E2 - 9128

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Submitted to ТМФ

ОБЪЕДИНЕННЫЙ
ИНСТИТУТ
ЯДЕРНЫХ
ИССЛЕДОВАНИЙ

ДУБНА

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1975

1. *Introduction*

By analogy with the geometrical limit of optics the high-energy potential scattering theory can be constructed due to existence of a scatterer characteristic scale. We shall assume the scattering of a particle with the rest mass m and the energy $E = k^2 (\hbar = 2m = 1)$ occurs on a potential $V(\vec r)$ with the effective range of interaction a and amplitude V_0.

The eikonal approximation for the potential scattering amplitude /1-3/,

$$f(\vec k_1, \vec k_2) = (-4\pi)^{-1} \int d\vec r\, e^{i\vec q\vec b} \cdot V(\vec r) \cdot \exp[\, i\chi_k(\vec r, \vec q)\,] \, , \qquad (1)$$

where the eikonal phase χ_k is

$$\chi_k(\vec r, \vec q) = -\frac{1}{2k} \cdot \int_{-\infty}^{z} dz' \cdot V(\vec b, z') \, , \qquad (2)$$

is the most popular one at present.

The z-direction in (2) is chosen parallel to the average momentum $\vec k = \frac{1}{2}(\vec k_1 + \vec k_2)$. In this coordinate system the longitudinal components of the incident $\vec k_1$ and the final $\vec k_2$ momenta are equal to $t_0 = k \cdot \cos\frac{1}{2}\theta$ (θ is the scattering angle) and their transversal components are: $\vec k_{1\perp} = -\vec k_{2\perp} = \frac{1}{2}\vec q \equiv \vec s_0$, $s_0 = k \cdot \sin\frac{1}{2}\theta$, where $\vec q = \vec k_1 - \vec k_2$. To accentuate this choice of reference frame the notation $\chi(\vec r, \vec k_1) \equiv \chi_k(\vec r, \vec q)$ in (1) is used.

The eikonal approximation is extensively discussed in the nonrelativistic quantum mechanics /4-10/ and in quantum-field theory /11-1[illegible]/. The well-founded result is its validity for the high-energy scattering on a smooth potential and for small angles $\theta \lesssim (ka)^{-1/2}$. In this connection the problem of the generalisation of (1,2) to large momentum transfers arises.

The eikonal amplitude (1,2) permits [3] a simple representation in the form of a two-dimensional Fourier transform (an impact parameter representation). But one has to use the representation (1) if one wants to carry out a generalisation of the eikonal approximation to large angles region on the basis of a potential smoothness. Such a consideration was first given by Schiff [4]. His approximation has been extensively used in analysis of electron-nucleon collisions [5,16]. The Schiff formula is well-founded for the so-called dynamically large scattering angles $\theta \gg (ka)^{-1/2}$. Its refinement has been made in papers [14]. The discussion and comparisons of various approximations for the scattering amplitude at large angles was given in survey [6]. The need is pointed out for a further investigation of the large-momentum-transfer region.

The present work develops a new approach to the high-energy potential theory which is based on the natural generalisation of the phase function method [17-20]. In Sect. 2 we discuss basic equations of the theory: an equation for the potential scattering amplitude and an equation for the so-called total phase function. In Sect. 3 an extension of the eikonal approximation to the large-angle region is derived. This approximation for the scattering amplitude is compared with well-known ones. Section 4 contains the systematic method for calculation of non-linear in potential corrections. Their role for large scattering angles is discussed. In sect. 5 the main results are summarized. Some possibilities of the present approach in potential theory and its relativistic generalisation are pointed out.

2. *An Equation for the Total Phase Function*

In the spirit of the phase method [17] the elastic scattering amplitude $f(\vec{k}_1, \vec{k}_2)$ can be considered as the asymptotic limit of the so-called scattering function

$f_S(\vec{k}_1, \vec{k}_2)$. This function is the scattering amplitude for a part of the potential $V_S(\vec{r})$ contained in a surface S. The scattering amplitude $f(\vec{k}_1, \vec{k}_2)$ is the amplitude $f(\vec{k}_1, \vec{k}_2)$ when the surface S cuts the potential at infinity where $V(\vec{r}) \to 0$. This approach was suggested in [18], where an integral equation for the scattering function $f(R, \vec{k}_1, \vec{k}_2)$ was obtained. $f(R, \vec{k}_1, \vec{k}_2)$ is the scattering amplitude for a part of the potential $V_R(\vec{r})$ contained in a sphere of radius R.

The eikonal approximation can be most naturally investigated in this approach on the basis of an equation for the scattering amplitude $f(\xi, \vec{k}_1, \vec{k}_2)$ for the potential

$$V_\xi(\vec{r}) = V(\vec{r}) \cdot \theta(\xi - |z|) , \tag{3}$$

where $\theta(x)$ is the ordinary θ-function.

Thus (3) is a part of the potential $V(\vec{r})$ contained between two planes. These planes are perpendicular to some straight line, which we choose as the z-axis, and are symmetrically at distances $\pm\xi$ from the origin.

