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# CANONICAL REALIZATIONS OF THE LIE ALGEBRA sp(2n,R)

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# ON THE "NEAR TO MINIMAL" CANONICAL REALIZATIONS OF THE LIE ALGEBRA $C_n$

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## *Introduction*

The object of this paper is to present a large class of realizations of the Lie algebra of the real symplectic group in the Weyl algebra, i.e., through polynomials in quantum canonical variables $q_i$ , $p_i$ with various good properties.

For physical relevance of canonical realizations of Lie algebras in general we refer to the review articles /1/ and /2/ and the references therein. As to the symplectic group, we remember only that it occurs in physics as a subgroup of general canonical transformations, namely, of the group $ISp(2n, R)$ of inhomogeneous linear transformations which leave the commutation relations of $n$ canonical pairs $[p_i, q_j] = \delta_{ij} z$ , $[q_i, q_j] = [p_i, p_j] = 0$, $i, j = 1, 2, \ldots, n$ invariant /2, 3/. . The Lie algebra $sp(2n,R)$ is the dynamical algebra of the $n$-dimensional harmonical oscillator /1/.

The proposed canonical realizations have common features with those of real forms of the other classical Lie algebras $A_n$, $B_n$, $D_n$ presented in /4/ and /5/. The realizations are recurrently defined by means of $2n-1$ canonical pairs and a canonical realization of the algebra $sp(2n-2, R)$ with one free real parameter. Using, for realization of the auxiliary Lie algebra $sp(2n-2,R)$ ,either the trivial one or the realization defined by the same formulas e.t.c. we obtain a set of realizations $sp(2n,R)$ . Realizations of this set are in one-to-one correspondence with the sequences $(d; 0, \ldots, 0, a_{n-d+1}, \ldots, a_n)$, $d = 1, 2, \ldots, n$, $a_i \in R$; these sequences we call signatures. The generators of $sp(2n, R)$ in a realization with signature $(d; 0, \ldots, 0, a_{n-d+1}, \ldots, a_n)$ lie in the Weyl algebra $W_{2N(d)}$, where $N(d) = d(2n-d)$, i.e., they are polynomials in $N(d)$ canonical pairs. All realizations are Schur realizations which means that every Casimir operator is realized by complex multiple of the identity element and all realizations are skew-hermitean with respect to an involution defined on the Weyl algebra. Two realizations characterized by different signatures cannot be transformed from one to another by means of endomorphisms of the Weyl algebra.

The number $N(d) = d(2n-d)$ of pairs used in the construction of the realizations with signature $(d; 0, \dots, 0, \alpha_{n-d+1}, \dots, \alpha_n)$ is the smallest for $d=1$ when $N(1) = 2n-1$. Of course, this is not the minimal number of canonical pairs which allows a faithful realization of $sp(2n, R)$. The well-known minimal realization $\tau_1$ of $sp(2n, R)$ is given by the following expressions

$$q_i p_j + \frac{1}{2}\delta_{ij}, \quad i q_i q_j, \quad i p_i p_j, \quad i,j = 1, \dots, n, \tag{1}$$

where canonical pairs are used. On the basis of JOSEPH's result( /7/ , Lemma 1) it could be proved for $n \geq 2$ that in any realization $\tau$ of $sp(2n,R)$ in the quotient division ring $D_{2(2n-2)}$ of $W_{2(2n-2)}$ (i.e., by means of rational functions in $2n-2$ canonical pairs)

$$\tau(z) = \tau_1(z) = \lambda_z, \quad \lambda_z \in \mathbf{C}$$

holds for any Casimir operator $z$ of $sp(2n,R)$ /12/.

