92 - 107

объединенный
институт
ядерных
исследований
дубна

E2-92-107

T.A.Ivanova, A.D.Popov

# ON A CONNECTION BETWEEN THE YANG — MILLS EQUATIONS IN DIMENSION GREATER THAN FOUR AND THE CLASSICAL YANG — BAXTER EQUATIONS

Submitted to "Letters in Mathematical Physics"

1992

1. Introduction. In this paper we will show that the solutions of the classical Yang-Baxter (YB) equations may be used in constructing the solutions of the Yang-Mills (YM) equations in $\mathbb{R}^n$. Our goal is to find some solutions of the equations for a pure classical YM theory in the Euclidean space $\mathbb{R}^n$ with the metric $\delta_{ab}$, $a,b,\ldots=1,\ldots,n$. Let $A_a$ be the YM potentials with values in the semisimple Lie algebra $\mathfrak{g}$ of the Lie group G and $F_{ab}=\partial_a A_b-\partial_b A_a+[A_a,A_b]$ be the curvature tensor for $A_a$.

The YM equations for the gauge potentials $A_a$ have the form

$$\partial_a F_{ab}+[A_a,F_{ab}]=0. \tag{1}$$

The Einstein summation convention is used throughout, if not stated otherwise.

Some solutions of Eqs.(1) in the spaces $\mathbb{R}^7$, $\mathbb{R}^8$ and $\mathbb{R}^{4k}$ were obtained in papers [1-5]. In particular, in [4,5] it has been shown that new solutions of the YM equations in n=7 and n=8 may be obtained from solutions of the classical YB equations. In what follows we shall show that it is possible to obtain other classes of solutions of the YM equations in the spaces of dimension $n\geq 4$ from the solutions of the classical YB equations.

2. Rouhani-Ward equations. In $\mathbb{R}^n$ let us consider gauge fields $A_a$ depending only on $t\equiv x_n$ (cosmological solutions). We can always transform to a gauge in which $A_n=0$ (gauge fixing). As a result, we have

$$F_{\alpha\beta} = [A_\alpha, A_\beta] \quad , \qquad F_{n\alpha} = \dot{A}_\alpha, \tag{2}$$

where $\dot{A}_\alpha \equiv dA_\alpha/dt$, $\alpha,\beta,\ldots=1,\ldots,n-1$. Then, the YM equations take the form

$$\ddot{A}_\alpha - [A_\beta,[A_\alpha,A_\beta]] = 0, \tag{3a}$$

$$[\dot{A}_\alpha, A_\alpha] = 0 . \tag{3b}$$

These equations generalize the equations of Corrigan, Wainwright and Wilson [6] who have considered the case n=4.

Consider the Lie algebra $\mathcal{H}$ of the simple compact Lie group H. We put $n=1+\dim\mathcal{H}$. Considering $\mathcal{H}$ as a vector space, we obtain $\mathbb{R}^n=\mathcal{H}\oplus\mathbb{R}$ and $\delta_{ab} = \{\delta_{\alpha\beta},\delta_{nn}\}$, $a,b,\ldots=1,\ldots,n$. Let the structure constants $f_{\alpha\beta\gamma}$ of the Lie algebra $\mathcal{H}$ be normalized to $f_{\alpha\gamma\delta}\, f_{\beta\gamma\delta} = 2\,\delta_{\alpha\beta}$.

In $\mathbb{R}^n=\mathcal{H}\oplus\mathbb{R}$ let us introduce the following antisymmetric 4-index tensor $T_{abcd}$:

$$T_{\alpha\beta\gamma\delta} = 0 \quad , \qquad T_{\alpha\beta\gamma n} = f_{\alpha\beta\gamma} \quad . \tag{4}$$

With the help of $T_{abcd}$ one can introduce the H-invariant self-duality equations (cf.[7,1]):

$$T_{abcd}\, F_{cd} = 2\, F_{ab} . \tag{5}$$

Using (4), one can rewrite Eqs.(5) in the form

$$f_{\alpha\beta\gamma}\, F_{\gamma n} = F_{\alpha\beta} , \tag{6}$$

$$f_{\delta\alpha\beta}\, F_{\alpha\beta} = 2\, F_{\delta n} . \tag{7}$$

Multiplying Eqs.(6) by $f_{\delta\alpha\beta}$, one obtains Eqs.(7). Therefore, the self-duality equations (5) are equivalent to Eqs.(6).