The exact nonlinear integral equation for the scattering amplitude $f(\xi, \vec{k}_1, \vec{k}_2)$ can be derived [19,20] in the form:

$$\frac{\partial}{\partial \xi} f(\xi, \vec{k}_1, \vec{k}_2) = -\frac{1}{2\pi} \cdot \int d^2\vec{b} \cdot V(r) \times$$
$$\times \{ e^{i\vec{k}_1\vec{r}} + \frac{i}{2\pi} \cdot \int d^2\vec{s} \cdot \frac{e^{i\vec{p}\vec{r}}}{t} \cdot f(\xi, \vec{p}, \vec{k}_1) \} \times \tag{4}$$
$$\times \{ e^{-i\vec{k}_2\vec{r}} + \frac{i}{2\pi} \int d^2\vec{s} \cdot \frac{e^{i\vec{p}\vec{r}}}{t} \cdot f(\xi, \vec{p}, -\vec{k}_2) \} ,$$

with the boundary condition

$$f(0, \vec{k}_1, \vec{k}_2) = 0 . \tag{5}$$

In eq. (4): $r = (\vec{b}, \xi)$, $p = (\vec{s}, t)$, where $\vec{s}$ and t are connected as

$$t = \begin{cases} \sqrt{k^2 - s^2}, & k > s, \\ i\sqrt{s^2 - k^2}, & k < s. \end{cases} \tag{6}$$

It was suggested in the derivation of equation (4) that the potential is spherically symmetric and the z-axis is parallel to the average momentum $\vec{k}$. With this choice eq. (4) has the most simple form [20]. The asymptotic limit $f(\infty, \vec{k}_1, \vec{k}_2)$ is the scattering amplitude for the total potential $V(r)$.

It is well known that the scattering amplitude $f(\vec{k}_1, \vec{k}_2)$ can be evaluated if the wave function of the scattered particle is known only within the range of the potential. In the present approach one can avoid the necessity of finding out the wave function in the whole space in the following way. Let us introduce a new function $\chi_k(\xi, \vec{r}, \vec{q})$ which is defined only within the potential range $V_\xi(r)$. We represent scattering amplitude in the form (compare (1))

$$f(\xi, \vec{k}_1, \vec{k}_2) = (-4\pi)^{-1} \cdot \int d\vec{r} \cdot e^{i\vec{b}\vec{q}} \cdot V_\xi(r) \cdot \exp[\, i\chi_k(\xi, \vec{r}, \vec{q})\,] \,. \tag{7}$$

For an unambiguous determination of the function $\chi_k(\xi, \vec{r}, \vec{q})$ we put the following boundary condition

$$\chi_k(\xi, \vec{r}, \vec{q})\,\big|_{z = \pm\xi} = 0 \,. \tag{8}$$

The boundary condition (8) makes it possible [20] to eliminate, from the consideration, the well-known Born approximation.

The quantity $\chi_k(\xi, \vec{r}, \vec{q})$ with the boundary condition (8) by analogy with (2), can be treated as the exact phase shift of the wave function ψ_ξ for the potential V_ξ. The asymptotic limit $\chi_k(\infty, \vec{r}, \vec{q})$ is the phase shift of the wave function ψ for the total potential $V(r)$. Hence the function $\chi_k(\xi, \vec{r}, \vec{q})$, in accordance with the phase method [17], can be called the total phase function.

By substituting (7) into (4) an exact non-linear integro-differential equation for the total phase function can be obtained [20]. This equation with the boundary condition (8) can be reduced to one integral equation,

$$\chi_k(\xi, \vec{r}, \vec{q}) = \phi_k(\xi, \vec{r}, \vec{q}) + \int_{|z|}^{\xi} d\eta \cdot \int d\vec{r}\,' \cdot w_2(\eta, k, \vec{r}, \vec{r}\,', \vec{q}) \times$$

$$\times \exp[\, i\chi_k(\eta, \vec{r}, \vec{q})\,] \cdot V_\eta(r) \,, \tag{9}$$

where

$$\phi_k(\xi, \vec{r}, \vec{q}) = \int_{|z|}^{\xi} d\eta \cdot w_1(\eta, k, \vec{r}, \vec{q}) \,. \tag{10}$$

The functions w_1 and w_2 in eqs. (9-10) are

$$w_1(\eta, k, \vec{r}, \vec{q}) = w_1^{(+)}(\eta, k, \vec{r}, \vec{s}_0) + w_1^{(-)}(\eta, k, \vec{r}, \vec{s}_0) \,, \quad \vec{s}_0 = \vec{q}/2 \,, \tag{11}$$

and

$$w_2(\eta, k, \vec{r}, \vec{r}\,', \vec{q}) = -\frac{1}{2(2\pi)^2} \cdot \int d^2\vec{s} \cdot \frac{e^{-i(\vec{b} - \vec{b}\,')(\vec{s} + \vec{s}_0) - i(z + z')(t - t_0)}}{t} \times$$

$$\times\, w_1^{(+)}(\eta, k, \vec{r}\,', \vec{s}) \,, \tag{12}$$

where

$$w_1^{(\pm)}(\eta, k, \vec{r}, \vec{s}) = -\frac{1}{(2\pi)^2} \cdot \int d^2\vec{s}\,' \cdot \frac{e^{i\vec{b}(\vec{s}\,' - \vec{s}) + i(\eta \pm z)(t' \pm t)}}{t'} \times \tag{13}$$

$$\times\, V(\vec{s} - \vec{s}\,', \eta) \,.$$

The quantities $\vec{s}$ and t ($\vec{s}\,'$ and t') are defined in (6), $t_0 = \sqrt{k^2 - s_0^2} = k \cdot \cos\frac{1}{2}\theta$ and

$$V(\vec{s}, z) = \int d^2\vec{b} \cdot e^{i\vec{b}\cdot\vec{s}}\, V(r) \tag{14}$$

is the two-dimensional Fourier transform of the potential. The expressions (11) and (12) can be derived from exps. (30) and (31) and from the Sommerfeld representation (12) for the free Green function of ref. [20]. Such a representation of the functions w_1 and w_2 is most convenient for their analysis in the high-energy limit.

Equation (9) is the basic one for the subsequent consideration. The existence of the exact phase equation makes it possible to extend the eikonal approximation to large angles and to investigate its corrections in a systematic manner.