So, the possibility to obtain realizations of $sp(2n,R)$ in which Casimir operators are realized by expressions other than in realization $\tau_1$ would appear only in $W_{2N}$ or $D_{2N}$ with $N \geq 2n-1$. The mentioned one-parameter set of realizations with signatures $(1; 0, \dots, 0, \alpha_n)$ in $W_{2(2n-1)}$ shows that $N$ equals just $2n-1$ and that canonical realizations are given by polynomials. Further, in these realizations, e.g., the quadratic Casimir operator $C^{(2)}$ depends on the parameter $\alpha_n$, $\tau(C^{(2)}) = -2(\alpha_n^2 + n^2)$, whereas for realization $\tau_1$ one finds $\tau_1(C^{(2)}) = -n^2 - \frac{1}{2}n$.

The fact that these realizations are still Schur-realizations is not accidental as it could be proved that in



$W_{2(2n-1)}$ any realization of $sp(2n,R)$ is Schur-realization /12/.

In the Conclusion we show how this "minimal" one-parameter set of realization of $sp(2n,R)$ can be obtained by means of the one-parameter set of minimal realizations of $gl(2n,R)$ given in our paper /5/. We discuss a formula very useful to construct canonical realizations of any finite dimensional Lie algebra.

Some considerations determine, for any compact classical Lie algebra, the minimal of canonical pairs needed for skew-hermitean realizations.

### *Preliminaries*

In the Lie algebra of the symplectic group, i.e. the group of linear transformations of the $2n$-dimensional vector space which left invariant the bilinear form

$$\sum_{i=1}^{n} (x^i y^{-i} - x^{-i} y^i) \tag{2}$$

we choose a basis consisting of $n(2n+1)$ generators $X^\alpha_\beta = -\epsilon_\alpha \epsilon_\beta X^{-\beta}_{-\alpha}$ $\alpha, \beta = -n, \dots, -1, 1, \dots, n$ satisfying the commutation rules

$$[X^\alpha_\beta X^\gamma_\delta] = \delta^\gamma_\beta X^\alpha_\delta - \delta^\alpha_\delta X^\gamma_\beta + \epsilon_\alpha \epsilon_\beta \delta^{-\beta}_\delta X^\gamma_{-\alpha} + \epsilon_\beta \epsilon_\gamma \delta^\gamma_{-\alpha} X^{-\beta}_\delta, \tag{3}$$

$$\epsilon_\alpha = \operatorname{sgn} \alpha .$$

A $2n \times 2n$ -matrix representation is

$$(X^\alpha_\beta)_{\gamma\delta} = \delta_{\gamma\alpha}\delta_{\beta\delta} - \epsilon_\alpha \epsilon_\beta \delta_{-\alpha\delta}\delta_{-\beta\gamma} . \tag{4}$$

A ***canonical realization*** of a Lie algebra $L$ is a homomorphism of $L$ in the ***Weyl algebra*** $W_{2N}$, the associative algebra over $\mathbf{C}$ with identity generated by $2N$ elements $q_i, p_i$, $i = 1, 2, \dots, N$ with commutation relations

$$[p_i, q_j] = \delta_{ij} \mathbf{1} .$$

The homomorphism $\tau$ extends naturally to a homomorphism (denoted by the same symbol $\tau$) of the enveloping algebra $\mathcal{U}L$ of $L$ into $W_{2N}$.

If in a realization of $L$ every Casimir operator, i.e., every element from the center of the enveloping algebra of $L$, is realized by a multiple of the identity element, then the realization is called to be a *Schur-realization.* Two realizations $\tau$ and $\tau'$ of $L$ in $W_{2N}$ are called to be *related* if an endomorphism $\theta$ of $W_{2N}$ exists such that either $\theta\cdot\tau = \tau'$ or $\theta\cdot\tau' = \tau$.

In $W_{2N}$ we define an *involution* induced by

$$
\begin{aligned}
p_i^+ &= -p_i\,, \\
q_i^+ &= q_i\,.
\end{aligned}
\tag{5}
$$

A canonical realization $\tau$ of the real Lie algebra $L$ is called *skew-hermitean* iff $\tau(x)^+ = -\tau(x)$ for all $x \in L$.

