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APPROXIMATIONS OF STRONG AND WEAK COUPLINGS IN THE TWO-POLARON PROBLEM

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Recently the attempts have been made to construct the dynamical models of the extended particles.

These attempts taking roots long ago have resulted recently in certain success within the models of strongly coupled particles and fields.

A consistent theory of strong coupling was developed in papers [1,2] where the problem was considered on a particle coupled with the scalar quantum field. The method used there is based on the canonical transformation introduced by N.N.Bogolubov in studying the quantum problem of polaron. These problems then have been studied in subsequent papers (see, e.g., [3,4]).

When considering the case of two and more interacting particles a number of interesting problems arises. As is known, the strong interaction of a particle with a quantum field can be described by the effective potential due to the polarization field around the particle. The overlapping of the polarization regions can significantly change the character of forces between different particles at comparatively small distances.

A criterion for the problem to be solved correctly is the requirement of the "correlation weakening" stated in paper [5]. By this requirement, the multiparticle Green function transforms into the product of the one-particles ones with increasing relative distance between particles.

In this paper we are studying the problem on bound states of two polarons by methods we have developed in papers /4,6/. In the first section we obtain the functional integrals representing the Green function of the problem. In the second section these integrals are calculated approximately and then the expressions for the particle interaction potentials are found. Using those potentials we calculate the energy of the ground state in sect. 3 and the effective mass of the bound state of polarons in sect. 4. The strong and weak coupling are studied in detail; the account for translational invariance makes the presentation more consistent.

1. FUNCTIONAL REPRESENTATIONS

The Hamiltonian of the system under consideration is

$$H = -\frac{1}{2\mu_1}\nabla_1^2 - \frac{1}{2\mu_2}\nabla_2^2 + \sum_k [a_k(A_k e^{-i\vec{k}\vec{r}_1} + B_k e^{-i\vec{k}\vec{r}_2}) + a_k^+(A_k^* e^{i\vec{k}\vec{r}_1} + B_k^* e^{i\vec{k}\vec{r}_2})] + \sum_k \omega_k a_k^+ a_k + V(r_1 - r_2), \qquad (1.1)$$

where μ_1 and μ_2 are the masses of two nonrelativistic particles, A_k and B_k are the Fourier components of the source densities, a_k^+ and a_k are the creation and annihilation operators of the scalar field with momentum $\vec{k}$ and energy $\omega_k : [a_k, a_{k'}^+] = \delta_{kk'}$.

We change the particle coordinates $\vec{r}_1$ and $\vec{r}_2$ by the coordinates of the mass center $\vec{R}$ and relative distance $\vec{r}$

$$\vec{R} = \frac{\mu_1\vec{r}_1 + \mu_2\vec{r}_2}{\mu_1 + \mu_2}, \quad \vec{r} = \vec{r}_1 - \vec{r}_2,$$
$$\vec{\nabla}_1 = \vec{\nabla}_R \frac{\mu_1}{\mu_1 + \mu_2} + \vec{\nabla}_r, \quad \vec{\nabla}_2 = \vec{\nabla}_R \frac{\mu_2}{\mu_1 + \mu_2} - \vec{\nabla}_r. \qquad (1.2)$$

Then introduce the notation for the total mass of two particles m and for the reduced mass μ

$$m = \mu_1 + \mu_2, \quad \mu = \frac{\mu_1\mu_2}{\mu_1 + \mu_2}. \qquad (1.3)$$

In terms of eqs. (1.2), (1.3) the Hamiltonian (1.1) takes the form

$$H = -\frac{1}{2m}\vec{\nabla}_R^2 - \frac{1}{2\mu}\vec{\nabla}_r^2 + \sum_k [a_k(A_k \exp(-i\vec{k}\vec{R} - i\vec{k}\vec{r}\frac{\mu}{\mu_1}) + B_k \exp(-i\vec{k}\vec{R} + i\vec{k}\vec{r}\frac{\mu}{\mu_2})) + \text{h.c.}] + \sum_k \omega_k a_k^+ a_k + V(\vec{r}).$$

Now we are able to make the Bogolubov canonical transformation with respect to the variable $\vec{R}$:

$$a_k \to \zeta_k = a_k e^{-i\vec{k}\vec{R}},$$
$$a_k^+ \to \zeta_k^+ = a_k^+ e^{i\vec{k}\vec{R}}, \quad [\zeta_k, \zeta_{k'}^+] = \delta_{kk'},$$
$$-i\vec{\nabla}_R \to \vec{P} - \sum_k \vec{k}\zeta_k^+\zeta_k$$

and to represent the Hamiltonian in the form

$$H = \frac{1}{2m}(\vec{P} - \sum_k \vec{k}\zeta_k^+\zeta_k)^2 - \frac{1}{2\mu}\vec{\nabla}_r^2 + \sum_k [\zeta_k C_k(\vec{r}) + \zeta_k^+ C_k^*(\vec{r})] +$$

$$+ \sum_k \omega_k \zeta_k^+ \zeta_k + V(\vec{r}) ,$$

where the total momentum of the system $\vec{P}$ is

$$\vec{P} = -i\vec{\nabla}_R + \sum_k \vec{k}\zeta_k\zeta_k ,$$

which may be put c - number due to the conservation law $[\vec{P}, H] = 0$.

