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объединенный
институт
ядерных
исследований
дубна

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G.N.Afanasiev

THE STATIC AND NONSTATIC ELECTRICAL SOLENOIDS

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§ 1. Introduction

The magnetic solenoids have a broad application in physics and technics (see, e.g., review their properties in /1,2/). The electrical solenoids (ES) are much less known objects. Under them we mean the system of charges and currents which generate electrical field (EF) confined to the space region S of finite extension. Although EF disappears outside S the scalar electric and magnetic vector potentials could be different from zero there. As far as we know there are only few references treating this subject. In ref. /3/ electrical toroidal solenoid (ETS) was constructed in terms of nonphysical magnetic monopoles current. The nonstatic point-like and cylindrical ES were studied in refs. /4/,/5/, resp. In both of them nonvanishing charge and current densities were presented. In ref. /6/ there was suggested ES with nonvanishing current density (that is in the absence of the charge density). It is the aim of present consideration to treat systematically ES. The plan of our exposition is as follows. In § 2 the main facts concerning magnetic toroidal solenoid (MTS) with the constant current in its windings are presented. In § 3 the alternative viewpoint on the MTS is given. It turns out that magnetic dipoles properly distributed inside the torus exactly reproduce the vector potential (VP) of the MTS with the constant current. The boundary conditions satisfied by the magnetic field strength and induction are discussed in § 4. In § 5 we change the magnetic dipoles inside the torus by the electrical ones thus obtaining ETS. The nonstatic ES are investigated in § 6. These ES emit the waves of electromagnetic potentials. They propagate off the solenoid with the velocity of light.

§2. Some facts concerning magnetic toroidal solenoids

Consider the torus T

$$(\rho-d)^2+z^2=R^2. \tag{2.1}$$

Introduce coordinates $\tilde{R}$, Ψ: $\rho=d+\tilde{R}\cos\Psi$, $z=\tilde{R}\cdot\sin\Psi$. The value of $\tilde{R}=R$ corresponds to the torus T. Let the constant poloidal current (fig.1) flow over its surface. The density of this current is

$$\vec{j}=-\frac{gc}{4\pi}\frac{\delta(\tilde{R}-R)}{d+R\cos\Psi}\vec{n}_\Psi. \tag{2.2}$$

Here $g=2NI/c$, N is the total number of turns in the poloidal coil, I is the current flowing in a particular turn, $\vec{n}_\Psi$ is the unit vector defining the current direction on the torus surface $\vec{n}_\Psi=\vec{n}_z\cos\Psi-(\vec{n}_x\cos\varphi+\vec{n}_y\sin\varphi)\sin\Psi$. The constant g may be also expressed through the magnetic flux Φ inside T: $g=\Phi\cdot[2\pi\cdot(d-\sqrt{d^2-R^2})]^{-1}$. Magnetic field (MF) $\vec{B}=\vec{H}=0$ outside T and $\vec{B}=\vec{H}=\vec{n}_\varphi\cdot g/\rho$ inside it. Here ρ is the distance of the particular point inside T from the torus symmetry axis ($\rho=d+\tilde{R}\cos\Psi$). The vector potential (VP) of the MTS was obtained in ref. /7/. Its properties were discussed in /8/. In the integral form the nonvanishing cylindrical components of VP are

$$A_z=\frac{g\sqrt{R}}{2\pi}\int_0^{2\pi}d\varphi\,\frac{d-\rho\cos\varphi}{q^{3/2}}\cdot Q_{\frac{1}{2}}(\operatorname{ch}\mu),$$

$$A_\rho=\frac{g\sqrt{R}}{2\pi}z\int_0^{2\pi}d\varphi\,\frac{\cos\varphi}{q^{3/2}}\,Q_{\frac{1}{2}}(\operatorname{ch}\mu) \tag{2.3}$$

($\operatorname{ch}\mu=(r^2+d^2+R^2-2d\rho\cos\varphi)/2Rq$, $q^2=(\rho\cos\varphi-d)^2+z^2$, $r^2=\rho^2+z^2$. $Q_\nu(x)$ is the Legendre function of the 2nd kind). For the infinitely thin TS ($R\ll d$) these integrals can be taken in a closed form (outside the solenoid)

$$A_z=\frac{gR^2}{2\cdot(d\rho)^{3/2}}\frac{1}{\operatorname{sh}\mu_1}\left[\rho\, Q^1_{\frac{1}{2}}(\operatorname{ch}\mu_1)-d\cdot Q^1_{-\frac{1}{2}}(\operatorname{ch}\mu_1)\right],$$

$$A_\rho=-\frac{gR^2}{2(d\rho)^{3/2}}\frac{1}{\operatorname{sh}\mu_1}Q^1_{\frac{1}{2}}(\operatorname{ch}\mu_1),\quad \operatorname{ch}\mu_1=\frac{r^2+d^2}{2d\rho}. \tag{2.4}$$

At large distances this VP falls like r^{-3}

$$A_z\sim\frac{1}{8}\pi gdR^2\,\frac{1+3\cos2\theta}{r^3},\quad A_\rho\sim\frac{3}{8}\pi gdR^2\,\frac{\sin2\theta}{r^3}.$$