Substitute (2) into (6) and obtain

$$f_{\alpha\beta\gamma}\, \dot{A}_\gamma + [A_\alpha, A_\beta] = 0 . \tag{8}$$

It is easy to see that each solution of the self-duality equations (8) satisfies Eqs.(3). Indeed, multiply Eqs.(8) by $f_{\alpha\beta\delta}$ and rename the indices, then obtain $\dot{A}_\alpha+\frac{1}{2}f_{\alpha\beta\gamma}[A_\beta, A_\gamma]=0$. If one differentiates these equations once more, then obtains $\ddot{A}_\alpha = -f_{\alpha\beta\gamma}[A_\beta,\dot{A}_\gamma]$. At the same time from (8) it follows that $[A_\beta,[A_\alpha,A_\beta]]= -f_{\alpha\beta\gamma}[A_\beta,\dot{A}_\gamma]$. Thus, if $A_\alpha$ satisfy Eqs.(8), then $A_\alpha$ satisfy Eqs.(3a), too. And finally, from Eqs.(8) and from the Jacobi identities for matrices $A_\alpha$ one obtains that $A_\alpha$ satisfy Eqs.(3b).

Equations (8) were introduced by Rouhani [8] for the case when $f_{\alpha\beta\gamma}$ are structure constants of the Lie algebra $sl(N,\mathbb{R})$. For a more general case, when $f_{\alpha\beta\gamma}$ are structure constants of an arbitrary simple Lie algebra $\mathcal{H}$, these equations were introduced and investigated by Ward [9]. Ward has also supposed [9] that Eqs.(8) can be obtained from the self-duality equations for gauge fields in higher dimensions. We shall call Eqs.(8) the Rouhani-Ward (RW) equations.

**3. Yang-Baxter equations.** In [8,9] the connection of Eqs.(8) with the classical Yang-Baxter equations was shown. The detailed discussion of the classical YB equations and the description of the explicit form of their solutions may be found in [10-13]. The classical YB equations on the Lie algebra $\mathcal{H}$ can be written in the form [8,9]

$$f_{\alpha\varepsilon\delta}W_{\varepsilon\beta}(u)W_{\delta\gamma}(u+v)+f_{\beta\varepsilon\delta}W_{\alpha\varepsilon}(u)W_{\delta\gamma}(v)+f_{\gamma\varepsilon\delta}W_{\alpha\varepsilon}(u+v)W_{\beta\delta}(v)=0. \quad (9)$$

Here functions $W_{\alpha\beta}$ $(\alpha,\beta,\ldots=1,\ldots,\dim\mathcal{H})$ depend on complex variables $u$ and $v$, $f_{\alpha\beta\gamma}$ are structure constants of the Lie algebra $\mathcal{H}$.

Assume that the functions $W_{\alpha\beta}(z)$ have a simple pole at 0 with residue of the form $\zeta\delta_{\alpha\beta}$, $\zeta=\mathrm{const}\neq 0$. Following [11], we shall call these functions the nondegenerate functions. As it has been proved in [8,9], when $v\longrightarrow 0$ the functional equations (9) for nondegenerate functions $W_{\alpha\beta}$ reduce to the following differential equations:

$$f_{\alpha\beta\gamma}\,\dot{W}_\gamma + [W_\alpha, W_\beta] = 0\ , \quad (10)$$

where $\dot{W}_\gamma\equiv dW_\gamma/du$, $W_\alpha\equiv(1/\zeta)W_{\alpha\beta}(u)I_\beta$, $I_\beta$ are generators of the Lie algebra $\mathcal{H}$, i.e. $[I_\alpha, I_\beta]=f_{\alpha\beta\gamma}I_\gamma$. Therefore, each nondegenerate solution of the classical YB equations satisfies the differential equations (10).

Comparing (8) with (10), we see that in the case $\mathcal{G}=\mathcal{H}$ the RW equations (8) coincide with Eqs.(10). Hence, each nondegenerate solution of the classical YB equations on $\mathcal{H}$ is a solution of the RW equations (8) if $\mathcal{G}=\mathcal{H}$. So, we have

**PROPOSITION 1.** *The classical Yang-Baxter equations on the Lie algebra $\mathcal{H}$ are equivalent to the self-duality equations (5) for gauge fields $A_\alpha$ of the Lie group H in the space $\mathbb{R}^n=\mathcal{H}\oplus\mathbb{R}$, reduced to one dimension. Each nondegenerate solution $W_\alpha$ of the classical YB equations on $\mathcal{H}$ gives a cosmological solution $A_\alpha$ of the YM equations in $\mathbb{R}^n=\mathcal{H}\oplus\mathbb{R}$, if $A_\alpha=W_\alpha(x_n)$.*



Proof follows from (2)-(10).

A lot of nondegenerate solutions of the classical YB equations are known (see, e.g., [10-13]). For simple Lie algebras $\mathcal{H}$ each solution $W_{\alpha\beta}$ of these equations is either an elliptic function, or a trigonometric function, or a rational function [11,12]. In [11,12] the detailed description of all nondegenerate elliptic and trigonometric solutions is given, and in [11,13] a vast family of rational solutions is constructed.