3. *High-Energy Representation for the Scattering Amplitude at Large Angles*

We shall assume now that the nonlinear term in Eq. (9) can be neglected. The use of this approximation will be discussed below (Sect. 4). Consider the short-wave limit $(\lambda = ka \to \infty)$ for the function ϕ_k (10). From (10) and (11) it follows that the behaviour of the functions $w_1^{(\pm)}$ in this limit must be found. One can derive the asymptotic representation (the leading term of an asymptotic expansion) of (13) by the following simple generalisation of the Laplace method [21]. More close consideration is done in Appendix. Neglecting the unimportant variables we represent (13) in the form

$$w_1^{(\pm)}(\vec{s}_0) = \int d^2\vec{s} \cdot h(\vec{s},\vec{s}_0) \cdot e^{g(\vec{s},\vec{s}_0)}, \tag{15}$$

where

$$h(\vec{s},\vec{s}_0) = -\frac{1}{(2\pi)^2} \cdot \frac{1}{t}, \tag{16}$$

and

$$g(\vec{s},\vec{s}_0) = \ln[V(\vec{s}-\vec{s}_0,\eta)] + i\vec{b}(\vec{s}-\vec{s}_0) + i(\eta \pm z)(t \pm t_0). \tag{17}$$

The two-dimensional Fourier transform of a smooth potential $V(\vec{s}-\vec{s}_0,\eta)$ has a pronounced maximum at the

point $\vec{s} = \vec{s}_0$, when $\lambda \to \infty$. For example, for the Gaussian potential we have $V(\vec{s}-\vec{s}_0,\eta) \sim \exp[-(\vec{s}-\vec{s}_0)^2/a^2]$. The asymptotic expansion of (15) is determined by the behaviour of the integrand around this point. Note, the imaginary part of (17) gives rise to an explicit dependence of the maximum point $\vec{s}(\lambda)$ on the asymptotic parameter λ. However, this dependence is weak for a smooth potential $(\vec{s}(\lambda) = \vec{s}_0 + O(\frac{1}{\lambda}))$ and can be allowed for by the iteration. Thus, to obtain the asymptotic representation for (17) it is necessary to expand the functions h and g around $\vec{s}_0$ and to retain only the first term of h and two first terms of g in their expansions. As a result, we have

$$w_1^{(\pm)}(\eta,k,\vec{r},\vec{s}_0) = \frac{i}{k\cdot\cos\frac{1}{2}\theta} \cdot e^{i(1\pm1)k\cdot\cos\frac{1}{2}\theta(\eta\pm z)} \times \tag{18}$$

$$\times \hat{P}_{\vec{b},\vec{s}_0}(\eta \pm z)\, V(r_\eta),$$

where $r_\eta = (b^2+\eta^2)^{1/2}$ and the translation operator in the impact parameter plane $\hat{P}_{\vec{b},\vec{s}_0}(x)$ is

$$\hat{P}_{\vec{b},\vec{s}_0}(x) = \exp[-x\cdot \mathrm{tg}\frac{1}{2}\theta\, \hat{\vec{s}}_0 \cdot \nabla_{\vec{b}}], \quad \hat{\vec{s}}_0 = \vec{s}_0/s_0. \tag{19}$$

The derivation of (18) is free of the small-angle limit. Thus, it is valid for a wide angle region. Note, the function (16) has a singular point at $\theta = \pi$. Therefore the representation (18) is irregular with respect to the scattering angle θ as $\theta \to \pi$.

Taking into account eqs. (10), (11) an (18) we get

$$\phi_k(\xi,\vec{r},\vec{q}) = \mu_k(\xi,\vec{r},\vec{q}) + \nu_k(\xi,\vec{r},\vec{q}), \tag{20}$$

where the main slowly varying part of this phase is

$$\mu_k(\xi,\vec{r},\vec{q}) = -\frac{1}{k\cdot\cos\frac{1}{2}\theta}\int_z^{\xi} d\eta \cdot \hat{P}_{\vec{b},\vec{s}_0}(\eta - z)\cdot V(r_\eta). \tag{21}$$

and the rapidly oscillating component with respect to z is

$$\nu_k(\xi,\vec r,\vec q) = -\frac{1}{k\cdot\cos\frac{1}{2}\theta}\int_{|z|}^{\xi} d\eta\cdot e^{2ik\cos\frac{1}{2}\theta(\eta+z)} \times \tag{22}$$

$$\times \hat P_{\vec b,\vec s_0}(\eta+z)\cdot V(r_\eta)\ .$$

The asymptotic limit $\phi_k(\infty,\vec r,\vec q)$ is the linear in potential approximation for $\chi_k(\vec r,\vec q)$ In this limit the second terms in (20) can be neglected. Thus, as a result of the phase method (Ph), we obtain*

$$\chi_k^{Ph}(\vec r,\vec q) = -\frac{1}{k\cdot\cos\frac{1}{2}\theta}\cdot\int_{|z|}^{\infty} dz'\cdot\hat P_{\vec b,\vec s_0}(z'-z)\cdot V(r')\ . \tag{23}$$

Substitution of (23) into (1) gives our approximation for the scattering amplitude

$$f_{Ph}(q) = (-4\pi)^{-1}\int d\vec r\cdot e^{i\vec b\vec q}\cdot V(r)\ \times \tag{25}$$

$$\times\exp\{-\frac{i}{k\cos\frac{1}{2}\theta}\cdot\int_{|z|}^{\infty} dz'\hat P_{\vec b,\vec s_0}(z'-z)V(r')\}.$$

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*Note, the Schiff approximation [4], when z-axis is parallel to the average momentum k, can be represented in a similar form:

$$\chi_k^{Sch}(\vec r,\vec q) = -\frac{1}{2k\cos\frac{1}{2}\theta}\cdot\int_{-\infty}^{\infty} dz'[\theta(z'-z)\cdot\hat P_{\vec b,\vec s_0}(z'-z)\ + \tag{24}$$

$$+\ \theta(z-z')\hat P_{\vec b,\vec s_0}(z-z')]V(r'),$$

where $\theta(x)$ is the ordinary θ-function.