§3. An alternative viewpoint on the MTS

Instead of the poloidal current (2.2) one may equally use /9,10/ the magnetization $\vec{j}=c\operatorname{rot}\vec{M}$. It is confined completely inside TS and given by

$$\vec{M}=M\cdot\vec{n}_\varphi,\quad M=\frac{g}{4\pi}\frac{\Theta\left(R-\sqrt{(\rho-d)^2+z^2}\right)}{\rho} \tag{3.1}$$

Here $\Theta(x)$ is the step function. For the infinitely thin TS ($R \ll d$) $\mathcal{M}$ reduces to

$$\mathcal{M} = \frac{1}{4\pi} \Phi \, \delta(\rho - d) \, \delta(z) . \tag{3.2}$$

The VP is expressed through the magnetization as follows /9,10/

$$\vec{A}(\vec{r}) = \int \vec{M}(\vec{r}\,') \times \frac{\vec{r} - \vec{r}\,'}{|\vec{r} - \vec{r}\,'|^3} \, dV' . \tag{3.3}$$

What is the physical meaning of these relations? Eqs. (3.2) and (3.3) mean that infinitely thin MTS can be realized as the closed chain of magnetic dipoles (fig.2). In fact the value of VP at the point $\vec{r}$ produced by the magnetic dipole situated at $\vec{r}\,'$ is given by (see, e.g., /9/)

$$\vec{A}(\vec{r}) = m \, \frac{\vec{n} \times (\vec{r} - \vec{r}\,')}{|\vec{r} - \vec{r}\,'|^3} . \tag{3.4}$$

Here $\vec{n}$ and m are the direction and power of dipole, resp. Integrating this Eq. over the circumference of the radius d lying in the $z=0$ plane ($\vec{n} = \vec{n}_\varphi$, $\vec{r}\,' = d \cdot \vec{n}_\rho$, $\vec{n}_\varphi = \vec{n}_y \cdot \cos\varphi - \vec{n}_x \cdot \sin\varphi$, $\vec{n}_\rho = \vec{n}_x \cdot \cos\varphi + \vec{n}_y \cdot \sin\varphi$) we arrive to (2.4) with $g = 4m/R^2$ or $\Phi = 4\pi m/d$. Eqs. (3.1) and (3.3) mean that finite MTS may be realized as a closed spin tube of the radius R . In fact, integrating (3.4) over the volume of T (under m in (3.4) one should understand the spin density which coincides with the magnetization given by Eq.(3.1)) we get Eqs. (2.3). The closed spin tube (ferromagnetic ring

with the magnetization independent of applied fields) was used by the Japanese physicists /11/ for the experimental investigation of the Aharonov - Bohm effect. The simpler case presents the cylindrical solenoid. It may be realized (fig.2) as a linear spin chain (or tube). In fact integrating Eq.(3.4) over the z axis we arrive to the VP of the cylindrical solenoid: $\vec{A} = \vec{n}_\varphi \cdot \Phi / 2\pi\rho$, $\Phi = 4\pi m$.
Such spin chain (magnetized whisker) was used in earlier experiments testing AB effect (see their review in /12/).

§4. The magnetic field and boundary conditions

We write out the general equations defining $\vec{B}, \vec{H}, \vec{M}$ /9/:

$$\operatorname{div} \vec{B} = 0 , \ \operatorname{rot} \vec{H} = \frac{4\pi}{c} \vec{J} , \ \vec{B} = \vec{H} + 4\pi \vec{M} .$$

For the TS with the surface current considered in § 2 $\vec{M} = 0$ therefore $\vec{B} = \vec{H}$ both inside and outside TS. On its boundary the normal component of $\vec{B}$ is continuous whereas the tangential component of $\vec{H}$ suffers finite jump equal to the surface current density. On the other hand for the magnetized spin tube treated in § 3 $\vec{J} = 0, \vec{M} \neq 0$. As div $\vec{M} = 0$ for the particular choice of $\vec{M}$, so div $\vec{H} = \operatorname{rot} \vec{H} = 0$. From this and the continuity of B_n and H_t at the boundary of spin tube we obtain $\vec{H} = 0$ everywhere, $\vec{B} = \operatorname{rot} \vec{A}$, $\operatorname{div} \vec{A} = 0$, $\Delta \vec{A} = -4\pi \operatorname{rot} \vec{M}$. The solution of the latter equation is just Eq.(3.3).

§ 5. Electrical toroidal solenoids

Now we substitute the magnetic dipoles by the electric ones. Then all Eqs. obtained in §§ 3,4 remain the same. The electrical polarization is given by $\vec{P} = P \cdot \vec{n}_\varphi$ where $P = \Phi \cdot \delta(\rho - d) \cdot \delta(z) / 4\pi$ for the circular dipole chain and $P = g \cdot \Theta(R - \sqrt{(\rho - d)^2 + z^2}) / 4\pi\rho$

for the finite ETS . Here $g=\Phi\cdot[2\pi(d-\sqrt{d^2-R^2})]^{-1}$ and Φ is the electric induction flux through the cross section of the solenoid. In the absence of free charges we have the following Eqs. for $\vec{D}$, $\vec{E}$: $\operatorname{div}\vec{D}=0$, $\operatorname{rot}\vec{E}=0$, $\vec{D}=\vec{E}+4\pi\vec{P}$. Excluding $\vec{E}$ we obtain Eqs. for $\vec{D}$