The simplest rational solution of Eqs.(9) has the form [10,11]:

$$W_\alpha(u) = \frac{1}{u}\,I_\alpha\ . \quad (11)$$

A more complicated trigonometric solution can be obtained from the solution given in [12] (formula (IV.2.12)) if one puts $\lambda=iu$ ($u$ is real) and $\omega=2\pi$ in notation of Faddeev and Takhtajan. This solution has the form

$$W_\alpha(u) = \frac{1}{2r}\sum_{j=1}^{r-1}\Theta^j(I_\alpha)\ \mathrm{cth}\,\frac{(u-i2\pi j)}{2r}\ , \quad (12)$$

where $\Theta$: $\mathcal{H}\longrightarrow\mathcal{H}$ is the Coxeter automorphism of the simple Lie algebra $\mathcal{H}$ and $r$ is its order, i.e. $\Theta^r=\mathrm{Id}$. For all simple Lie algebras $\mathcal{H}$ the Coxeter numbers $r$ and the description of the automorphism $\Theta$ may be found in [14] (see, also, [11]).

The Coxeter automorphisms of the classical Lie algebras $\mathcal{H}$ have the form [11,14]: $\Theta(\mathcal{A}) = Q\mathcal{A}Q^{-1}$ or $\Theta(\mathcal{A}) = -Q\mathcal{A}^tQ^{-1}$, where $\mathcal{A}\in\mathcal{H}$. The explicit form of matrices $Q$ and the values of $r$ may be found in [11]. With the help of $Q$, all constant matrices

$\Theta^j(I_\alpha)$ can be explicitly written out, and one can show that solution (12) is real. Particular cases of this solution for $\mathcal{H}=so(7)$ and $\mathcal{H}=so(8)$ are described in [5]. Notice that for real $u$ the solution (12) is singular only when $u$=0. The other poles are on the imaginary axis at the points $u=i2\pi l$, $l=\pm1, \pm2, \ldots$ [11,12].

The explicit form of general nondegenerate trigonometric and elliptic solutions of the classical YB equations for any simple Lie algebra $\mathcal{H}$ is given in [11]. They are rather complicated, that is why we do not write out them here. When $u=x_n$, all these solutions give the cosmological solutions of the YM equations in $\mathbb{R}^n=\mathcal{H}\oplus\mathbb{R}$.

**4. Tensors $J^\alpha_{ab}$ and ansatz.** Now we show that the Rouhani-Ward and Yang-Baxter equations may appear in the Yang-Mills theory on the space $\mathbb{R}^n$ not only in the case when the gauge fields $A_a$ depend on one coordinate $x_n$. It turns out that the connection between dimension q of the algebra $\mathcal{H}$ and dimension n of the space will be different (i.e. $q\neq n-1$).

Let us suppose that in the space $\mathbb{R}^n$ with metric $\delta_{ab}$ there are q constant tensors $J^1_{ab},\ldots,J^q_{ab}$ that are antisymmetric in indices $a$ and $b$ and obey the relations

$$J^\alpha_{ac} \; J^\beta_{bc} = \delta^{\alpha\beta}\,\delta_{ab} + \Sigma^{\alpha\beta}_{ab} \quad , \tag{13}$$

where $\Sigma^{\alpha\beta}_{ab}$ are some constant antisymmetric in $a$ and $b$ tensors, $\alpha,\beta,\ldots=1,\ldots,q$. Examples of tensors $J^\alpha_{ab}$ satisfying (13) will be given later.



In $\mathbb{R}^n$ we consider gauge fields $A_a$ of the Lie group G. We shall look for solutions of the YM equations (1) in the form

$$A_a = -\,J^\alpha_{ac}\,T_\alpha(\varphi)\,\partial_c\varphi, \tag{14}$$

where the real antisymmetric tensors $J^\alpha_{ab}$ satisfy (13); $\varphi$ is an arbitrary function of coordinates $x^a\in\mathbb{R}^n$; $T_1,\ldots,T_q$ depend only on $\varphi$ and take values in the Lie algebra $\mathcal{G}$, i.e. they are matrix functions. If n=4 and q=3, as $J^\alpha_{ab}$ one may take the well-known 't Hooft tensors, and in this case ansatz (14) coincides with the ansatz of papers [15]. If n=4, q=3 and $\varphi=x_ax^a$ then (14) coincides with the ansatz of papers [16].

Substitute (14) into the definition of $F_{ab}$ and obtain

$$F_{ab} = J^\alpha_{ac}\Big\{T_\alpha\partial_b\partial_c\varphi + \dot T_\alpha\partial_b\varphi\;\partial_c\varphi\Big\} - J^\alpha_{bc}\Big\{T_\alpha\partial_a\partial_c\varphi + \dot T_\alpha\partial_a\varphi\;\partial_c\varphi\Big\} + $$
$$+\,J^\beta_{ac}\,J^\gamma_{be}\,[T_\beta,T_\gamma]\;\partial_c\varphi\;\partial_e\varphi\;, \tag{15}$$

where $\dot T_\alpha\equiv dT_\alpha/d\varphi$, $\partial_a\equiv\partial/\partial x^a$. Substituting (14) and (15) into Eqs.(1) and using relations (13), we obtain