Eq. (25) is one of the main results of the present work. This approximation is valid for wide-angle region. It holds when $\lambda\to\infty$ (see also inequality (42) below). The eikonal approximation is the simple limit of (25) as $\theta\to 0$.

The replacement of the operator $\hat P_{\vec b,\vec s_0}$ by the unit one gives the nonrelativistic limit for the scattering amplitude $f_{AI}(q)$ in [15]. Numerical comparisons of the differential cross section, the imaginary and real parts of the amplitude $f_{AI}(q)$ with the partial wave calculations were presented in [6-9]. A wrong behaviour of f_{AI} at large angles (see also Figs. 1 and 2) was indicated. Therefore the correct generalisation of the eikonal amplitude to large scattering angles can be reached if we replace $k\to k\cos\frac{1}{2}\theta$ in (2) and change the path of integration like in (25).

The scattering amplitude (25) and the Schiff one (1,24) give the same result for finite range smooth potentials when the momentum tramsfer $q\to 0$. Indeed, the asymptotic representation of the amplitude (1) for these potentials are defined [5] by the value of the phase $\chi_k(\vec r,\vec q)$ at $z=0$. The expressions (23) and (24) coincide at this point. Thus, the asymptotic representations coincide too.

For the numerical analysis let us make some simplification of (25). If $V_0/E \lesssim \lambda^{-1}$ one can approximately neglect, in (23), the dependence on the azimuthal angle ϕ when integrating over the impact parameter plane $d^2\vec b = b\,db\,d\phi$. Really, in this case it can be easily shown that the ϕ-depending part of (23) is coherently added to the quantity $\vec b\vec q$ and is $O(\frac{1}{\lambda})$ of it for those values of ϕ where the phase χ_k should not be overlooked.

Hence we obtain a Fourier-Bessel transform like integral

$$f_{Ph}(q) = -ik\cdot\int_0^{\infty} b\,db\cdot J_0(bq)\cdot T_{Ph}(b,q)\ , \tag{26}$$

where the impact parameter amplitude

$$T_{Ph}(b,q) = \frac{1}{2ik}\cdot\int_{-\infty}^{\infty} dz\cdot V(r)\cdot\exp\{-\frac{i}{k\cdot\cos\frac{1}{2}\theta}\cdot\int_{|z|}^{\infty} dz' V[(r'^2+(z-z')^2\operatorname{tg}^2\frac{1}{2}\theta)^{1/2}]\}.$$

Note, the amplitude (26) turns into the eikonal approximation when $\theta \lesssim (ka)^{-1/2}$. A similar simplification can be performed for the Schiff approximation [4].

On the basis of (26) we compare the scattering amplitude with partial wave calculations and with well known approximations. To compute the integral (26) and that for the amplitude f_{Sch} the standard subroutine [23] for calculation of multiple integrals was used. The evaluation was performed up to 160°. Figure 1 shows the differential cross section for scattering from the Gaussian potential $V(r) = V_0 \cdot \exp[-r^2/a^2]$. The partial wave calculation (f -curve 1) is compared with the phase method (f_{Ph} -curve 2), the eikonal approximation (f_E -curve 3), the Schiff approximation (f_{Sch}-curve 4) and with the Abarbanel-Itzykson approximation (f_{AI} -curve 5). The partial wave result was taken from [8]. The dimensionless parameters of the theory are: $\lambda = 4$ and $V_0/E = -0.25$. Figure 2 compares the imaginary parts of the exact scattering amplitude (f -curve 1), the phase method approximation (f_{Ph}-curve 2) and AI-approximation (f_{AI} -curve 3) for the Yukawa potential $V(r) = V_0 \cdot \exp(-r/a)$.

The expansion parameters are $\lambda = 5$ and $V_0/E = -0.2$. The partial wave result was taken from paper [9]. This example illustrates the essential role of the translation operator (19). Note, the values of the expansion parameters in both the examples are far from their asymprotic values. So, one must pay attention to the qualitative development of the curves.

Figures 1 and 2 show good agreement of the phase method approximation for the scattering amplitude with the partial wave calculations in a wide angle region. The difference for very large angles is a consequence of the irregularity in the scattering anlge of (18). It is somewhat surprising that f_{Sch} in Fig. 1 is accurate at forward angles and is not especially good at large angles (see also [6]). Probably, this gives evidence that for potentials like the Gaussian potential the improved variant [14] of Schiff formula is more adequate.