$$\operatorname{div}\vec{D}=0 \tag{5.1}$$

$$\operatorname{rot}\vec{D}=4\pi\operatorname{rot}\vec{P}. \tag{5.2}$$

To satisfy Eq. (5.1) we put $\vec{D}=\operatorname{rot}\vec{A}$, $\operatorname{div}\vec{A}=0$ and get the following Eq. for $\vec{A}$: $\Delta\vec{A}=-4\pi\operatorname{rot}\vec{P}$. Its solution is

$$\vec{A}=\int\frac{\operatorname{rot}\vec{P}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}dV'=\int\vec{P}(\vec{r}\,')\times\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}dV' \tag{5.3}$$

that coincides with (3.3). It follows from this that $\vec{D}=\operatorname{rot}\vec{A}=4\pi\vec{P}$ Hence, $\vec{E}=0$. The appearance of vector potential is somewhat unusual in electrostatic problems. In Appendix we use the electric VP for the solution of two known problems of electrostatics (polarized sphere and ellipsoid) and prove that it has the same status as the scalar one.

How to verify the existence of electric field inside ETS? The are the same means as for the MTS. We briefly enumerate them

1) the electromagnetic field appears outside ETS when it moves uniformly in the medium with $\epsilon\mu\neq 1$ /13/;
2) electromagnetic field appears outside the accelerated ETS (both in vacuum or medium) /1/;

3) the interaction of the external electric field (EF) with the electric dipoles confined inside ETS. is given by

$$U=-\int\vec{E}_{ext}\cdot\vec{P}\,dV. \tag{5.1}$$

At large distances from the source of the external EF (or for small dimensions of ETS) this Eq. reduces to

$$U=-\frac{1}{2}\vec{\epsilon}_t\cdot\operatorname{rot}\vec{E}_{ext}=\frac{1}{2c}\frac{\partial H_{ext}(\vec{r})}{\partial t}\vec{\epsilon}_t. \tag{5.2}$$

Here $\vec{\epsilon}_t=\int\vec{r}\times\vec{P}\,dV$ is the so-called toroidal electric moment /3/. For the polarization $\vec{P}$ given above $\vec{\epsilon}_t$ is directed along ETS symmetry axis and is equal to $\epsilon_t=\frac{1}{2}\pi g d R^2$. Eq.(5.2) means that at large distances ETS interacts with time varying MF /3/. This Eq. was used in ref. /14/ to explain the observed rotation /15/ of nonmagnetic molecules in the uniform MF slowly varying with time. The situation looks much simpler for the cylindrical electric solenoid which is obtained by inserting the linear electric dipole chain into the cylinder (fig.3). This solenoid tends to be oriented along the external EF.

The question arises on the technical realization of ETS. There exist neutral dielectrics called electrects that carry nonzero static electric dipole moment /16/. Among the different types of electrets the most suitable seems to be the ferroelectrics which are the electric analogues of ferromagnetics. From these substances the electrified ring can be manufactured exactly in the same way as the magnetized ring in Tonomura experiments /11/. It is rather curious that superposing the electric and magnetic dipole distributions inside the torus we get electromagnetic toroidal solenoid. The electromagnetic inductions differ from zero only inside T. Outside it there are nonvanishing electric and magnetic VP.

In the examples considered so far we have either forced the EMF to come out of the ETS by putting it into the motion or permitted the external EF to penetrate inside ETS and interact with electrical dipoles confined there. There is non-vanishing electric VP outside it. This VP cannot be eliminated by the gauge transformation as $\oint A_e dl = \Phi$ for the closed contours passing through the torus hole. Can we prove the existence of electric VP outside ETS without penetrating inside it? (a suitable screen can be used to obtain impenetrability of the incoming particles). We do not see the obvious answer. In fact, the analogue of AB effect for this case is the quantum scattering of free magnetic charges on the electric VP outside ETS. However, these particles (monopoles) have not been found in Nature up to now.

§ 6. Nonstatic electrical solenoids

To satisfy continuity equation $\mathrm{div}\,\vec{j} + \frac{\partial\rho}{\partial t} = 0$ automatically we choose ρ and $\vec{j}$ in the form

$$\rho = \exp(-i\omega t)\,\Delta f, \quad \vec{j} = i\omega\exp(-i\omega t)\,\vec{\nabla} f. \tag{6.1}$$

It is suggested therefore that these densities are the periodical functions of time. In what follows we shall frequently omit the common factor $\exp(-i\omega t)$. It should be restored when the time differentiation is performed or in final expressions from which the real part should be taken. The electromagnetic potentials and field strengths corresponding to the densities (6.1) are

$$\begin{gathered} \varphi = -4\pi f - k^2 \int G_k(\vec{r},\vec{r}\,')\, f(\vec{r}\,')\, dV', \quad k = \omega/c \\ \vec{A} = ik\vec{\nabla}\int G_k(\vec{r},\vec{r}\,')\, f(\vec{r}\,')\, dV', \quad \mathrm{div}\vec{A} + \frac{1}{c}\frac{\partial\varphi}{\partial t} = 0, \\ \vec{E} = 4\pi\vec{\nabla} f, \quad \vec{H} = 0, \quad G_k(\vec{r},\vec{r}\,') = \exp(ik|\vec{r}-\vec{r}\,'|)/|\vec{r}-\vec{r}\,'|. \end{gathered} \tag{6.2}$$