$$\partial_aF_{ab} + [A_a,F_{ab}] = T_\alpha\;J^\alpha_{ab}\;\partial_a(\Box\varphi) + [\dot T_\alpha,T_\alpha]\;\partial_c\varphi\;\partial_c\varphi\;\partial_b\varphi\;+$$
$$+\Big(\;\ddot T_\alpha - [\;T_\beta\;,[T_\alpha,T_\beta]]\Big)\;J^\alpha_{ab}\;\partial_a\varphi\;\partial_c\varphi\;\partial_c\varphi\;-\;\partial_a\varphi\Big\{2\dot T_\alpha\;J^\alpha_{bc}\;\partial_c\partial_a\varphi+$$
$$+\;[T_\alpha,T_\beta]\;\Sigma^{\alpha\beta}_{ac}\;\partial_c\partial_b\varphi\;+\;2[T_\alpha,T_\beta]J^\alpha_{ac}J^\beta_{be}\partial_c\partial_e\varphi\;-\;\dot T_\alpha\;J^\alpha_{ab}\;\Box\varphi\Big\}, \tag{16}$$

where $\ddot T\equiv d^2T_\alpha/d\varphi^2$, $\Box\equiv\partial_c\partial_c$.

**5. Equations for $T_\alpha(\varphi)$ and $\varphi$.** The indices $\alpha,\beta,\ldots,\varepsilon$ range over $1,\ldots,q$. Let us assume that in the space $\mathbb{R}^q$ there is a constant totally antisymmetric 3-index tensor $f_{\alpha\beta\gamma}$ satisfying

$f_{\alpha\gamma\delta}\, f_{\beta\gamma\delta} = 2\,\delta_{\alpha\beta}$. For example, if q=7, then as $f_{\alpha\beta\gamma}$ one may take the octonionic structure constants (see [4,5]). If q coincides with dimension of simple compact Lie algebra $\mathcal{H}$, then as $f_{\alpha\beta\gamma}$ one may take the structure constants of $\mathcal{H}$.

Using the antisymmetric tensor $f_{\alpha\beta\gamma}$, one can rewrite Eqs.(16) in the following way:

$$\partial_a F_{ab} + [A_a, F_{ab}] = T_\alpha\, J^{\alpha}_{ab}\, \partial_a(\Box\varphi) + [\dot T_\alpha, T_\alpha]\, \partial_c\varphi\, \partial_c\varphi\, \partial_b\varphi +$$
$$+\Big(\ddot T_\alpha - [\,T_\beta\,,[T_\alpha, T_\beta]]\Big)\, J^{\alpha}_{ab}\, \partial_a\varphi\, \partial_c\varphi\, \partial_c\varphi - (f^{\ \ \gamma}_{\alpha\beta}\dot T_\gamma + [T_\alpha, T_\beta])\times$$
$$\times\Big\{\Sigma^{\alpha\beta}_{ac}\, \partial_c\partial_b\varphi + 2J^{\alpha}_{ac}J^{\beta}_{be}\partial_c\partial_e\varphi\Big\}\partial_a\varphi + \dot T_\alpha\, \Big\{f^{\ \alpha}_{\beta\gamma}\, \Sigma^{\beta\gamma}_{ac}\, \partial_c\partial_b\varphi -$$
$$- 2J^{\alpha}_{bc}\partial_c\partial_a\varphi + 2f^{\ \alpha}_{\beta\gamma}J^{\beta}_{ac}J^{\gamma}_{be}\partial_c\partial_e\varphi + J^{\alpha}_{ab}\,\Box\varphi\Big\}\partial_a\varphi\,. \qquad (17)$$

Let q coincide with a dimension of simple compact Lie algebra $\mathcal{H}$ with structure constants $f_{\alpha\beta\gamma}$. Assume that $T_\alpha(\varphi)$ satisfy the RW equations (8):

$$f_{\alpha\beta\gamma}\, \dot T_\gamma + [T_\alpha, T_\beta] = 0 \qquad (18)$$

and function $\varphi$ obeys the following system of linear equations

$$f^{\ \alpha}_{\beta\gamma}\, \Sigma^{\beta\gamma}_{ac}\, \partial_c\partial_b\varphi - 2J^{\alpha}_{bc}\partial_c\partial_a\varphi + 2f^{\ \alpha}_{\beta\gamma}J^{\beta}_{ac}J^{\gamma}_{be}\partial_c\partial_e\varphi + J^{\alpha}_{ab}\,\Box\varphi = 0. \qquad (19)$$

As was shown in Sect.2, from Eqs.(18) it follows that $[\dot T_\alpha, T_\alpha] = \ddot T_\alpha - [T_\beta, [T_\alpha, T_\beta]] = 0$. If we differentiate (19) with respect to $x^a$ then obtain $J^{\alpha}_{ab}\, \partial_a(\Box\varphi) = 0$. Thus, if $T_\alpha(\varphi)$ and $\varphi$ satisfy the system of Eqs.(18), (19), then the right-hand side of (17) will be equal to zero and gauge field (14) will be the solution of the YM equations (1).