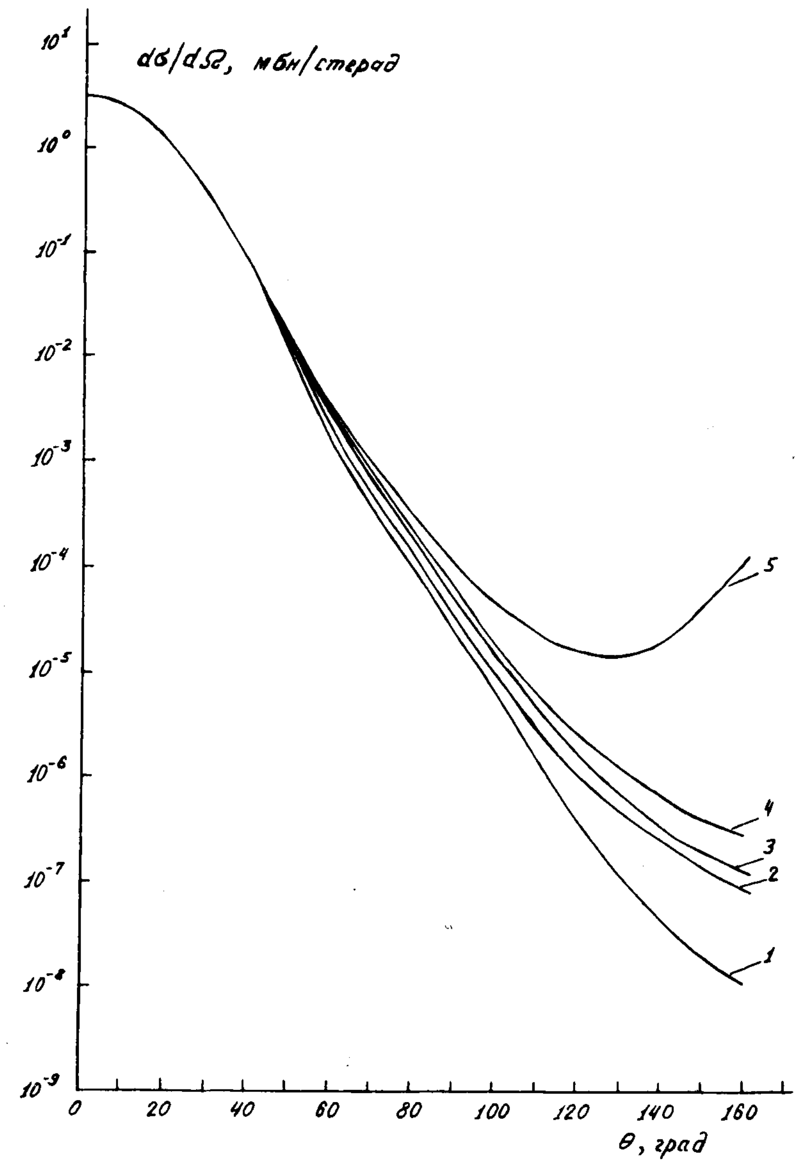

Fig. 1. Comparison of differential cross section for the Gaussian potential scattering. Curve 1 - the partial wave result, curve 2 - the phase method approximation (f_{Ph}), curve 3 - the eikonal approximation (f_E), curve 4 - the Schiff approximation (f_{Sch}) and curve 5 - the AI approximation (f_{AI}).

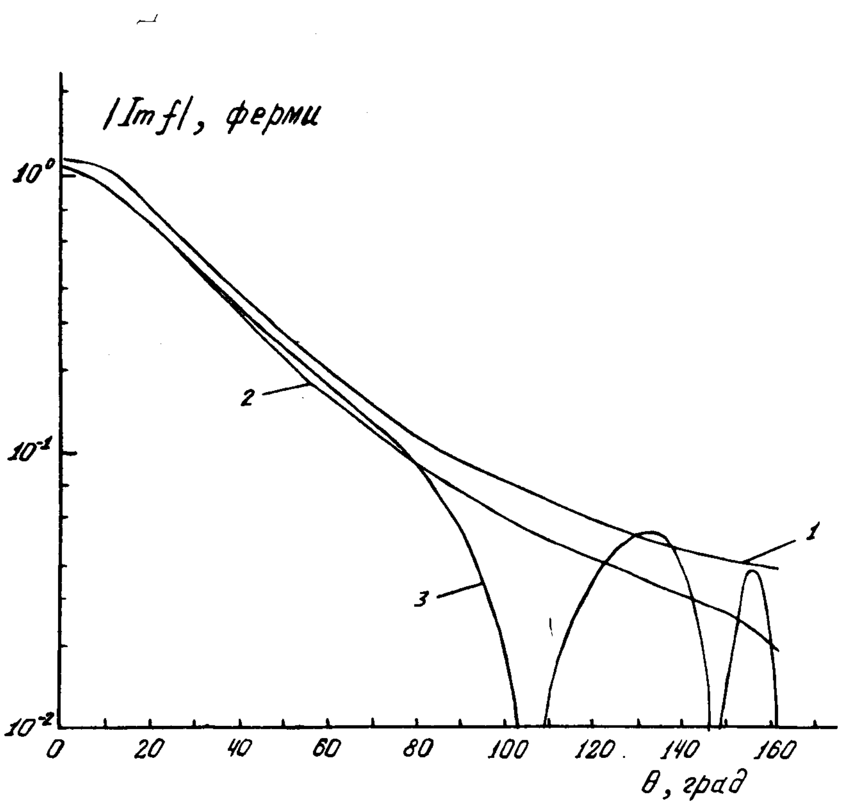

Fig. 2. The imaginary part of the scattering amplitude for the Yukawa potential. Curve 1 - the partial wave result, curve 2 - the phase method approximation (f_{Ph}) *and curve 3 - the AI approximation* (f_{AI}).

4. Non-Linear in Potential Corrections

Now consider the non-linear term in the phase equation (9) let us use the notation $\tau_k(\xi,\vec r,\vec q)$ for it and represent this term in the form

$$\tau_k(\xi,\vec r,\vec q) = \int_{|z|}^{\xi} d\eta \cdot \int_{-\eta}^{\eta} dz' \cdot W(\eta,k,z',\vec r,\vec q)\,, \tag{28}$$

$$W(\eta,k,z',\vec r,\vec q) = \int d\, b' \cdot w_2(\eta,k,\vec r,\vec r\,',\vec q)\cdot \Omega(\eta,k,\vec r\,',\vec q)\,, \tag{29}$$

where

$$\Omega(\eta,k,\vec r,\vec q) = V(r)\cdot \exp[i\phi_k(\eta,\vec r,\vec q) + i\tau_k(\eta,\vec r,\vec q)]\,. \tag{30}$$

The function w_2 is defined in (12). As the phase ϕ_k is well-known (10) the expressions (28-29) are an equation for the function $\tau_k(\xi,\vec r,\vec q)$.