The functions of the relative distance $C_k(\vec{r})$ are defined by the formula

$$C_k(\vec{r}) = A_k \exp(-i\vec{k}\vec{r}\frac{\mu}{\mu_1}) + B_k \exp(i\vec{k}\vec{r}\frac{\mu}{\mu_2}) .$$

To find out the energy levels of the system it suffices to construct the quantity $\exp(-\tau H)$. In terms of the Feynman ordering variable s it is

$$\exp(-\tau H) = T\exp(-\int_0^\tau ds H_s) . \tag{1.4}$$

Then, following paper [4] we can obtain the functional representation for (1.4).

Indeed, performing the continual integration with the Gaussian measure we linearize the operators of the kinetic energy of particles

$$\exp[-\frac{1}{2m}\int_0^\tau ds (P - \sum_k k\zeta_k^+\zeta_k)_s^2] = \int [\delta\nu]_0^\tau \exp[-i\sqrt{\frac{2}{m}}\int_0^\tau ds\,\nu(s)(P - \sum_k k\zeta_k^+\zeta_k)_s]$$

$$\exp[\int_0^\tau ds \frac{(\vec{\nabla}_r^2)_s}{2\mu}] = \int [\delta\vec{a}]_0^\tau \exp[-\sqrt{\frac{2}{\mu}}\int_0^\tau ds\,\vec{a}(s)\,(\vec{\nabla}_r)_s] ,$$

where

$$[\delta\vec{\nu}]_0^\tau = \delta\vec{\nu}\exp(-\int_0^\tau \vec{\nu}^2) / \int \delta\vec{\nu}\exp(-\int_0^\tau \vec{\nu}^2) .$$

After some calculations (see ref. [4].) we arrive at the representation

$$\exp(-\tau H) = \int [\delta\vec{\nu}]_0^\tau [\delta\vec{a}]_0^\tau \exp[-i\sqrt{\frac{2}{m}}\vec{P}\int_0^\tau \vec{\nu}(s)ds - \sqrt{\frac{2}{\mu}}\vec{\nabla}_r \int_0^\tau \vec{a}(s)ds] \times$$

$$\times f f^+ f^- \exp\{-\int_0^\tau ds\, V(\vec{r} + \sqrt{\frac{2}{\mu}}\int_0^s d\eta\,\vec{a}(\eta)) + \sum_k \int_0^\tau ds_1 \int_0^{s_1} ds_2 \times \tag{1.5}$$

$$\times C_k(r + \sqrt{\frac{2}{\mu}}\int_0^{s_1} a(\eta)\,d\eta)\, C_k^*(r + \sqrt{\frac{2}{\mu}}\int_0^{s_2} a(\eta)\,d\eta) \exp[-\omega_k(s_1 - s_2) +$$

$$+ i\sqrt{\frac{2}{m}}\vec{k}\int_{s_2}^{s_1} \vec{\nu}(\eta)\,d\eta\,]\} ,$$

where

$$f = \exp[\sum_k \zeta_k^+\zeta_k \int_0^\tau d\eta\,(-\omega_k + i\sqrt{\frac{2}{m}}\vec{k}\vec{\nu}(\eta))] ,$$

$$f^+ = \exp\{-\sum_k \zeta_k^+ \int_0^\tau ds_1 C_k^*(\vec{r} + \sqrt{\frac{2}{\mu}}\int_0^{s_1}\vec{a})\exp[\omega_k s_1 - i\sqrt{\frac{2}{m}}\vec{k}\int_0^{s_1}\vec{\nu}]\} , \tag{1.6}$$

$$f^- = \exp\{-\sum_k \zeta_k \int_0^\tau ds_1 C_k(\vec{r} + \sqrt{\frac{2}{\mu}}\int_0^{s_1}\vec{a})\exp[-\omega_k s_1 + i\sqrt{\frac{2}{m}}\vec{k}\int_0^{s_2}\vec{\nu}]\} .$$

To calculate the characteristics of the process when the field oscillators in the initial and final states are unexcited one should compute the vacuum expectation value of the operator $\exp(-\tau H)$. Since according to (1.6) $\langle 0 | ff^+f^- | 0 \rangle = 1$ then $\langle 0 | \exp(-\tau H) | 0 \rangle$ is given by (1.5), where the product ff^+f^- should be put unity.