It follows from this that for the function f being confined to the finite region of space S the same is valid for the electric strength $\vec{E}$. On the other hand the electromagnetic potentials φ, $\vec{A}$ are certainly different from zero both inside and outside S. Thus, densities (6.1) realize nonstatic electrical solenoid. Particularly, function f may be taken to be nonvanishing inside the torus $(\rho-d)^2 + z^2 = R^2$ only. For this one may simply put $f = \mathcal{D}\cdot\Theta(R - \sqrt{(\rho-d)^2+z^2})$. There is known /4/ point-like realization of electrical TS. It is obtained if the δ-function choice of function f is made:

$$\begin{gathered} \rho = \mathcal{D}\cdot\Delta\cdot\delta^3(\vec{r}), \quad \vec{j} = i\omega\mathcal{D}\vec{\nabla}\delta^3(\vec{r}), \\ \varphi = -\mathcal{D}[4\pi\delta^3(\vec{r}) + k^2\exp(ikr)/r], \\ \vec{A} = ik\mathcal{D}\vec{\nabla}\exp(ikr)/r, \\ \vec{E} = 4\pi\mathcal{D}\vec{\nabla}\delta^3(\vec{r}), \quad \vec{H} = 0. \end{gathered} \tag{6.3}$$

The realization of the cylindrical electrical TS via the cylindrical capacitor was proposed in ref. /5/. Instead of this we choose function f in such a way as to obtain the spherical nonstatic capacitor:

$$\rho = \frac{e}{4\pi r^2}\left[\delta(r-r_1) - \delta(r-r_2)\right],$$
$$\vec{j} = i\omega e\frac{\vec{r}}{r^3}\,\Theta(r-r_1)\,\Theta(r_2-r). \qquad (6.4)$$

This means that the charge density differs from zero only on the spherical shells ($r = r_1$ and $r = r_2$) where it periodically changes with time (take into account the omitted factor $\exp(-i\omega t)$). The periodical current $\vec{j}$ flows between these shells in the radial direction. The scalar and vector potentials (only the radial component of $\vec{A}$ differs from zero) corresponding to these densities are

$$\varphi = ike\, h_0^{(1)}(kr)\left[j_0(1) - j_0(2)\right],$$
$$A_r = -ek\, h_1^{(1)}(kr)\left[j_0(1) - j_0(2)\right] \quad \text{for} \quad r > r_2$$
$$\varphi = ike\, j_0(kr)\left[h_0^{(1)}(1) - h_0^{(1)}(2)\right], \qquad (6.5)$$
$$A_r = -ek\, j_1(kr)\left[h_0^{(1)}(1) - h_0^{(1)}(2)\right] \quad \text{for} \quad r < r_1,$$
$$\varphi = ike\left[h_0^{(1)}(kr)\, j_0(1) - j_0(kr)\, h_0^{(1)}(2)\right],$$
$$A_r = ek\left[j_1(kr)\, h_0^{(1)}(2) - h_1^{(1)}(kr)\, j_0(1)\right] - ie/kr^2$$

for $r_1 < r < r_2$. Here $j_l(x) = \sqrt{\frac{\pi}{2x}}\, J_{l+\frac{1}{2}}(x)$ and $h_l^{(1)}(x) = \sqrt{\frac{\pi}{2x}}\, H^{(1)}_{l+1/2}(x)$ are the spherical Bessel functions. Further $j_l(1) \equiv j_l(kr_1)$ etc. The magnetic strength $\vec{H} = 0$ everywhere, while the electric one differs from zero only inside the spherical capacitor: $\vec{E} = e\vec{r}/r^3$. From Eqs. (6.2), (6.3) and (6.5) it follows that for the nonstatic electric solenoid there exist the waves of the electromagnetic potentials. They propagate off the source with the light velocity. As $\vec{E} = \vec{H} = 0$ inside them, they do not carry the electromagnetic energy. This means that they can be observable

(if ever) on the quantum level only (the electromagnetic potentials, not the field strengths, enter into the Schroedinger equation). Let the space region S where $\vec{E} \neq 0$ is inaccessible for the observer (a suitable screen can be used). Can he prove the existence of electromagnetic potentials operating entirely outside S? It seems at first that it is impossible. In fact, the transformation

$$\vec{A} \to \vec{A}' = \vec{A} - \operatorname{grad}\chi, \quad \varphi \to \varphi' = \varphi + \frac{1}{c}\frac{\partial\chi}{\partial t},$$
$$\psi \to \psi' = \psi\exp(-ie\chi/\hbar c), \quad \chi = ik\int G_k(\vec{r},\vec{r}\,')\,\rho(\vec{r}\,')\,dV'$$

eliminates electromagnetic potentials outside S. If the function χ is single-valued (continuous) outside S then ψ and ψ' describe the same physical situation. This takes place for the point like solution of ref. /4/, for the cylindrical capacitor /5/ and for the spherical one. In all these cases the electromagnetic potentials can be removed without spoiling the singlevaluedness properties of the wave function and, thus, they are not observable. The accessible space region (lying outside S) should be multiconnected as multivalued wave functions are not permissible in the simply connected regions of space. If **this condition is fulfilled (that is multivaluedness of ψ' outside S)** then particle waves travelling along different paths may acquire different phases before their subsequent recombining (like in magnetic or electrostatic Aharonov – Bohm effect). This in principle could be observed experimentally.