**PROPOSITION 2.** *If tensors $J^{\alpha}_{ab}$ satisfy the relations (13) and q=dim$\mathcal{H}$, then to each solution of system {(18),(19)} one may correspond the solution (14) of the YM equations (1) for gauge fields $A_a$ of an arbitrary semisimple Lie group G in the Euclidean space $\mathbb{R}^n$.*



Proof follows from formulas (14)-(19).

In [8,9] it has been shown that if $\mathcal{G}=\mathcal{H}$ then each nondegenerate solution of the classical YB equations on the Lie algebra $\mathcal{H}$ will be the solution of Eqs.(18) and from Proposition 2 it follows:

**PROPOSITION 3.** *If tensors $J^{\alpha}_{ab}$ satisfy equations (13), q=dim$\mathcal{H}$ and $\mathcal{G}=\mathcal{H}$, then to each nondegenerate solution $T_\alpha(\varphi)$ of the classical YB equations on the Lie algebra $\mathcal{H}$ with $\varphi$ satisfying Eqs.(19) one may correspond the solution (14) of the YM equations (1) for gauge fields $A_a$ of a simple Lie group H in the Euclidean space $\mathbb{R}^n$.*

Equations (19) have a particular solution

$$\varphi = p_a x_a, \qquad (20)$$

where $p_a$=const. Then, from Proposition 2 it follows that to each solution of the RW equations (18) with $\varphi$ from (20) the plane-wave solution (14) of the YM equations in $\mathbb{R}^n$ may be corresponded, and from Proposition 3 we obtain

**PROPOSITION 4.** *To each nondegenerate solution $T_\alpha(\varphi)$ of the classical YB equations on the Lie algebra $\mathcal{H}$ one may correspond the plane-wave solution (14) with $\varphi = p_a x_a$ of the YM equations (1) for gauge fields $A_a$ of the Lie group H in the Euclidean space $\mathbb{R}^n$.*

In particular, to the trigonometric solution (12) of Eqs.(9) one may correspond the plane-wave solution of Eqs.(1) in the Euclidean space with arbitrary dimension n.

**6. Explicit form of** tensors $J^{\alpha}_{ab}$. To find more complicated than (20) solutions of Eqs.(19), one should give the concrete expressions to the tensors $J^{\alpha}_{ab}$ and $\Sigma^{\alpha\beta}_{ab}$. The theory of Clifford algebras gives the examples of such tensors.

Let us denote by Cl(0,q) the Clifford algebra for the space $\mathbb{R}^q$ with the metric $g_{\alpha\beta} = -\delta_{\alpha\beta}$, $\alpha,\beta,\ldots=1,\ldots,q$. It has been known for a long time that the algebra Cl(0,q) can be realized in terms of matrices. In particular, $Cl(0,6)\cong M(8,\mathbb{R})$ and $Cl(0,8)\cong M(16,\mathbb{R})$ (see, e.g., [17]), where through $M(s,\mathbb{R})$ the full $s\times s$ matrix algebra over $\mathbb{R}$ is denoted. Let us give some examples of tensors $J^{\alpha}_{ab}$.

**Example 1.** Consider the algebra Cl(0,2) with generators $\gamma^1$ and $\gamma^2$. It is well-known [17] that Cl(0,2) is isomorphic to the algebra of quaternions $\mathbb{H}$, and elements $\gamma^1$, $\gamma^2$, $\gamma^3\equiv\gamma^1\gamma^2$ can be realized in terms of real antisymmetric 4×4 matrices $\eta^1$, $\eta^2$, $\eta^3$ with components: $\eta^{\alpha}_{\beta\gamma} = \varepsilon^{\alpha}_{\beta\gamma}$, $\eta^{\alpha}_{\mu 4} = -\eta^{\alpha}_{4\mu} = \delta^{\alpha}_{\mu}$, where $\varepsilon_{\alpha\beta\gamma}$ are structure constants of SU(2), $\alpha,\beta,\gamma,\delta=1,2,3$; $\mu,\nu,\ldots=1,\ldots,4$. Tensors $\eta^1_{\mu\nu}$, $\eta^2_{\mu\nu}$ and $\eta^3_{\mu\nu}$ coincide with the well-known 't Hooft tensors that obey the relations (13) with $\Sigma^{\alpha\beta}_{\mu\nu} = \varepsilon^{\alpha\beta\gamma}\eta^{\gamma}_{\mu\nu}$.

Now, let us introduce the tensors

$$J^{\alpha}_{(\mu i)(\nu j)} = \delta_{ij}\,\eta^{\alpha}_{\mu\nu} \qquad (21)$$



with the double indices $(\mu i)$, $(\nu j)$, ..., where $i,j,\ldots=1,\ldots,p$. If we denote the double indices by $a,b,\ldots=1,\ldots,4p$, then it is not difficult to verify that the tensors $J^{\alpha}_{ab}$ will satisfy the relations (13) with $\Sigma^{\alpha\beta}_{ab}=\varepsilon^{\alpha\beta\gamma}J^{\gamma}_{ab}$. Thus in the spaces $\mathbb{R}^{4p}$ one may always introduce three tensors $J^{\alpha}_{ab}$ satisfying (13).