2. APPROXIMATE CALCULATION OF THE FUNCTIONAL INTEGRALS

With the aim to calculate the energy of the ground state we put $P = 0$. Instead of integration over $\vec{\nu}$ we will integrate over the Feynman paths changing the variables $\vec{\nu} \to \vec{x}$ where

$$\vec{x}(\eta) = \sqrt{2}\int_0^{\eta} d\eta' \vec{\nu}(\eta') .$$

Then we obtain

$$\langle 0|\exp(-\tau H)|0\rangle = \int [\delta \vec{a}]_0^{\tau} \exp\Big(-\sqrt{\frac{2}{\mu}}\vec{\nabla}_r \int_0^{\tau} \vec{a}\Big) \times \\ \times \exp\Big[-\int_0^{\tau} ds\, V\Big(\vec{r} + \sqrt{\frac{2}{\mu}}\int_0^{s} \vec{a}\Big)\Big] G(\vec{a}) , \tag{2.1}$$

where

$$G(\vec{a}) = \int \delta\vec{x}\, e^{S[x,\vec{a}]} ,$$

$$S[\vec{x},\vec{a}] = -\frac{1}{2}\int_0^{\tau} \dot{\vec{x}}^2(\eta)\, d\eta + \sum_k \int_0^{\tau} ds_1 \int_0^{s_1} ds_2\, C_k\Big(\vec{r} + \sqrt{\frac{2}{\mu}}\int_0^{s_1}\vec{a}\Big) \times \\ \times C_k^*\Big(\vec{r} + \sqrt{\frac{2}{\mu}}\int_0^{s_2}\vec{a}\Big) \exp\Big[-\omega_k(s_1 - s_2) + \frac{i}{\sqrt{m}}\vec{k}(\vec{x}(s_1) - \vec{x}(s_2))\Big] . \tag{2.2}$$

To calculate the quantity $G(a)$ we introduce, following Feynman /7/, the approximating action

$$S'[\vec{x}] = -\frac{1}{2}\int_0^{\tau} \dot{\vec{x}} - \frac{C}{2}\int_0^{\tau} ds_1 ds_2\, e^{-\omega|s_1 - s_2|}[\vec{x}(s_1) - \vec{x}(s_2)] . \tag{2.3}$$

The action S' approximates S in the sense that the functional variables $x(s)$ describe the quantum vibrations after separating the motion along the classical trajectory (in this section it is $x(s) = 0$).

Since the time intervals are limited by the exponential damping the distance $|\vec{x}(s_1) - \vec{x}(s_2)|$ cannot be very large and we expand $\exp\{\frac{i}{\sqrt{m}}\vec{k}[\vec{x}(s_1) - \vec{x}(s_2)]\}$ in (2.2) up to the quadratic terms. The constant C defines the attraction force between two polarons (see ref. /7/). It is considered as a variational parameter. As we shall see below, its value is the same as for one polaron. From the approximate formula

$$G(\vec{a}) \approx \int \delta\vec{x}\, e^{S'[\vec{x}]} \exp\{\int \delta\vec{x} e^{S'} (S - S') / \int \delta\vec{x}\, e^{S'}\} .$$

with the use of formulae of paper /7/ (see also /6/) we arrive at the following result

$$G(\vec{a}) = \exp\Big\{-\frac{3\tau}{4}\frac{(V-\omega)^2}{V} + \sum_k \int_0^{\tau} ds_1 \int_0^{s_1} ds_2\, C_k\Big(\vec{r} + \sqrt{\frac{2}{\mu}}\int_0^{s_1}\vec{a}\Big) \times \\ \times C_k^*\Big(\vec{r} + \sqrt{\frac{2}{\mu}}\int_0^{s_2}\vec{a}\Big) \exp\Big[-\omega_k(s_1 - s_2) - \frac{\vec{k}^2}{2mV^2}F(|s_1 - s_2|)\Big]\Big\} , \tag{2.4}$$

where

$$V^2 = \omega^2 + \frac{4C}{\omega} , \quad F(\sigma) = \omega^2\sigma + \frac{V^2 - \omega^2}{V}(1 - e^{-V\sigma}) .$$