*Canonical Realization of* $sp(2n, R)$

*Theorem 1:* Let $Z^i_j$ be a canonical realization of $sp(2n-2, R)$ in $W_{2m}$. Then the generators

$$
\begin{aligned}
X^i_j &= q_i p_j - \epsilon_i \epsilon_j q_{-j} p_{-i} + Z^i_j\,, \\
X^j_n &= q_j(q\cdot p + n - i\alpha) - \epsilon_j q_0 p_{-j} + Z^j_k q_k\,, \\
X^n_j &= -p_j - \epsilon_j q_{-j} p_0\,, \\
X^n_{-n} &= -2p_0\,, \\
X^{-n}_n &= 2q_0(q\cdot p + n - i\alpha) + \epsilon_\ell Z^{-\ell}_k q_\ell q_k\,, \\
X^n_n &= -q_0 p_0 - q\cdot p - (n - i\alpha)\cdot 1\,,
\end{aligned}
\tag{6}
$$

$$i,j,k,\ell = -(n-1),\dots,-1,1,\dots,n-1\,,$$

where $\alpha \in \mathbb{C}$ and $q\cdot p = q_0 p_0 + q_k p_k$, define a realization of $sp(2n, R)$ in $W_{2(2n-1+m)}$. This realization has the following properties:

- i The realization is skew-hermitean if $\alpha$ is real and if $Z^i_j$ is skew-hermitean.
- ii The realization is a Schur-realization if $Z^i_j$ is a Schur-realization.
- iii Two realizations (6) with different parameters are non-related.
- iv Two realizations (6) differing only in the realizations of $sp(2n-2, R)$ are related if and only if these realizations of $sp(2n-2, R)$ are related.

Proof: The verification that the generators (6) fulfil the commutation relations (3) of $sp(2n, R)$ and that they are skew-hermitean under the involution defined by (5) is straightforward and will be omitted here.

In the proof of (ii)-(iv) we use two assertions which are easily probable using the relation $[qp, q^k p^s] = (k-s)\, q^k p^s$ for each canonical pair occuring in $W_{2N}$.

*Assertion 1:* If $x \in W_{2N}$ commutes with $p_i$ (resp. $q_i$) then $x$ does not depend on $q_i$ (resp. $p_i$).

*Assertion 2:* Assume that for $x \in W_{2N}$ there holds $[q_1 p_1 + \dots + q_{N'} p_{N'}, x] = m\cdot x$ for some $m = 0, \pm 1, \pm 2, \dots$ where $N' \leq N$. Then

$$x = \sum_{\substack{k,\ell \\ k-\ell = m}} \alpha_{k\ell}\cdot q^k p^\ell\,,$$

where $\alpha_{k\ell} q^k p^\ell \equiv \alpha_{k_1\dots k_{N'}\ell_1\dots\ell_{N'}} \cdot q_1^{k_1}\dots q_{N'}^{k_{N'}} p_1^{\ell_1}\dots p_{N'}^{\ell_{N'}}$, $k - \ell \equiv k_1 + \dots + k_{N'} - \ell_1 - \dots - \ell_{N'}$ and $\alpha_{k\ell}$ do not depend on $q_1, \dots, q_{N'}, p_1, \dots, p_{N'}$.

(ii) Let $Y$ be an element from the center of the enveloping algebra of $sp(2n, R)$ in its realization induced by (6). By definition $Y$ commutes with all generators of $sp(2n, R)$. First, we consider the consequence of this fact using only generators which do not depend on $Z^i_j$:

$$[Y, X^n_{-n}] = 0, \text{ i.e., } [Y, p_0] = 0\,, \tag{7}$$

$$[Y, X^{n}_{-j}] = 0, \text{ i.e., } [Y, p_j] + \epsilon_j [Y, q_{-j}\, p_0] = 0, \tag{8}$$

$$[Y, X^{n}_{n}] = 0, \text{ i.e., } [Y, 2q_0p_0 + q_jp_j] = 0 . \tag{9}$$

From (7) it follows, due to Assertion 1, that $Y$ does not depend on $q_0$. Therefore $Y$ can be written in the form