**Example 2.** Let us consider the algebra Cl(0,6) with generators $\gamma^1,\ldots,\gamma^6$ and also introduce $\gamma^7=\gamma^1\gamma^2\gamma^3\gamma^4\gamma^5\gamma^6$. It is known [17] that $\gamma^{\alpha}$ $(\alpha=1,\ldots,7)$ can be realized in terms of real antisymmetric 8×8 matrices. The components $\gamma^{\alpha}_{\mu\nu}$ $(\mu,\nu,\ldots=1,\ldots,8)$ of these matrices satisfy the relations (13) with $\Sigma^{\alpha\beta}_{\mu\nu} = \frac{1}{2}\gamma^{[\alpha}_{\mu\lambda}\gamma^{\beta]}_{\nu\lambda} \equiv \frac{1}{2}(\gamma^{\alpha}_{\mu\lambda}\gamma^{\beta}_{\nu\lambda} - \gamma^{\beta}_{\mu\lambda}\gamma^{\alpha}_{\nu\lambda})$.

We now introduce the tensors

$$J^{\alpha}_{(\mu i)(\nu j)} = \delta_{ij}\,\gamma^{\alpha}_{\mu\nu}, \qquad (22)$$

where $\mu,\nu,\ldots=1,\ldots,8$; $i,j,\ldots=1,\ldots,p$. Numbering the components of these tensors by the indices $a,b,\ldots=1,\ldots,8p$, in the space $\mathbb{R}^{8p}$ we obtain seven tensors $J^{\alpha}_{ab}$ satisfying (13) with $\Sigma^{\alpha\beta}_{ab} = \frac{1}{2}J^{[\alpha}_{ac}J^{\beta]}_{bc}$. It is clear that for ansatz (14) one can choose not all seven tensors but only q of them with $4\le q\le 7$.

**Example 3.** Let us consider now the algebra Cl(0,8) with generators $\gamma^{\alpha}$, $\alpha,\beta,\ldots=1,\ldots,8$. It ts known [17] that $\gamma^{\alpha}$ can be realized in terms of real antisymmetric 16×16 matrices. The components $\gamma^{\alpha}_{\mu\nu}$ $(\mu,\nu,\ldots=1,\ldots,16)$ of these matrices satisfy (13) with $\Sigma^{\alpha\beta}_{\mu\nu} = \frac{1}{2}\gamma^{[\alpha}_{\mu\lambda}\gamma^{\beta]}_{\nu\lambda}$. Let us also introduce the tensors $J^{\alpha}_{(\mu i)(\nu j)}$ defined by (22) but with $\mu,\nu,\ldots=1,\ldots,8$;

$i,j,\ldots=1,\ldots,p$. Numbering the components of these tensors by the indices $a,b,\ldots=1,\ldots,16p$, we obtain eight tensors $J_{ab}^{\alpha}$. In the space $\mathbb{R}^{16p}$ all these tensors satisfy the relations (13) with $\Sigma_{ab}^{\alpha\beta} = \frac{1}{2} J_{ac}^{[\alpha} J_{bc}^{\beta]}$ and can be used in constructing of the ansatz (14).

And finally, we point out that in the spaces $\mathbb{R}^n$ one may introduce q tensors $J_{ab}^{\alpha}$ satisfying (13) in the following cases:

$$n = p\, 2^{2+4m} \quad \Rightarrow \quad 1+8m \le q \le 3+8m, \tag{23a}$$

$$n = p\, 2^{3+4m} \quad \Rightarrow \quad 4+8m \le q \le 7+8m, \tag{23b}$$

$$n = p\, 2^{4+4m} \quad \Rightarrow \quad q=8+8m, \tag{23c}$$

where $m= 0,1,2,\ldots$; $p=1,2,\ldots$ . Proof may be obtained with the help of formula [17]:

$$Cl(0,s+8m) = Cl(0,s)\otimes \underbrace{Cl(0,8)\otimes\ldots\otimes Cl(0,8)}_{m \text{ times}}, \tag{24}$$

where $1 \le s \le 8$. Using the recurrence relations given in [17], one can easily obtain the explicit form of tensors $J_{ab}^{1},\ldots, J_{ab}^{q}$ in the spaces of dimension n indicated in (23).

**7. Constructing of solutions.** Substituting the explicit form of $J_{ab}^{\alpha}$ into Eqs.(19), one may try to find solutions, different from solutions (20). Such solutions exist. Rather than make an exhaustive study of all the possibilities we shall restrict ourselves to the case n=4p and q=3. The cases of other n from (23) will be considered in a separate paper.