The expression (2.4) for G should be inserted into (2.1) and integrated over $\vec{a}$. As a result we obtain some operator. The approximate form of the operator may be taken as follows

$$\langle 0|\exp(-\tau H)|0\rangle \approx \exp(-\tau H_{eff}) , \quad \tau \to \infty ; \\ H_{eff} = -\frac{1}{2\mu}\vec{\nabla}_r^2 + V_{eff}(\vec{r}) . \tag{2.5}$$

In other words, in this approximation the relative motion of "dressed" particles is described by the Schrödinger equation, all the quantum effects due to the scalar field being taken into account by the effective potential $V_{eff}(\vec{r})$. If the relation (2.5) were exact the functional representation (2.1) in the limit $\tau \to \infty$ would be of the form:

$$\langle 0|e^{-\tau H}|0\rangle = \int [\delta\vec{a}]_0^{\tau} \exp\Big(-\sqrt{\frac{2}{\mu}}\vec{\nabla}_r \int_0^{\tau}\vec{a}\Big) \exp\Big[-\int_0^{\tau} ds\, V_{eff}\Big(\vec{r} + \sqrt{\frac{2}{\mu}}\int_0^{s}\vec{a}\Big)\Big] .$$

The deviations from the representation (2.6) arise due to the terms of the type $\int_{s_2}^{s_1} \vec{a}(\eta)\,d\eta$ in G. From the equation (2.4) it is clear that large time intervals $(s_1 - s_2)$ are suppressed and the main contribution comes from the region $s_1 \approx s_2$. However, to put the terms of the type $\int_{s_2}^{s_1} \vec{a}(\eta)\,d\eta$ zero would be rather rough approximation. We proceed therefore in the following way: change the terms $\int_{s_2}^{s_1} \vec{a}(\eta)\,d\eta$ by $\int_{s_2}^{s_1} \vec{\beta}(\eta)\,d\eta$ and consider $\vec{\beta}(\eta)$ to be a new functional variable, then average the resulting functional over $\beta(\eta)$ with the Gaussian measure

$$\langle 0|e^{-\tau H}|0\rangle = \int [\delta\vec{a}]_0^\tau \exp\Big[-\frac{3\tau}{4}\frac{(V-\omega)^2}{V} - \sqrt{\frac{2}{\mu}}\,\vec{V}_r \int_0^\tau \vec{a}\Big] \times$$

$$\times \exp\Big[-\int_0^\tau ds\, V\Big(\vec{r} + \sqrt{\frac{2}{\mu}}\int_0^s \vec{a}\Big)\Big] \int [\delta\vec{\beta}]_0^\tau \exp\Big\{\sum_k \int_0^\tau ds_1 \int_0^{s_1} ds_2\, C_k\Big(\vec{r} +$$

$$+ \sqrt{\frac{2}{\mu}}\int_0^{s_2}\vec{a} + \sqrt{\frac{2}{\mu}}\int_{s_2}^{s_1}\vec{\beta}\Big) C_k^*\Big(\vec{r} + \sqrt{\frac{2}{\mu}}\int_0^{s_2}\vec{a}\Big) \exp\Big[-\omega_k(s_1-s_2) - \frac{\vec{k}^2}{2mV^2}F(s_1-s_2)\Big]\Big\}.$$

The integral over $\vec{\beta}(\eta)$ is calculated with the use of the approximating action (2.3). In doing so, the parameter C may be accepted the same as in calculating (2.2), due to symmetry of the problem.

As a result, we have

$$\langle 0|e^{-\tau H}|0\rangle = \int [\delta\vec{a}]_0^\tau \exp\Big[-\frac{3\tau}{2}\frac{(V-\omega)^2}{V} - \sqrt{\frac{2}{\mu}}\,\vec{V}_r \int_0^\tau \vec{a}\Big] \exp\Big\{-\int_0^\tau V\Big(\vec{r} +$$

$$+ \sqrt{\frac{2}{\mu}}\int_0^s \vec{a}\Big)ds + \int_0^\tau ds \sum_k \int_0^\infty d\sigma\, e^{-\omega_k\sigma} C_k^*\Big(\vec{r} + \sqrt{\frac{2}{\mu}}\int_0^s \vec{a}\Big)\Big[A_k \exp\Big(-i\frac{\mu}{\mu_1}\vec{k}\Big(\vec{r} + \sqrt{\frac{2}{\mu}}\int_0^s \vec{a}\Big)\Big) \times$$

$$\times \exp\Big(-\frac{\vec{k}^2}{2\mu_1 V^2}F(\sigma)\Big) + B_k \exp\Big(i\frac{\mu}{\mu_2}\vec{k}\Big(\vec{r} + \sqrt{\frac{2}{\mu}}\int_0^s \vec{a}\Big) - \frac{\vec{k}^2}{2\mu_2 V^2}F(\sigma)\Big]\Big\},$$