We assume that the phase ϕ_k gives the main contribution to the total phase function $\chi_k(\xi,\vec r,\vec q)$. In the high-energy limit $\lambda \to \infty$ the eq. (28-29) is essentially simplified. Substituting (18) into (12) the asymptotic representation of the function w_2 can be obtained. Then substituting it into (29) we get

$$W(\eta,k,z',\vec r,\vec q) \approx -\frac{1}{2}\cdot \exp[2it_0(\eta+z')]\times$$

$$\times \frac{1}{(2\pi)^2}\cdot\int d^2\vec b\,'\int d^2\vec s\cdot \frac{e^{i(\vec b-\vec b\,')(\vec s-\vec s_0)+i(2\eta+z'-z)(t-t_0)}}{t}\times$$

$$\times\Lambda(\eta,k,\vec r\,',\vec s,\vec s_0)\,, \tag{31}$$

where

$$\Lambda(\eta,k,\vec r,\vec s,\vec s_0) = \Omega(\eta,k,\vec r,\vec q)\cdot \hat P_{\vec b,\vec s_0}(\eta+z)\cdot V(r_\eta),\; r_\eta = (b^2+\eta^2)^{1/2}. \tag{32}$$

Using the approximation (20) it can be easily shown that (32) is a slowly varying function in the impact parameter $\vec b$-plane. The scale of variation is

$$d = \min \{ a \cdot \cos\frac{1}{2}\theta \,,\; a \cdot \cos\frac{1}{2}\theta / (V_0 a/k) \} . \tag{33}$$

The quantity d refines the scale a. Hence, when $\lambda = kd \to \infty$ eq. (32) is the integral of type (15) and its asymptotic representation (in accordance with (18)) is

$$W(\eta, k, z', \vec{r}, \vec{q}) \approx -\frac{1}{2t_0^2} \cdot \exp[\, 2it_0(\eta + z') \,] \times \tag{34}$$

$$\times \hat{P}_{\vec{b},\vec{s}_0}(\eta - z)\,[\, V(r_\eta) \cdot \hat{P}_{\vec{b},\vec{s}_0}(\eta + z') \cdot \Omega(\eta, k, \vec{r}', \vec{q})\,], \quad \vec{r}' = (\vec{b}, z') .$$

Now inserting (34) into (28) and taking account of (20) and (30) we obtain

$$\tau_k(\xi, \vec{r}, \vec{q}) \approx \frac{i}{2t_0^2} \int_{|z|}^{\xi} d\eta \cdot \hat{P}_{\vec{b},\vec{s}_0}(\eta - z)\,[\, V(r_\eta) \cdot J(\eta, k, \vec{r}, \vec{q})\,] , \tag{35}$$

where

$$J(\eta, k, \vec{r}, \vec{q}) = \int_{-\eta}^{\eta} dz' \cdot \exp[\, 2it\,(\eta + z')\,] \times \tag{36}$$

$$\times \hat{P}_{\vec{b},\vec{s}_0}(\eta + z') \cdot \{ V(r') \cdot \exp[\, i\mu_k(\eta, \vec{r}', \vec{q}) + i\nu_k(\eta, \vec{r}', \vec{q}) + i\tau_k(\eta, \vec{r}', \vec{q})\,] \} .$$

The quantities μ_k (21) and τ_k are slowly varying functions of z'. In estimation of (36) a difficulty due to the rapidly varying function ν_k arises. Indeed, the integration by parts of (22), gives, when $\lambda \to \infty$.

$$\nu_k(\eta, \vec{r}, \vec{q}) \approx -\frac{i}{2t_0^2}[\, e^{2it_0(z + |z|)} \times \tag{37}$$

$$\times \hat{P}_{\vec{b},\vec{s}_0}(z + |z|) \cdot V(r) - e^{2it_0(z+\eta)} \cdot \hat{P}_{\vec{b},\vec{s}_0}(z + \eta)\, V(r_\eta)\,] .$$

and $\nu_k \ll |\mu_k|$ for $|z| \approx \eta$.

The evaluation of (36) can be reduced to estimation of the Fourier-integral in the following way. We expand $\exp(i\nu_x)$ into the power series, substitute (37) in each term of it and decompose every binomial. We substitute the final result of this procedure into (36) and change the order of summation and integration. Then the double sum arises each term of which is the Fourier-type integral. Using the Erdélyi theorem /21/ we obtain the asymptotic representation of (36). Consequently, in the limit $\lambda \to \infty$ the equation for the function τ_k is

$$\tau_k(\xi, \vec{r}, \vec{q}) = -\frac{1}{2t_0^2} \cdot \int_{|z|}^{\xi} dz' \cdot \hat{P}_{\vec{b},\vec{s}_0}(z' - z) \cdot \{ V^2(r') \cdot \sum_{n=0}^{\infty} [\frac{V(r')}{2t_0^2}]^n \times$$

$$\times \sum_{\ell=0}^{n} \frac{(-1)^\ell}{(n-\ell)!\,(\ell+1)!} \cdot [\, 1 - \frac{1}{2t_0^2(\ell+1)} \cdot \hat{P}_{\vec{b},\vec{s}_0}(2z') \cdot V(r') +$$