So, let us substitute (21) into (19) where $\varepsilon_{\alpha\beta\gamma}$ are taken instead of $f_{\alpha\beta\gamma}$ and $\Sigma_{ab}^{\alpha\beta}=\varepsilon^{\alpha\beta\gamma}J_{ab}^{\gamma}$. We use the following identities for $\eta_{\mu\nu}^{\alpha}$ [18]:

$$\eta_{\mu\lambda}^{\alpha}\,\eta_{\nu\lambda}^{\beta} = \delta^{\alpha\beta}\,\delta_{\mu\nu} + \varepsilon^{\alpha\beta\gamma}\,\eta_{\mu\nu}^{\gamma}, \tag{25a}$$

$$\varepsilon_{\beta\gamma}^{\alpha}\,\eta_{\mu\lambda}^{\beta}\,\eta_{\nu\sigma}^{\gamma} = \delta_{\mu\nu}\eta_{\lambda\sigma}^{\alpha} - \delta_{\mu\sigma}\eta_{\lambda\nu}^{\alpha} - \delta_{\lambda\nu}\eta_{\mu\sigma}^{\alpha} + \delta_{\lambda\sigma}\eta_{\mu\nu}^{\alpha}, \tag{25b}$$

and obtain the equations:

$$2\,\eta_{\mu\lambda}^{\alpha}\,(\partial_{\lambda i}\partial_{\nu j}\varphi - \partial_{\lambda j}\partial_{\nu i}\varphi) - 2\,\eta_{\nu\lambda}^{\alpha}\,(\partial_{\lambda j}\partial_{\mu i}\varphi - \partial_{\lambda i}\partial_{\mu j}\varphi) +$$
$$+\delta_{\mu\nu}\eta_{\lambda\sigma}^{\alpha}(\partial_{\lambda i}\partial_{\sigma j}\varphi-\partial_{\lambda j}\partial_{\sigma i}\varphi) + \eta_{\mu\nu}^{\alpha}(2\partial_{\lambda i}\partial_{\lambda j}\varphi+\delta_{ij}\Box\varphi)=0, \tag{26}$$

where $\partial_{\lambda i}\equiv\partial/\partial x^{\lambda i}$. It is clear that Eqs.(26) are equivalent to the equations

$$\partial_{\mu i}\partial_{\nu j}\varphi = \partial_{\mu j}\partial_{\nu i}\varphi, \qquad \partial_{\lambda i}\partial_{\lambda j}\varphi = 0, \tag{27}$$

where $\mu,\nu,\ldots=1,\ldots,4$; $i,j,\ldots=1,\ldots,p$.

Eqs.(27) are simpler than Eqs.(19) and appear in study of the hyperkähler manifolds of dimension 4p (see [19]). In principle, for Eqs.(27) one may write a general solution (see [19]), but we shall not do this here. As an example, we write out one of the particular solutions of Eqs.(27) (and (26)):

$$\varphi = 1 + \sum_{I=1}^{N} \frac{B_I^2}{(X_\mu-C_\mu^I)\,(X_\mu-C_\mu^I)}, \tag{28}$$

where $X_\mu=x_{\mu i}p_i$, $p_i$=const, N is any integer number, $B_I$ and $C_\mu^I$ are arbitrary constants. For a special case of the space $\mathbb{R}^8$ and group G=SU(2) the solution of this type was obtained by Ward [1].

Eqs.(18) with q=3 and $\mathcal{H}=su(2)$ coincide with the well-known Nahm equations (see [6,8,9,20]). These equations appeared in constructing the solutions of the YM equations in $\mathbb{R}^4$ [6,15,16] and of the model of chiral fields in $\mathbb{R}^2$ [21]. Nahm's equations have a Lax type representation with a spectral parameter, and in terms of theta functions one can write a general solution of Nahm's equations for any semisimple Lie algebra $\mathcal{G}$ (see [20,9]). The explicit form of particular solutions of Nahm's equations may be found in [15,16]. We shall not write it here.

**8. Conclusion.** An example for n=4p and q=3 shows that Eqs.(19) may have not only solution (20) linear on coordinates $x^a$, but also more complicated solutions. It is interesting to study Eqs.(19) in the spaces $\mathbb{R}^n$ with q tensors $J^{\alpha}_{ab}$ and n>4p from (23) in the case when q coincides with the dimension of some simple Lie algebra $\mathcal{H}$. In this case, as $f_{\alpha\beta\gamma}$ in (18) one may take structure constants of $\mathcal{H}$. We have considered the case of Example 1 when n=4p, q=3 and $\mathcal{H}=su(2)$. If one takes eight tensors $J^{\alpha}_{ab}$ in $\mathbb{R}^{16p}$ from Example 3, then as $f_{\alpha\beta\gamma}$ one may choose the structure constants of the Lie algebra $su(3)$. In particular, from (23c) it follows that in spaces of dimension n=4096p one may introduce 24 tensors $J^{\alpha}_{ab}$ satisfying the relations (13), and as $f_{\alpha\beta\gamma}$ one may take the structure constants of the Lie algebra $su(5)$. All these cases need a special investigation.