Is it possible to construct nonstatic solenoid using only the changing with time current density (i.e. for the vanishing charge density)? Such a construction was suggested in ref. /6/ (fig.4). The torus $T\ ((\rho-d)^2+z^2=R^2)$ is densily covered by the infinitely thin toroidal solenoids ts_i. It is claimed in /6/ that for the periodical current in the winding of ts_i the electromagnetic fields is confined to the interior of the torus T. We prove now that this does not take place. For this aim we consider the case of infinitely thin torus T. The set of toroidal solenoids ts_i (being infinitely small now) can be treated as a circular chain of toroidal moments /17/ :

$$\vec{t}=\exp(-i\omega t)\cdot t\cdot\vec{n}_\varphi\cdot\delta(\rho-d)\cdot\delta(z) \qquad (6.6)$$

(see fig.2 where arrows now mean toroidal moments $\vec{t}$). To this toroidal moment corresponds the current $\vec{j}$ given by $\vec{j}=c\cdot\mathrm{rot}\,\mathrm{rot}\,\vec{t}$ /17/ . The vector potential generated by the chain of toroidal moments is

$$\vec{A}=\int G_K(\vec{r},\vec{r}')\cdot\mathrm{rot}\,\mathrm{rot}\,\vec{t}(\vec{r}')\,dV'$$

(the factor $\exp(-i\omega t)$ is again omitted). The integration here is performed along the circumference of the radius d. Integrating twice by parts we get

$$\vec{A}=\mathrm{rot}\,\mathrm{rot}\int G_K(\vec{r},\vec{r}')\,\vec{t}(\vec{r}')\,dV'=$$

$$=(\mathrm{grad}\,\mathrm{div}+K^2)\,\vec{n}_\varphi\,I+4\pi\,\vec{t}(\vec{r}). \qquad (6.7)$$

Here

$$I=\int_0^{2\pi}\frac{\exp(ikZ)}{Z}\cos\varphi\,d\varphi\ ,\quad Z=(r^2+d^2-2d\rho\cos\varphi)^{1/2}.$$

As I is independent of φ , so $\mathrm{div}(\vec{n}_\varphi\cdot I)=0$ and

$$\vec{A}=4\pi\,\vec{t}(\vec{r})+K^2\cdot I\cdot\vec{n}_\varphi\ ,\quad K=\omega/c. \qquad (6.8)$$

To find the value of I we use the well known expansion /18/

$$\frac{\exp(ik\tilde{Z})}{\tilde{Z}}=\frac{i\pi}{(\tilde{d}\tilde{r})^{1/2}}\cdot\sum_{m=0}^{\infty}(m+\tfrac{1}{2})\cdot J_{m+\frac{1}{2}}(k\tilde{d})\cdot H^{(1)}_{m+\frac{1}{2}}(k\tilde{r})\cdot$$

$$\cdot P_m(\cos\varphi)\ ,\quad \tilde{Z}=(\tilde{r}^2+\tilde{d}^2-2\tilde{d}\tilde{r}\cos\varphi)^{1/2},\ \tilde{r}>\tilde{d}$$

Thus

$$\int_0^{2\pi}\frac{\exp(ik\tilde{Z})}{\tilde{Z}}\cos\varphi\,d\varphi=\frac{i\pi^2}{(\tilde{d}\tilde{r})^{1/2}}\cdot$$

$$\sum_{n=0}^{\infty}(2n+\tfrac{3}{2})\,\frac{1}{2^{4n}}\,\frac{2n+1}{n+1}\cdot\binom{2n}{n}^2\cdot J_{2n+\frac{3}{2}}(k\tilde{d})\cdot H^{(1)}_{2n+\frac{3}{2}}(k\tilde{r}).$$

Now we compare $\tilde{Z}$ with Z . We try to adjust parameters $\tilde{d}$ and $\tilde{r}$ in such a way as to equalize $\tilde{Z}$ and Z . This takes place if

$$\tilde{r}=\tfrac{1}{2}(r_1+r_2)\ ,\quad \tilde{d}=\tfrac{1}{2}(r_1-r_2)$$

$$r_1=[(\rho+d)^2+z^2]^{1/2}\ ,\quad r_2=[(\rho-d)^2+z^2]^{1/2}. \qquad (6.9)$$

It follows from this that

$$I = \int_0^{2\pi} \frac{\exp(ikZ)}{Z} \cos\varphi \, d\varphi = \frac{i\pi^2}{(\tilde{d}\,\tilde{r})^{1/2}} \cdot \qquad (6.10)$$