$$Y = \sum_r \gamma_r p_0^r, \tag{10}$$

with

$$\gamma_r = \sum_{k,\ell} \alpha^r_{k\ell} q^k p^\ell ,$$

where $\alpha^r_{k\ell}$ are polynomials in $Z^i_j$. We show that only $\alpha^0_{00}$ can differ from zero. For $\gamma_r$ relation (9) gives

$$[q_jp_j, \gamma_r] = 2r\gamma_r , \quad r = 0, 1, \dots . \tag{9'}$$

Taking (8) for the 0-th power in $p_0$ we obtain further

$$[\gamma_0, p_j] = 0, \; j = -(n-1), \dots, -1, 1, \dots, n-1$$

and, due to Assertion 1, $\gamma_0$ does not depend on $q_i$,

$$\gamma_0 = \sum_\ell \alpha^0_{0\ell} p^\ell .$$

Equation (9') in combination with Assertion 2 leads immediately to

$$\gamma_0 = \alpha^0_{00}(z^i_j)$$

since the condition $k - \ell = -\ell_{-(n-1)} - \dots - \ell_{n-1} = 0$ necessary for $\alpha^0_{0\ell}$ to be nonzero is fulfilled only for $\ell_{-(n-1)} = \dots = \ell_{n-1} = 0$. If we take (8) for the first power in $p_0$ we obtain

$$[\gamma_1, p_j] + \epsilon_j [\gamma_0, q_{-j}] = 0 ,$$

which, due to Assertion 1, implies

$$\gamma_1 = \sum_\ell \alpha^1_{0\ell} p^\ell . \tag{10 }$$



Equation (9') and Assertion 2 give, for the difference $k - \ell = -\ell_{-(n-1)} - \dots - \ell_{n-1}$ the condition $\ell_{-(n-1)} + \dots + \ell_{n-1} = -2$, which cannot be fulfilled for non-negative integers $\ell$'s . Consequently all the coefficients $\alpha^1_{0\ell}$ in (10) must be zero, i.e.,

$$\gamma_1 = 0.$$

Putting $\gamma_1$ in (8) taken for the 2-nd power in $p_0$ we get by the same arguments $\gamma_2 = 0$ and so on. Thus we can show that $Y = \alpha^0_{00}$ is a polynomial in the generators $Z^i_j$ of $sp(2n-2, R)$ only. Hence from the condition that $Y$ commutes with the remaining three types of generators $X^i_j$ , $X^i_n$, $X^{-n}_n$ which contain the generators $Z^i_j$ follows that $Y = \lambda \cdot 1, \lambda \in C$ if $Z^i_j$ form a Schur-realization and (ii) is proved.

In the proof of (iii) and (iv) we use very similar arguments. Let $\theta$ be an endomorphism of the Weyl algebra $W_{2(2n-1+m)}$ which connects two realizations (6)

$$\theta(\tilde{\tilde{X}}{}^\alpha_\beta) = X^\alpha_\beta ; \; \alpha, \beta = -n, \dots, -1, 1, \dots, n, \tag{11}$$

where the realization $\tilde{\tilde{X}}{}^\alpha_\beta$ depends on $\tilde{\tilde{\alpha}}$ and $\tilde{\tilde{Z}}{}^i_j$ and the realization $X^\alpha_\beta$ depends on $\alpha$ and $Z^i_j$.
From the equations (11) we use first only three types

$$\theta(\tilde{\tilde{X}}{}^n_{-n}) = X^n_{-n} \text{ i.e., } \theta(p_0) = p_0, \tag{12}$$

$$\theta(\tilde{\tilde{X}}{}^n_j) = X^n_j \text{ i.e., } \theta(p_j + \epsilon_j q_{-j} p_0) = p_j + \epsilon_j q_{-j} p_0 , \tag{13}$$

$$\theta(\tilde{\tilde{X}}{}^n_n) = X^n_n \text{ i.e., } \theta(q_0p_0 + q \cdot p + \tilde{\tilde{\alpha}} 1) = q_0 p_0 + q \cdot p + \alpha \cdot 1 . \tag{14}$$