which give the following effective potential of interaction

$$V_{eff}(\vec{r}) = V(\vec{r}) - \sum_k \int_0^\infty d\sigma\, e^{-\omega_k\sigma}\Big\{|A_k|^2 \exp\Big[-\frac{\vec{k}^2}{2\mu_1 V^2}F(\sigma)\Big] + |B_k|^2 \times$$

$$\times \exp\Big[-\frac{\vec{k}^2}{2\mu_2 V^2}F(\sigma)\Big] + A_k B_k^* \exp\Big[-i\vec{k}\vec{r} - \frac{\vec{k}^2}{2\mu_1 V^2}F(\sigma)\Big] + \tag{2.7}$$

$$+ A_k^* B_k \exp\Big[i\vec{k}\vec{r} - \frac{\vec{k}^2}{2\mu_2 V^2}F(\sigma)\Big]\Big\} + \frac{3}{2}\frac{(V-\omega)^2}{V}.$$

Provided the particles are the same: $\mu_2 = \mu_1$, $A_k = B_k$, the potential (2.7) is somewhat simplified

$$V_{eff}(\vec{r}) = \frac{3}{2}\frac{(V-\omega)^2}{V} + V(\vec{r}) -$$

$$- 2\sum_k |A_k|^2 (1 + \cos\vec{k}\vec{r}) \int_0^\infty d\sigma \exp\Big[-\omega_k\sigma - \frac{\vec{k}^2}{2\mu_1 V^2}F(\sigma)\Big]. \tag{2.8}$$

In the case of strong coupling the parameter V, as shown in [7], is large and $F(\sigma) \approx V$. Then the potential (2.8) takes the form

$$V_{eff}^{(S)}(\vec{r}) = \frac{3}{2}V + V(\vec{r}) - 2\sum_k \frac{|A_k|^2}{\omega_k}(1 + \cos\vec{k}\vec{r}) \exp\Big(-\frac{\vec{k}^2}{2\mu_1 V}\Big). \tag{2.9}$$

In the case of weak coupling $V \approx \omega$ and $F(\sigma) \approx \omega^2\sigma$ therefore

$$V_{eff}^{(W)}(\vec{r}) = V(\vec{r}) - 2\sum_k \frac{|A_k|^2}{\omega_k + \vec{k}^2/2\mu_1}(1 + \cos\vec{k}\vec{r}). \tag{2.10}$$

3. A CALCULATION OF THE ENERGY OF THE GROUND STATE

Let us calculate the lowest level for the polaron coupling and neglecting the direct interaction of particles:

$$V(\vec{r}) = 0, \quad \omega_k = \omega,$$

$$\sum_k |A_k|^2 \rightharpoondown = \frac{\omega^{3/2} a}{2\sqrt{2}\,\pi^2\sqrt{\mu_1}} \int \frac{d\vec{k}}{\vec{k}^2}. \qquad (3.1)$$

Consider first the case of strong coupling. The equation (2.9) together with (3.1) brings to the potential

$$V_{eff}^{(S)}(\vec{r}) = \frac{3}{2}V - \frac{2a\sqrt{\omega V}}{\sqrt{\pi}} - \frac{2a\sqrt{\omega}}{\sqrt{2\mu_1}} \frac{\phi\left(r\sqrt{\frac{\mu_1 V}{2}}\right)}{r}, \qquad (3.2)$$

where $\phi(r)$ is the probability integral. When $r \to \infty$

$$V_{eff}^{(S)}(\infty) = E_\infty = \frac{3}{2}V - 2a\sqrt{\frac{\omega V}{\pi}}. \qquad (3.3)$$

The minimum in the expression (3.3) is reached when

$$\sqrt{V} = \frac{2a}{3}\sqrt{\frac{\omega}{\pi}} \qquad (3.4a)$$

and it equals

$$E_\infty = -\frac{2a^2\omega}{3\pi}, \qquad (3.4b)$$

which is twice as large as the energy of one polaron.