$$\sum_{n=0}^{\infty} \left(2n+\frac{3}{2}\right) \frac{1}{2^{4n}} \, \frac{2n+1}{n+1} \cdot \binom{2n}{n}^2 \cdot J_{2n+\frac{3}{2}}(k\tilde{d}) \cdot$$

$$H^{(1)}_{2n+\frac{3}{2}}(k\tilde{r})$$

where Z, $\tilde{r}$ and $\tilde{d}$ are defined by Eqs. (6.7) and (6.9). It follows from this that I is certainly different from zero. Then outside the chain of toroidal moments there are nonvanishing field strengths

$$E_\varphi = i k^3 \cdot I$$

$$H_r = \frac{k^2}{r\sin\theta} \frac{\partial}{\partial\theta} (\sin\theta \cdot I), \quad H_\theta = -\frac{k^2}{r} \frac{\partial}{\partial r} (r \cdot I).$$

Thus $\vec{E}$ and $\vec{H}$ go beyond the interior of the torus and this complete our proof. It is valid for the infinitely thin torus T. But the measurements described in ref. /6/ were performed for the torus of finite radius. It is known that for the cylindrical solenoid with periodical current in its winding the vector potential $\vec{A}$ disappears outside the solenoid for the definite values of KR ($K = \omega/c$, R is the radius of the solenoid). Suppose that the same property takes place for the current configuration of fig.4. Then the disappearance of the electromagnetic field outside T observed in /6/ may be

attributed to the proximity of torus radius to the specific value mentioned above.

7. Conclusion

We briefly summarize the main results obtained:

1. A realistic construction of the static electric toroidal solenoid is presented. Outside it there is electric vector potential which cannot be eliminated by the gauge transformation. By solving the known electrostatic problems in terms of the electric vector potential we find that it has the same footing as the scalar one.

2. The nonstatic electrical solenoids are considered. Both scalar and vector potentials differ from zero outside these solenoids while inside them only electrical field strength is nonvanishing.

We feel that present paper raises more questions than answers on them. We rephrase the question posed by Aharonov and Bohm in their famous 1959 paper /19/ in the following way: Do the electric vector potential of the static electrical solenoid and the electromagnetic potentials of the nonstatic solenoid have the physical meaning? How their existence can be verified experimentally?

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Appendix

We feel that the physical meaning of the electrical vector potential should be stated more clearly. For this we consider the polarized sphere of the radius a. Let the polarization $\vec{P}$ has the constant value and is directed along z axis

$$\vec{P} = P_0 \cdot \vec{n}_z \cdot \Theta(a-r). \tag{A.1}$$

In the absence of free charges the Maxwell equations are

$$\operatorname{div} \vec{D} = 0 \tag{A.2}$$

$$\operatorname{rot} \vec{E} = 0 \tag{A.3}$$

$$\vec{D} = \vec{E} + 4\pi \vec{P}. \tag{A.4}$$

The usual solution of these equations proceeds of follows /16/. Excluding $\vec{D}$ from Eq.(A.2) we get Eqs. for $\vec{E}$

$$\operatorname{rot} \vec{E} = 0 \tag{A.5}$$

$$\operatorname{div} \vec{E} = -4\pi \operatorname{div} \vec{P}. \tag{A.6}$$

Eq. (A.5) is automatically satisfied if

$$\vec{E} = -\operatorname{grad} \varphi. \tag{A.7}$$

Substituting this in Eq.(A.6) we get equation

$$\Delta \varphi = 4\pi \operatorname{div} \vec{P}, \qquad \operatorname{div} \vec{P} = -P_0 \cdot \cos\theta \cdot \delta(r-a). \tag{A.8}$$

Its solution is $\varphi = -\int \frac{1}{|\vec{r}-\vec{r}'|} \operatorname{div} \vec{P}(\vec{r}')\, dV'$

or explicitly

$$\varphi = \frac{4}{3}\pi a^3 P_0 \frac{\cos\theta}{r^2} \quad \text{for} \quad r > a \quad \text{and}$$

$$\varphi = \frac{4}{3}\pi \cdot P_0 \cdot r \cdot \cos\theta \quad \text{for} \quad r < a. \tag{A.8}$$

Using (A.7) and (A.4) we obtain $\vec{E}$, $\vec{D}$

$$E_r = \frac{8}{3}\pi a^3 P_0 \cos\theta / r^3,$$

$$E_\theta = \frac{4}{3}\pi a^3 P_0 \sin\theta / r^3, \qquad \vec{D} = \vec{E} \tag{A.9}$$

outside the sphere and

$$E_r = -\frac{4}{3}\pi \cdot P_0 \cos\theta, \qquad E_\theta = \frac{4}{3}\pi \cdot P_0 \cdot \sin\theta,$$

$$D_r = \frac{8}{3}\pi P_0 \cdot \cos\theta, \qquad D_\theta = -\frac{8}{3}\pi \cdot P_0 \cdot \sin\theta \tag{A.10}$$

inside it.