Now we turn to $\theta(q_i)$ and determine it from its commutation realizations with (12)-(14). Because of (12)

$$[\theta(q_i) - q_i, p_0] = 0$$

holds which, due to Assertion 1, implies that $(\theta(q_i) - q_i)$ does not depend on $q_0$. We denote this element again by $Y$ and we can write

$$Y = \theta(q_i) - q_i = \sum_r \gamma_r \cdot p^r_0$$

with

$$\gamma_r = \sum_{k,\ell} \alpha^r_{k\ell} \cdot q^k p^\ell,$$

where $\alpha^r_{k\ell}$ are polynomials in $Z^i_j$. From equation (13) we obtain

$$[Y, p_j + c_j q_{-j} p_0] = 0 \tag{13'}$$

and from (14) it follows that

$$[q_k p_k, \gamma_r] = (2r+1)\,\gamma_r\,. \tag{14'}$$

A comparison with (8) and (9') shows that we have the same problem as in the proof of (ii). The only difference is the factor $(2r+1)$ in the r.h.s. of (14') instead of $2r$ in (9'). Using the same argument as in the proof of (ii) one finds $Y = 0$ since with the factor $(2r+1)$ instead of $2r$ the necessary condition for $\alpha^r_{k\ell} \neq 0$ reads $k - l = 2r+1$ which cannot be fulfilled even for $\alpha^0_{00}$. So we get $\theta(q_i) = q_i$ and it follows then immediately from (12) and (13) that $\theta(p_i) = p_i$. Therefore (14) turns into

$$(\theta(q_0) - q_0)\,p_0 = (\alpha - \tilde{\tilde{\alpha}}) \cdot 1, \tag{15}$$

which in the Weyl algebra, where negative powers in $p_0$ do not occur, can be fulfilled only if $\tilde{\tilde{\alpha}} = \alpha$. This proves (iii). From (15) we get further that $\theta(q_0) = q_0$ because of the absence of nonzero zero divisors in the Weyl algebra.

So, we assume $\tilde{\tilde{\alpha}} = \alpha$ and show (iv) as follows. Since the canonical pairs from the subalgebra $W_{2m}$ commute with the remaining $2n-1$ canonical pairs $W_{2(2n-1+m)}$, $\theta(a)$ for $a \in W_{2m}$ cannot depend on these remaining $2n-1$ pairs because of Assertion 1. Thus $\theta$ restricted to $W_{2m}$ must be an endomorphism $\hat{\theta}$ of $W_{2m}$. Therefore the relations (11) $\theta(\tilde{\tilde{X}}{}^\alpha_\beta) = X^\alpha_\beta$ taken for $X^i_j$ imply

$$\hat{\theta}(\tilde{\tilde{Z}}{}^i_j) = Z^i_j\,. \tag{16}$$



On the other hand if $\tilde{\tilde{Z}}{}^i_j$ and $Z^i_j$ are related, i.e., there exists an endomorphism $\hat{\theta}$ of $W_{2m}$ such that (16) holds, then the identical extension of $\hat{\theta}$ to an endomorphism $\theta$ of $W_{2(2n-1+m)}$ yields (11), i.e., $X^\alpha_\beta$ and $\tilde{\tilde{X}}{}^\alpha_\beta$ are related, the proof is completed.

The obviously inducing character of Theorem 1 gives rise to the construction of $d$-parameter sets of canonical realizations of $sp(2n, R)$. For this purpose let us define "signatures" as the $(n+1)$-tuples $(d; 0, \dots, 0, \alpha_{n-d+1}, \dots, \alpha_n)$, where $d = 1, 2, \dots, n$ and $\alpha_i$ are real numbers.