Inserting the parameter (3.4a) into (3.2) we find the ultimate result for the effective potential in the case of strong coupling (the particle mass μ_1 is expressed

through the reduced mass $\mu = \mu_1/2$):

$$V_{eff}^{(S)}(\vec{r}) = -\frac{2}{3\pi}a^2\omega - a\sqrt{\frac{\omega}{\mu}} \frac{\phi\left(r\frac{2a}{3}\sqrt{\frac{\mu\omega}{\pi}}\right)}{r}. \qquad (3.5)$$

To estimate the lowest level we introduce the Pöschl-Teller approximating potential [8]:

$$V_a(\vec{r}) = -\frac{2}{3\pi}a^2\omega - \frac{c^2}{2\mu}\frac{\lambda(\lambda-1)}{ch^2 cr}, \qquad (3.6)$$

where c and λ are certain parameters. One of them will be defined from the condition of coincidence of the potentials (3.5) and (3.6) at the point $r = 0$ (as $r \to \infty$ these also coincide). Then we obtain

$$\lambda = \frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{32a^2\omega\mu}{3\pi c^2}}. \qquad (3.7)$$

Assuming (3.6) to be leading term of interaction we find parameter c from the condition for the first correction to the lowest level, by perturbation theory, to be zero. The known solutions [8] of the Schrödinger equation with potential (3.6) are

$$E_0 = -\frac{2}{3\pi}a^2\omega - \frac{c^2(\lambda-2)^2}{2\mu}, \quad \lambda \ge 2,$$

$$\chi_0(r) = A\, sh\, cr \cdot ch^{1-\lambda} cr, \quad A = \sqrt{c/2\pi B(\tfrac{3}{2}, \lambda-2)}, \quad \int_0^\infty |\chi_0|^2 dr = \frac{1}{4\pi}. \qquad (3.8)$$

These are the energy of the ground state E_0, the radial wave function $\chi_0(r)$, and the condition of existence of the bound states.

The first correction to energy δE_0 is given by the formula

$$\delta E_0 = 4\pi\int_0^\infty dr\,\delta V|\chi_0(r)|^2 = 4\pi|A|^2 \int_0^\infty dr\, sh^2 cr \cdot ch^{2(1-\lambda)} cr \times$$

$$\times\left[\frac{4a^2\omega}{3\pi\, ch^2\, cr} - a\sqrt{\frac{\omega}{\mu}} \frac{\phi\left(r\frac{2a}{3}\sqrt{\frac{\mu\omega}{\pi}}\right)}{r}\right]. \qquad (3.9)$$

Changing variables in (3.9) $r \to r/c$ and equating δE_0 to zero we get the equation

$$\int_0^\infty dr \frac{sh^2 r}{ch^{2(1-\lambda)} r}\left[\frac{1}{ch^2 r} - \frac{\sqrt{\pi}}{2}\frac{\phi(rx)}{rx}\right] = 0, \qquad (3.10)$$

where we introduce the parameter

$$x = \frac{2\alpha}{3c}\sqrt{\frac{\mu\omega}{\pi}}. \qquad (3.11)$$

Substituting (3.11) into (3.7) and (3.8) we obtain λ and E_0 as functions of x:

$$\lambda = \frac{1}{2} + \frac{1}{2}\sqrt{1+24x^2}, \quad x \geq 1/\sqrt{3};$$

$$E_0 = -\frac{2\alpha^2\omega}{3\pi}\,\frac{5 + 18x^2 - 3\sqrt{1+24x^2}}{6x^2}. \qquad (3.12)$$

The numerical integration produces the following value of x

$$x = 1.911. \qquad (3.13)$$

At this value of x the energy of the ground state is

$$E_0 = -\frac{2\alpha^2\omega}{3\pi}\,1.939 = -0.411\alpha^2\omega. \qquad (3.14)$$

As is seen, the effective interaction potential gives the energy of the ground state smaller by about a factor of two as compared to the minimum energy of two free polarons (cf. (3.4) and (3.14)).

In the case of weak coupling, as we shall see below, the interaction of polarons via the scalar field leads only to small corrections to the energy of noninteracting polarons.

For the weak coupling the potential is

$$V_{eff}^{(W)}(\vec{r}) = -2\alpha\omega - 2\alpha\omega\,\frac{1-\exp(-2\sqrt{\mu\omega}\,r)}{2\sqrt{\mu\omega}\,r}. \qquad (3.15)$$

It is not difficult to see that the energy of the ground state differs from the doubled polaron energy $-2\alpha\omega$ by a value of order α^2. Indeed, for q, the mean momentum in the ground state, the value of r in that state is defined from the uncertainty relation $r \approx 1/q$ and the energy then is as follows

$$E(q) = \frac{q^2}{2\mu} - 2\alpha\omega - 2\alpha\omega\,\frac{1-\exp(-2\sqrt{\mu\nu}/q)}{2\sqrt{\mu\nu}/q}.$$