The other way to solve Eqs. (A.2)-(A.4) is to introduce electric vector potential. Excluding $\vec{E}$ from Eqs. (A.2)-(A.4) we get equations for $\vec{D}$

$$\operatorname{div} \vec{D} = 0 \tag{A.11}$$

$$\operatorname{rot} \vec{D} = 4\pi \cdot \operatorname{rot} \vec{P}; \qquad \operatorname{rot} \vec{P} = P_0 \cdot \sin\theta \cdot \delta(r-a) \cdot \vec{n}_\varphi. \tag{A.12}$$

To satisfy Eq. (A.11) automatically we put

$$\vec{\mathfrak{D}} = \mathrm{rot}\,\vec{A} \quad , \quad \mathrm{div}\,\vec{A} = 0 \tag{A.13}$$

and find the following equation for $\vec{A}$

$$\Delta \vec{A} = -4\pi\, \mathrm{rot}\, \vec{P}. \tag{A.14}$$

Its solution is

$$\vec{A} = \int \frac{\mathrm{rot}\, \vec{P}(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|}$$

or in explicit form

$$\vec{A} = \frac{4}{3}\pi a^3 \cdot P_0 \frac{\sin\theta}{r^2} \cdot \vec{n}_\varphi \qquad \text{outside the sphere and}$$

$$\vec{A} = \frac{4}{3}\pi P_0 r \sin\theta \cdot \vec{n}_\varphi \qquad \text{inside it .} \tag{A.15}$$

Using Eqs. (A.13) and (A.4) we arrive to Eqs. (A.9) and (A.10) . Thus, there are two equivalent ways to find $\vec{E}$, $\vec{\mathfrak{D}}$ for the permanently polarized sphere. They are expressed through the electrical scalar potential in the first case and the vector electrical potential in the second case. As the space outside the sphere is simply -connected, the potentials φ and $\vec{A}$ may be expressed through $\vec{E}$, $\vec{\mathfrak{D}}$ and thus they both have auxiliary meaning.

Now we consider the prolate auxially symmetric ellipsoid

$$\frac{x^2+y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad ; \quad c > b. \tag{A.16}$$

Let this ellipsoid has the constant polarization directed along the z axis. This electrostatic problem is again easily solved in terms of either electric scalar /16/ or vector potentials.

Introduce the spheroidal coordinates

$$x = a\cdot \mathrm{sh}\mu \sin\theta\cos\varphi, \quad y = a\, \mathrm{sh}\mu \sin\theta \sin\varphi, \quad z = a\,\mathrm{ch}\mu\cos\theta$$

Let the value $\mu = \mu_0$ corresponds to the ellipsoid (A.16)

$$b = a\cdot \mathrm{sh}\mu_0 \quad , \quad c = a\cdot\mathrm{ch}\mu_0 . \tag{A.17}$$

Then polarization is given by

$$\vec{P} = P_0 \cdot \vec{n}_z \; \Theta(\mu_0 - \mu). \tag{A.18}$$

Equations (A.2) - (A.4) with the polarization (A.18) may be solved in terms of either electric scalar or vector potentials. In the first case one has

$$\varphi = 4\pi P_0\, a\, \mathrm{sh}^2\mu_0 \cdot \cos\theta \cdot f_{10}(\mu,\mu_0)$$

$$\vec{E} = \frac{4\pi P_0 \mathrm{sh}^2\mu_0}{(\mathrm{ch}^2\mu - \cos^2\theta)^{1/2}} \Big[\sin\theta \cdot f_{10}(\mu,\mu_0)\cdot \vec{n}_\theta \tag{A.19}$$

$$- \cos\theta \cdot \frac{d f_{10}(\mu,\mu_0)}{d\mu} \cdot \vec{n}_\mu \Big].$$

Here

$$f_{em}(\mu;\mu_0) = \begin{cases} Q_e^m(\mathrm{ch}\mu)\cdot P_e^m(\mathrm{ch}\mu_0) & \text{for } \mu > \mu_0 \\ P_e^m(\mathrm{ch}\mu)\cdot Q_e^m(\mathrm{ch}\mu_0) & \text{for } \mu < \mu_0 \end{cases}$$

P_e^m and Q_e^m are the Legendre functions.

On the other hand the same equations may be solved in terms of electrical vector potential. The answer is

$$\vec{A} = A \cdot \vec{n}_\varphi$$

$$A = -2\pi P_0 a\, \mathrm{ch}\mu_0\, \mathrm{sh}\mu_0 \sin\theta \cdot f_{11}(\mu,\mu_0). \tag{A.20}$$

Substituting $\vec{A}$ into Eq. (A.13) we arrive to $\vec{\mathfrak{D}}, \vec{E}$ obtained via the electric scalar potential. Thus, as for the case of polarized sphere there are two alternative ways to solve the treated electrostatic problem. Let the major semi-axis c of the ellipsoid (A.16) tends to infinity, while the minor one (b) remains the same. For this we put $a = b/\varkappa\mu_0$ in Eqs. (A.19) and (A.20) and then take the limit $\mu_0 \to 0$. It turns out that $\varphi \to 0$, $\vec{E} \to 0$ everywhere. Further $\vec{\mathfrak{D}} \to 0$, $A \to 2\pi P_0 b^2/\rho$ outside the ellipsoid and $\vec{\mathfrak{D}} \to 4\pi\vec{P}$, $A \to 2\pi P_0 \rho$ inside it. Thus, we recover the vector potential of the electrical cylindrical solenoid. The same situation (that is the disappearance of $\vec{E}, \varphi$ and surviving of $\vec{A}$, $\vec{\mathfrak{D}}$) takes place for the toroidal solenoid.