*Theorem 2:* To every signature $(d; 0, \dots, 0, \alpha_{n-d+1}, \dots, \alpha_n)$ there corresponds a canonical realization of $sp(2n, R)$ in $W_{2N(d)}$, $N(d) = d(2n-d)$ defined as follows

a) $(1; 0, \dots, 0, \alpha_n)$ denotes the realization (6) of $sp(2n, R)$, where $\alpha = \alpha_n$ and $Z^i_j = 0$.

b) $(d; 0, \dots, 0, \alpha_{n-d+1}, \dots, \alpha_n)$ $d > 1$ denotes the realization (6) where $\alpha = \alpha_n$ and the realization of $sp(2n-2, R)$ has the signature $(d; 0, \dots, 0, \alpha_{n-d+1}, \dots, \alpha_{n-1})$. For these realizations it holds that:

i The realizations are skew-hermitean.

ii The realizations are Schur-realizations.

iii Two realizations are related if and only if their sugnatures are the same.

Theorem 2 follows immediately from Theorem 1. The number of canonical pairs is

$$\sum_{k=n-d+1}^{n} (2k-1) = d(2n-d)\,.$$

*Remark:* For property (iii) it is of course assumed that $W_{2N(d)}$ is naturally embedded in $W_{2N(d')}$ if $d < d'$.

## *Concluding Remarks*

A. If we consider the generators $X^\alpha_\beta$ given by (6) as the basis of a linear space over $C$, i.e., if we replace

$sp(2n,R)$ by its complexification $sp(2n,C)$, then all assertions of Theorems 1 and 2 are also true with exception of skew-hermiticity, which has no sense for complex Lie algebras. The parameters $a_i$, $i = n-d+1,\dots,n$, then can be taken from C since the restriction to real parameters was caused only by skew-hermiticity and other parts of the proof do not depend on this restriction. So, together with the results from /4/ and /5/ series of canonical realizations with the same properties described here for $sp(2n,C)$ are given for all the four fundamental series $A_n$, $B_n$, $C_n$, $D_n$, of the Cartan classification of complex simple Lie algebras.

B. A very well-known method of getting, for an arbitrary Lie algebra L with a basis $x^1,\dots,x^n$, a canonical realization which is bilinear in $q_i$ and $p_i$ starts with a $N\times N$ -matrix representation $X^\alpha = (X^\alpha_{ij})$ of the generators $x^\alpha$ of L and uses the formula

$$\tau(x^\alpha) \equiv \tilde{\tilde{X}}^\alpha = \sum_{i,j=1}^{N} q_i X^\alpha_{ij} p_j . \qquad (17)$$

Formula (17) already used by SCHWINGER /8/ in 1952 for the Lie algebra $su(2)$ was generalized and used for Lie algebras of non-compact groups $(U(6,6))$ in hadron classification by DOTHAN, GELL-MANN, NE'EMAN /9/ with the help of HERMANN and FEYNMAN (see related remarks in /9/ ). As is pointed out by CORDERO and GHIRARDI in the review article /1/ some properties of formula (17) restric strongly its generality. So, the number N of canonical pairs depends on the existence of a $N\times N$ -matrix representation of the Lie algebra L . Further, the realizations by bilinear expressions in $q_i$ and $p_i$ give only a small subset of all possible canonical realizations and the minimal realizations are usually not of this type.

It is possible to generalize formula (17): The commutation relations among $\tau(X^\alpha)$ will be conserved if we substitute $q_i p_j$ by $E_{ij}$ satisfying the commutation relations of the Lie algebra $g\ell(N)$, , i.e., if

$$[E_{ij}, E_{k\ell}] = \delta_{jk} E_{i\ell} - \delta_{\ell i} E_{kj}, \qquad /18/$$

then



$$\tau(x^\alpha) = \sum_{i,j=1}^{N} X^\alpha_{ij} E_{ij} \qquad (19)$$

fulfil the commutation relations of the algebra L.