Solving the equation $dE(q)/dq = 0$ for small α we find that the $E(q)$ minimum is reached at $q \approx \alpha$, i.e., the energy of the ground state E_0 is given by the expression

$$E_0 = -2\alpha\omega + O(\alpha^2).$$

To obtain the terms of the order α^2 it suffices to calculate them in the part of the effective potential (3.15) independent of r. However, as has been mentioned, this part of the potential represents the doubled polaron energy. Using the results of [7] we obtain the same potential (3.15) up to terms of the order α^2

$$V_{eff}^{(W)}(\vec{r}) = -2\alpha\omega - \frac{2\alpha^2\omega}{81} - \alpha\sqrt{\frac{\omega}{\mu}}\,\frac{1-\exp(-2r\sqrt{\mu\omega})}{r}. \qquad (3.16)$$

The considerations on the basis of the uncertainty relation have shown that for the ground state the mean value of r is of the order $1/\alpha$, i.e., we may neglect the exponentially small terms in (3.16). In this way we arrive at the Coulomb potential

$$V_{eff}^{(W)}(r) = -2\alpha\omega - \frac{2\alpha^2\omega}{81} - \alpha\sqrt{\frac{\omega}{\mu}}\,\frac{1}{r},$$

for which the energy of the ground state is

$$E_0 = -2\alpha\omega - \frac{2\alpha^2\omega}{81} - \frac{\alpha^2\omega}{2} = -2\alpha\omega - 52.47\left(\frac{\alpha}{10}\right)^2\omega . \quad (3.17)$$

Thus, the interaction of polarons in the case of weak coupling results in the appearance, in the energy, of terms small as compared to the energy of two noninteracting polarons. However, this interaction has to be taken into account when calculating with an accuracy to terms of the order of the squared coupling constant α since these terms are almost entirely due to the presence of that interaction.

4. EFFECTIVE MASS OF THE BOUND STATE OF TWO PARTICLES

In the previous section we have put total momentum of the system zero in calculations of the energy of the ground state. To calculate the effective mass of the system one should determine the effective potential up to terms of the order $\vec P^2$. To begin with, we follow paper [6] and obtain the classical trajectory $\vec\nu_0(\eta, \vec r)$. On this trajectory one reaches the extremum in the variable $\vec\nu$ of the action in the exponent of (1.5). The equation for $\vec\nu_0$ is

$$\vec\nu_0(\eta) = -i\frac{\vec P}{\sqrt{2m}} + i\sum_k \frac{\vec k}{\sqrt{2m}}\int_\eta^\tau ds_1\int_0^\eta ds_2\, C_k\Big(\vec r + \sqrt{\tfrac{2}{\mu}}\int_0^{s_1}\vec a\Big)\times$$
$$\times\, C_k^*\Big(\vec r + \sqrt{\tfrac{2}{\mu}}\int_0^{s_2}\vec a\Big)\exp\Big[-\omega_k(s_1 - s_2) + i\sqrt{\tfrac{2}{m}}\,\vec k\int_{s_2}^{s_1}\vec\nu_0\Big] . \quad (4.1)$$

Note, first, that at $\vec P = 0$, $\vec\nu_0 = 0$ too. Since we are interested in small $\vec P$ we may expand the exponent dependent on $\vec\nu_0$. Second, we need the value of $\vec\nu_0(\eta)$ only at large values of η and for $\tau\to\infty$. In this case the main contribution to eq. (4.1) comes from the region $s_1 \approx s_2 \approx \eta$. As a result, we have the equation

$$\vec\nu_0(\eta) = -i\frac{\vec P}{\sqrt{2m}} - \frac{2}{m}\sum_k \frac{\vec k(\vec k\vec\nu_0(\eta))}{\omega_k^3}\Big|C_k\Big(\vec r + \sqrt{\tfrac{2}{\mu}}\int_0^\eta\vec a\Big)\Big|^2 . \quad (4.2)$$

From equation (4.2) it follows that the dependence of $\vec\nu_0$ on η is defined by the vector $\vec r + \sqrt{\frac{2}{\mu}}\int_0^\eta\vec a$:

$$\vec\nu_0(\eta) = \vec v\Big(\vec r + \sqrt{\tfrac{2}{\mu}}\int_0^\eta\vec a\Big) .$$