$$+ \frac{1}{2t_0(\ell+1)} \cdot (\frac{d}{dz''}\tau_k(z', \vec{r}'', \vec{q}))_{z''=-z'}\,]^{-1} , \tag{38}$$

where $t_0 = k \cdot \cos\frac{1}{2}\theta$, $\vec{r}' = (\vec{b}, z')$, $\vec{r}'' = (\vec{b}, z'')$ and the translation operator $\hat{P}_{\vec{b},\vec{s}_0}$ is defined in (19). In the derivation of (38) we take into account that $\mu_k|_{|z|=\eta} = \nu_k|_{|z|=\eta} = \tau_k|_{|z|=\eta} = 0$, $\frac{d}{dz}\mu_k|_{z=-\eta} = -\frac{1}{t_0}\hat{P}_{\vec{b},\vec{s}_0}(2\eta) \cdot V(r_\eta)$, $r_\eta = (b^2 + \eta^2)^{1/2}$ and

that all terms of order $O(\frac{1}{\lambda})$ in comparison with leading ones were omitted.

If $\frac{V_0}{E} \ll 1$ the solution of equation (38) can be obtained by interation setting $\tau_k = \sum_{n=0}^{\infty} \frac{1}{\lambda^n}\tau_k^{(n)}$ with $\tau_k^{(0)} = 0$. For example, the first two terms are

$$r_k^{(1)}(\xi,\vec{r},\vec{q}) = -\frac{1}{4t_0^3}\cdot\int_{|z|}^{\xi} dz'\cdot\hat{P}_{\vec{b},\vec{s}_0}(z'-z)\cdot V^2(r'), \tag{39}$$

$$r_k^{(2)}(\xi,\vec{r},\vec{q}) = -\frac{1}{4t_0^3}\cdot\int_{|z|}^{\xi} dz'\cdot\hat{P}_{\vec{b},\vec{s}_0}(z'-z)\{\frac{V^2(r')}{2t_0^2}[\hat{P}_{\vec{b},\vec{s}_0}(2z')V(r')+\frac{1}{2}V(r')]\}. \tag{40}$$

There are no difficulties in evaluation of the subsequent corrections.

The asymptotic quantities $r_k^{(n)}(\infty,\vec{r},\vec{q})$ $(n=1,2,3,\ldots)$ are the non-linear corrections for the phase χ_k^{Ph} (23). Taking into account only the first term (39) we thus obtain

$$\chi_k^{Ph}(\vec{r},\vec{q}) = -\frac{1}{k\cdot\cos\frac{1}{2}\theta}\int_{|z|}^{\infty} dz'\cdot\hat{P}_{\vec{b},\vec{s}_0}(z'-z)\,[V(r')+\frac{V^2(r')}{4k^2\cdot\cos^2\frac{1}{2}\theta}]. \tag{41}$$

Consequently, the nonlinear term in the phase equation (9) leads to corrections $O(\frac{V_0}{E})$ in comparison with (10). Note, thus corrections are analogous to those which arise when the eikonal phase (2) is replaced by the WKB-phase but do not reduce to them. It is pointed out in ref. /8/ that these of the WKB-phase not always improves (1,2).

Let us discuss the role of the correction $O(V_0/E)$ at large angles. From (41) it follows that this correction is irregular with respect to the scattering angle θ. Its degree of irregularity does not depend on a potential type and is defined by the factor $\cos^{-2}\frac{1}{2}\theta$. In Appendix it is noted that the correction $O(\frac{1}{\lambda})^2$ is irregular with respect θ too. But in this case the degree of irregularity depends on the potential form. For example, for Gaussian potential this factor is $\cos^{-5}\frac{1}{2}\theta$ and for the Yukawa potential $\cos^{-4}\frac{1}{2}\theta$. Thus for these types of potentials the degree of irregularity of the correction $O(\frac{1}{\lambda})$ is larger than that of the correction $O(V_0/E)$. So, if

$$\frac{V_0}{E} \lesssim \lambda \quad ; \tag{42}$$

the linear in potential phase χ_k^{Ph} (23) and, respectively, the approximation for the scattering amplitude f_{Ph} (25), dominates at all angles where the parameter λ can be considered as the asymptotical one.

5. Conclusion

We have presented the high-energy potential theory which is a generalisation of the phase method /17/. Within the framework of this approach the natural extention to large angles of the eikonal approximation is obtained. On the basis of equation (38) one can consistently evaluate the non-linear in potential corrections. It makes it possible to investigate the importance of the corrections $O(\frac{1}{\lambda})$ and $O(\frac{V_0}{E})$ at large angles. It has been established that the approximation (25) is valid for a wide class of potentials if the inequality (42) holds. The degree of angle irregularity of (25) in general depends on the potential shape. The simplification of (25) is obtained. The numerical calculations show its agreement with partial wave results in a wide-angle region.

The equation for the scattering amplitude in paper /18/ can also be considered in terms of the total phase function. The geometry of this equation seems to be more adequate for investigation of the large-angle scattering /24/.

This approach can be generalised to the relativistic case in the framework of the quasipotential equations /11-12/. The possibility of such an extension is supported by the recent relativistic generalisation /25/ of the standard phase method.

In conclusion the author is grateful to V.K.Lukyanov and R.M.Mir-Kasimov for permanent interest in the work. The author also thanks sincerely V.A.Alebastrov, V.R.Garsevanishvili, S.P.Kuleshov, V.V.Nesterenko, V.N.Pervushin, A.N.Sissakian, M.A.Smondyrev for discussions and A.I.Saltykov for consultations in carrying out the numerical calculations.