Whereas in the literature for $E_{ij}$ there were often used $q_i p_j$ or, because of skew-hermiticity $q_i p_j + \frac{1}{2}\delta_{ij}$ we shall stress the possibility to take other realizations of $g\ell(N)$ for $E_{ij}$ . So we can use, e.g., the canonical realizations of $g\ell(N)$ given in /5/ and among them the one-parameter set of minimal realizations in $W_{2(N-1)}$ * which reduce the number of canonical pairs in comparison to those used in (17).

The canonical realizations of $sp(2n,R)$ (6) with $Z^i_j = 0$ can be got by (19) using just the mentioned minimal one-parameter set of realizations of $g\ell(2n,R)$ and the matrix representation (4) of the generators of $sp(2n,R)$ .

As the matrix representation (4) of $sp(2n,R)$ is real and the realization of $g\ell(2n,R)$ is skew-hermitean, the realization of $sp(2n,R)$ defined by (19) is skew-hermitean, too.

C. To take a representation of L by real matrices is not, of course, the only possibility by means of (19) to get a skew-hermitean realization of L . The suitable chioce of a representation of L by $N\times N$ -dimensional complex matrices in combination with a suitable non-skew-hermitean realization of $g\ell(N,R)$ can lead also to a skew-hermitean realization of L. If we are interested, at the same time, in the realizations with the smallest number of canonical pairs, we have to use a matrix-representation with the smallest dimension N. For example, taking the fundamental n -dimensional ( 2n -dimensional) skew-her-

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* $E_{\mu\nu} = q_\mu p_\nu + \frac{1}{2}\delta_{\mu\nu}$, $E_{N\mu} = -p_\mu$, $E_{\mu N} = q_\mu (q_\nu p_\nu + \frac{N}{2} - i\alpha)$, $E_{NN} = -q_\nu p_\nu - \frac{N-1}{2} + i\alpha$, $\mu,\nu = 1,\dots,N-1$, $\alpha \in R$ (eq.(11) in /5/ ).

These realizations are skew-hermitean and they are Schur-realizations.

mitean representation of $L = su(n)\,(=sp(2n)$ *)and the realization of $g\ell(n,R)$ $(g\ell(2n,R))$ given by

$$E_{ij} = \frac{1}{2}(p_i p_j + q_i p_j - q_j p_i - q_i q_j)$$

for which $E^+_{ij} = E_{ji}$ holds, we obtain a skew-hermitean realization of $su(n)\,(sp(2n))$ in $W_{2n}\,(W_{2,2n})$.

These realizations are *minimal skew-hermitean* realizations of the Lie algebras $su(n)$ and $sp(2n)$, i.e. realizations with the smallest number of canonical pairs among all skew-hermitean realizations. The following considerations show that in $W_{2(n-1)}\, W_{2(2n-1)}$ no skew-hermitean realization of $su(n)\,(sp(2n))$ exists. Due to JOSEPH /10/ ,(Th 4.4) no skew-hermitean nontrivial realization of a compact Lie algebra is a Schur-realization. It was shown, however, that all realizations of the Lie algebra $A_{n-1}$ in $W_{2(n-1)}$ are Schur-realizations /10,11/. Consequently, the same assertion takes place for any real form of $A_{n-1}$ including $su(n)$ and therefore a skew-hermitean realization of $su(n)$ in $W_{2(n-1)}$ does not exist. The same assertion is valid for $sp(2n)$ in $W_{2(2n-1)}$ because as we mentioned in the Introduction, any realization of $sp(2n,R)$ in $W_{2(2n-1)}$ is a Schur-realization and $sp(2n)$ is another real form of the common complexification $sp(2n,C)$.

Since, it was proved in /6/ that minimal skew-hermitean canonical realizations of the Lie algebra $o(n)$ exist in $W_{2(n-1)}$, the problem of the existence of these realizations is solved completely for all compact classical Lie algebras.

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*The compact form of the algebra $C_n$ from the Cartan classification



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