The moral of these considerations is that in general electric vector potential has equal status with the scalar one. There are special situations in electrostatics for which survive either electrical scalar or vector potentials. The electrical toroidal and cylindrical solenoids are of the latter kind.

In §§ 3-5 we have constructed static magnetic and electric solenoids using the chains of magnetic and electric dipoles, resp. What happens if we change these dipoles by the static toroidal (magnetic or electric) moments (they are given by Eq.(6.6) with $\omega = 0$ in it)? It follows from Eq.(6.8) that VP vanishes outside these spin chains (or tubes). This means that complete self-screening takes place.

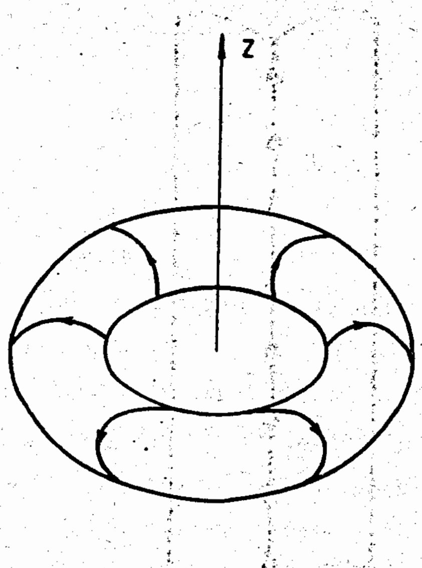

Fig.1. Poloidal current flowing on the surface of torus.

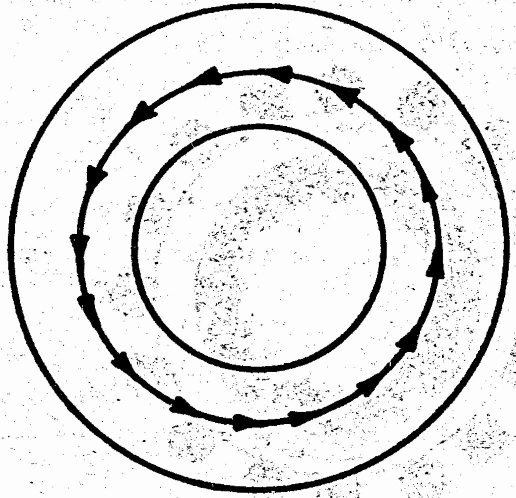

Fig.2. The explicit realization of magnetic (electric) toroidal solenoid by means of circular magnetic (electric) dipole chain.

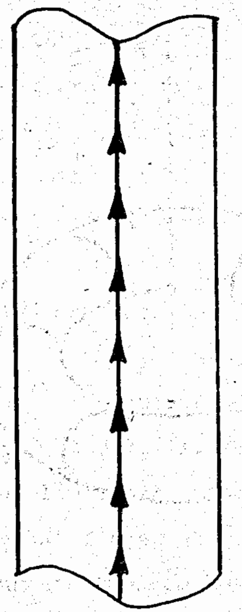

Fig.3. The explicit realization of magnetic (electric) cylindrical solenoid by means of linear magnetic (electric) dipole chain.

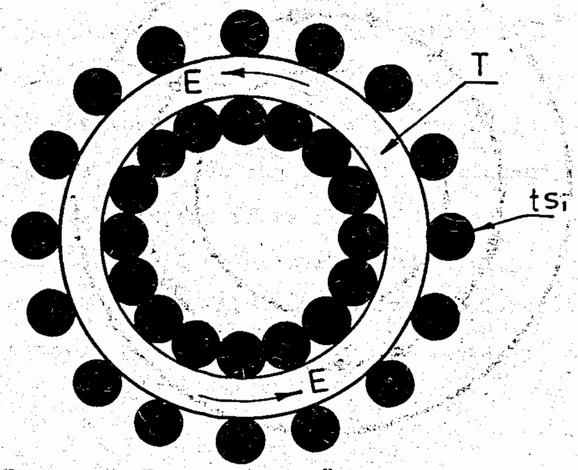

Fig.4. The torus T is "dressed" by the toroidal solenoids ts_i with alternate currents in their coil. For the very thin solenoids ts_i and dense covering of T by ts_i the electromagnetic field is confined inside T (according to ref. /6/).

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Статические и нестатические
электрические соленоиды

Предлагаются распределения зарядов и токов, которые реализуют статические и нестатические электрические соленоиды. Их свойства обсуждаются. Остается открытым вопрос о физическом смысле электромагнитных потенциалов вне соленоидов и об их экспериментальной проверке.

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Afanasiev G.N. E2-92-132
The Static and Nonstatic Electrical
Solenoids

We propose the set of charge and current densities which realize static and nonstatic electrical solenoids. Their properties are discussed. The question on the physical meaning of the electromagnetic potentials outside the solenoids and their experimental verification remains to be opened.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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