K 1974 *Phys.Rev.D* **10** 2445.

[11] Polyakov, A M *Gauge Fields and Strings* (Chur, Switzerland, Hardwood Academic Publishers GmbH).

[12] Prokhorov, L V 1990 *Lett.Math.Phys.* **19** 45.

Received by Publishing Department
on March 26, 1992.

Шабанов С.В.

E2-92-136

Собственное поле зарядов и калибровочно-инвариантные переменные в электродинамике

Согласно Дираку[1] из требования калибровочной инвариантности физических динамических переменных в электродинамике следует, что заряды должны рождаться и уничтожаться вместе со своим собственным (кулоновским) полем, которое индуцирует статическое (кулоновское) взаимодействие зарядов. Показано, что в электродинамике существует бесконечное число способов выбора калибровочно-инвариантных переменных. Конфигурация собственного поля зависит от этого выбора и может отличаться от кулоновской. Однако все такие конфигурации нестабильны. Их распад сопровождается излучением электромагнитных волн и образованием кулонова поля, которое является единственным стабильным собственным полем заряда. В качестве примера подробно рассмотрен распад "электродинамической струны".

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна 1992

Shabanov S.V.

E2-92-136

The Proper Field of Charges and Gauge-Invariant Variables in Electrodynamics

According to Dirac[1], it follows from the gauge invariance requirement for physical dynamical variables in electrodynamics that charges are created and annihilated together with their proper (Coulomb) field that induces the static (Coulomb) interaction of charges. It is shown that in electrodynamics there is an infinite number of ways to determine gauge-invariant variables. The proper field configuration of charges depends on the choice of these variables and may differ from the Coulomb one. However, all configurations of that kind are unstable. Their decay is accompanied by radiating electromagnetic waves and creating the Coulomb field configuration that is the only stable one. The decay of the "electromagnetic string" is analyzed in detail as an example.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna 1992