To obtain the asymptotic representation of (13) for $\lambda = ka \to \infty$ it is convenient to introduce new dimensionless variables $\vec{r} = \vec{r}/a$ and $\vec{p} = \vec{p}/k\,(p = k)$. Suppose that this has been done and keep the earlier notations. Thus the parameter λ arises. Now the problem is to obtain an asymptotic expansion of the integral (compare (15))

$$J(\lambda) = \iint_{(D)} d^2\vec{s} \cdot h(\vec{s}, \lambda) \cdot e^{-\lambda g(\vec{s}, \lambda)}. \tag{A.1}$$

If the functions g and h in (A.1) *are real and independent of the parameter* λ and if the function $g(\vec{s})$ has a minimum at the interior point $\vec{s}_0$ of the domain D of integration (the first-order critical point), the asymptotic series representation of J is given by the formula (111) in ref. /22/.

In the case of (13) both h and g are in general functions of λ and is complex. However the potential smoothness permits to take into account these complications by iteration. Let functions g and h be expanded into the series in powers of $\frac{1}{\lambda}$ around the point $\lambda = \infty$. Then one can easily show that the asymptotic representation of (A.1) is the following generalisation of eq. (111) /22/ :

$$J(\lambda) \approx \frac{2\pi}{\sqrt{\alpha_0\beta_0 - \gamma_0^2}} \cdot h^{(0)}(\vec{s}^{(0)}) \cdot \frac{1}{\lambda} \exp\{-\lambda[g^{(0)}(\vec{s}^{(0)}) + \frac{1}{\lambda} g^{(1)}(\vec{s}^{(0)}, \vec{s}^{(1)})]\}. \tag{A.2}$$

The first-order critical point $\vec{s}(\lambda)$ is defined by the equation

$$\frac{\partial}{\partial \vec{s}} g(\vec{s}, \lambda) = 0, \tag{A.3}$$

and can be represented in the series $\vec{s}(\lambda) = \sum_{n=0}^{\infty} \frac{1}{\lambda^n} \cdot \vec{s}^{(n)}$.

The functions g and h can be also represented at this point as follows: $g(\vec{s}(\lambda)) = \sum \frac{1}{\lambda^n} \cdot g^{(n)}$ and $h(\vec{s}, \lambda) = \sum_{n=0} \frac{1}{\lambda^n} h^{(n)}$;

The quantities α_0, β_0 and γ_0 are the asymptotic limits, respectively, of $\alpha(\lambda) = (\partial^2 g/\partial s_x^2)_{\vec{s}=\vec{s}(\lambda)}$, $\beta(\lambda) = (\partial^2 g/\partial s_y^2)_{\vec{s}=\vec{s}(\lambda)}$ and $\gamma(\lambda) = (\partial^2 g/\partial s_x \partial s_y)_{\vec{s}=\vec{s}(\lambda)}$ when $\lambda \to \infty$.

For example, consider the high-energy limit of (13) for the Gaussian potential $V(r) = V_0 \cdot \exp(-r^2/a^2)$. In dimensionless variables, in this case eq. (13) is the integral (A.1) with

$$g^{(\pm)}(\vec{s}, \lambda) = \frac{1}{4}\lambda(\vec{s} - \vec{s}_0)^2 - i[(\vec{s} - \vec{s}_0)\vec{b} + (\xi \pm z)(t \pm t_0)], \tag{A.4}$$

and

$$h(\vec{s}, \lambda) = -\frac{i}{(2\pi)^2} \cdot \pi \cdot a^2 \cdot V_0 \cdot \frac{e^{-\xi^2}}{t}. \tag{A.5}$$

We find the critical point $\vec{s}(\lambda)$ from eq. (A.3) and represent it in the power series the first terms of which are $\vec{s}^{(0)} = \vec{s}_0$, $\vec{s}^{(1)} = 2i[\vec{b} - (\xi \pm z) \cdot tg\frac{1}{2}\theta \cdot \hat{\vec{s}}_0]$, $\hat{\vec{s}}_0 = \vec{s}_0/s_0$. The evaluation of the quantities $g^{(0)}$, $g^{(1)}$ and $h^{(0)}$ in (A.2) at $\vec{s}(\lambda)$ gives

$$g^{(0)}(\vec{s}^{(0)}) = -i(\xi \pm z) \cdot \cos\frac{1}{2}\theta(1 \pm 1),$$

$$g^{(1)}(\vec{s}^{(0)}, \vec{s}^{(1)}) = -\frac{1}{4}\vec{s}^{(1)\,2},$$

$$h_0^{(0)}(\vec{s}^{(0)}) = -\frac{i}{(2\pi)^2} \cdot \pi a^2 V_0 \frac{e^{-\xi^2}}{k \cdot \cos\frac{1}{2}\theta}. \tag{A.6}$$

Finally the calculation of α_0, β_0 and γ_0 gives

$$(\alpha_0\beta_0 - \gamma_0^2)^{-1/2} = 2/\lambda. \tag{A.7}$$

On substituting (A.6-A.7) into (A.2) and passing to the old variables we obtain the asymptotic representation

(18) of the functions $w_1^{(\pm)}$. Note, the function (A.5) has a singularity at $s = 1$. That leads to an irregularity of the representation (18) with respect to θ as $\theta \to \pi$. Corrections $O(\frac{1}{\lambda})$ for (18) can be evaluated by a further generalisation of the formula (110) of ref. /22/. The investigation of these corrections for the Gaussian potential shows that for validity of (18) the quantity $\lambda \cos^5 \frac{1}{2}\theta$ must be asymptotically large. So, the factor of irregularity is $\cos^{-5}\frac{1}{2}\theta$. This factor is different for various potentials. For the Yukawa potential a similar consideration yields the factor $\cos^{-4}\frac{1}{2}\theta$.

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Received by Publishing Department
on August 25, 1975.