Thus, we have shown that the Rouhani-Ward equations and



the classical Yang-Baxter equations appear in constructing the solutions of the Yang-Mills equations in the spaces of dimension greater than four. Our results show strong evidence for detailed study of the integrability of the Rouhani-Ward equations (18) and Eqs.(19) for scalar field $\varphi$.

**References**

1. Ward R.S., *Nucl.Phys.* **B236**, 381 (1984).
2. Fairlie D.B. and Nuyts J., *J.Phys.* **A17**, 2867 (1984); Fubini S. and Nicolai H., *Phys.Lett.* **155B**, 369(1985).
3. Corrigan E., Goddard P. and Kent A., *Commun.Math.Phys.* **100**, 1 (1985).
4. Popov A.D., *Europhys.Lett.* **17**, 23 (1992); Ivanova T.A. Usp.Math.Nauk, in press.
5. Ivanova T.A. and Popov A.D., *Lett.Math.Phys.*, in press.
6. Corrigan E., Wainwright P.R. and Wilson S.M.J., *Commun. Math.Phys.* **98**, 259 (1985).
7. Corrigan E., Devchand C., Fairlie D.B. and Nuyts J., *Nucl. Phys.* **B214**, 452 (1983).
8. Rouhani S., *Phys.Lett.* **104A**, 7 (1984).
9. Ward R.S., *Phys. Lett.* **112A**, 3 (1985).
10. Kulish P.P. and Sklyanin E.K., Trudy LOMI 95, 129 (1980) (in Russian).
11. Belavin A.A., *Funct. Anal. Appl.* **14**, 18 (1980); Belavin A.A. and Drinfeld V.G., *Funct. Anal. Appl.* **16**, 159 (1982).
12. Faddeev L.D. and Takhtajan L.A., *Hamiltonian methods in the theory of solitons,* Springer, Berlin,1987.

13. Stolin A., *Commun.Math.Phys.* 141, 533 (1991).
14. Bourbaki N., *Groupes et algebres de Lie.* Chapitre IV-VI, Hermann, Paris, 1968.
15. Popov A.D., *JETP Lett.* 54, 69 (1991); *Theor.Math.Phys.* 89, 402 (1991).
16. Ivanova T.A. and Popov A.D., *Lett.Math.Phys.* 23, 29 (1991); Ivanova T.A., *Usp.Math.Nauk*, 46, 149 (1991).
17. Karoubi M., *K-Theory*, Springer, Berlin, 1979; Coguereaux R., *Phys.Lett.* 115B, 389 (1982); Salingaros N., *J.Math. Phys.* 22, 226 (1981); 23, 1 (1982); Kugo T. and Townsend P., *Nucl.Phys.* B221, 357 (1983).
18. Prasad M.K., *Physica* D1, 167 (1980).
19. Hitchin N.J., Karlhede A., Lindström U. and Roček M., *Commun.Math.Phys.* 108, 535 (1987); Karlhede A., Lindström U. and Roček M., *Commun Math.Phys.* 108, 529 (1987); Lindström U. and Roček M., *Commun Math.Phys.* 115, 21 (1988); Pedersen H. and Poon Y.S., *Commun.Math.Phys.* 117, 569 (1988).
20. Hitchin N.J., *Commun.Math.Phys.* 89, 145 (1983); Donaldson S.K., *Commun.Math.Phys.* 96, 387 (1984); Hartubise J., *Commun.Math.Phys.* 100, 191 (1985); 120, 613 (1989).
21. Popov A.D., *JETF Lett.* 54, 124 (1991).

Received by Publishing Department
on March 11, 1992.



Иванова Т.А., Попов А.Д. E2-92-107

О связи между уравнениями Янга — Миллса в размерности более четырех и классическими уравнениями Янга — Бакстера

Рассмотрены уравнения Янга — Миллса для калибровочных полей произвольной полупростой группы Ли G в евклидовых пространствах $R^n$ размерности $n \geqslant 4$. Для калибровочных полей $A_\alpha$ введены анзацы, редуцирующие уравнения Янга — Миллса в $R^n$ к системе нелинейных дифференциальных матричных уравнений, связанных с классическими уравнениями Янга — Бакстера. Связь с классическими уравнениями Янга — Бакстера позволяет выписать явный вид космологических, плосковолновых и некоторых других классов решений.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна 1992

Ivanova T.A., Popov A.D. E2-92-107

On a Connection Between the Yang — Mills Equations in Dimension Greater than Four and the Classical Yang — Baxter Equations

We consider the Yang — Mills equations for gauge fields of an arbitrary semisimple Lie group G in Euclidean spaces $R^n$ with dimension $n \geqslant 4$. For gauge fields $A_\alpha$ we introduce the ansätze which reduce the Yang — Mills equations in $R^n$ to a set of nonlinear differential matrix equations closely linked to the classical Yang — Baxter equations. The connection with the classical Yang — Baxter equations permits us to write out the explicit form of cosmological, plane-wave and other types of solutions.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna 1992