Changing now the variables in (1.5) $\vec\nu\to\vec\nu + \vec\nu_0$ we arrive at the representation

$$\langle 0|e^{-\tau H}|0\rangle = \int[\delta\vec\nu]_0^\tau[\delta\vec a]_0^\tau \exp\Big(-\sqrt{\tfrac{2}{\mu}}\vec\nabla_r\int_0^\tau\vec a\Big)\times$$
$$\times\exp\Big\{-\int_0^\tau d\eta\,\vec\nu(\eta)\Big[i\sqrt{\tfrac{2}{m}}\vec P + 2\vec\nu_0(\eta)\Big] - \int_0^\tau ds\Big[V\Big(\vec r + \sqrt{\tfrac{2}{\mu}}\int_0^s\vec a\Big) +$$
$$+\,\vec v^2\Big(\vec r + \sqrt{\tfrac{2}{\mu}}\int_0^s\vec a\Big) + i\sqrt{\tfrac{2}{m}}\,\vec P\vec v\Big(\vec r + \sqrt{\tfrac{2}{\mu}}\int_0^s\vec a\Big)\Big] + \sum_k\int_0^\tau ds_1\int_0^{s_1}ds_2\,\times \quad (4.3)$$
$$\times\, C_k\Big(\vec r + \sqrt{\tfrac{2}{\mu}}\int_0^{s_1}\vec a\Big)C_k^*\Big(\vec r + \sqrt{\tfrac{2}{\mu}}\int_0^{s_2}\vec a\Big)\exp\Big[-\omega_k(s_1 - s_2) +$$
$$+\, i\sqrt{\tfrac{2}{m}}\,\vec k\int_{s_2}^{s_1}d\eta\Big(\vec\nu(\eta) + \vec v\Big(\vec r + \sqrt{\tfrac{2}{\mu}}\int_0^\eta a\Big)\Big)\Big]\Big\} .$$

Then to the representation (4.3) we may apply the functional integral approximation presented in sect. 2. As a result, we obtain the following expression for the effective potential $\tilde{\tilde V}_{eff}(\vec r)$ (which is a generalization of (2.7)):

$$\tilde{\tilde{V}}_{eff}(\vec r) = V_{eff}(\vec r) + \tilde{\tilde{V}}(\vec r),$$

$$\tilde{\tilde{V}}(\vec r) = \vec v^2(\vec r) + i\sqrt{\frac{2}{m}}\vec P\vec v(\vec r) - \sum_k \int_0^\infty d\sigma \frac{\sigma^2}{2} e^{-\omega_k\sigma}\left[i\sqrt{\frac{2}{m}}\vec k\vec v(\vec r)\right]^2 \times \tag{4.4}$$

$$\times\left[|A_k|^2 \exp\left(-\frac{\vec k^2}{2\mu_1 V^2}F(\sigma)\right) + |B_k|^2 \exp\left(-\frac{\vec k^2}{2\mu_2 V^2}F(\sigma)\right) + A_k B_k^* \exp\left(-i\vec k\vec r - \frac{\vec k^2}{2\mu_1 V^2}F(\sigma)\right) + A_k^* B_k \exp\left(i\vec k\vec r - \frac{\vec k^2}{2\mu_2 V^2}F(\sigma)\right)\right],$$

where V_{eff} is given by (2.7).

For the identical particles the expression (4.4) is simplified:

$$\tilde{\tilde{V}}(\vec r) = \vec v^2(\vec r) + i\sqrt{\frac{2}{m}}\vec P\vec v(\vec r) + \frac{2}{m}\sum_k |A_k|^2 (\vec k\vec v(\vec r))^2 \times \tag{4.5}$$

$$\times (1+\cos\vec k\vec r)\int_0^\infty d\sigma\,\sigma^2 \exp\left[-\omega_k\sigma - \frac{\vec k^2}{2\mu_1 V^2}F(\sigma)\right],$$

where $\vec v(\vec r)$ is defined by the equation

$$\vec v(\vec r) = -i\frac{\vec P}{\sqrt{2m}} - \frac{2}{m}\sum_k |A_k|^2 (1+\cos\vec k\vec r)\,\vec k(\vec k\vec v(\vec r)) \times \tag{4.6}$$

$$\times \int_0^\infty d\sigma\,\sigma^2 \exp\left[-\omega_k\sigma - \vec k^2 f(\sigma)\right],$$

following from (4.2) at $a=0$. In the equation (4.6) we have introduced the cutoff in momenta on the upper limit to remove divergences. The cutoff function $f(\sigma)$ is obtained in the following way. In the limit $r\to\infty$ we have two independent particles therefore in that limit the ratio of the effective to the total mass should be equal to that in the case of one particle.

Just from that condition of coincidence we shall define the function $f(\sigma)$.

Further calculations will be performed for polarons, i.e., with allowing for the

relations (3.1). When $r\to\infty$, in eqs. (4.5) and (4.6) the rapidly oscillating cosine may be neglected and we obtain immediately the equation

$$\vec v(\vec r) = -i\frac{\vec P}{\sqrt{2m}}\left[1 + \frac{a\omega^{3/2}}{6\sqrt{2\pi}\,\mu_1^{3/2}}\int_0^\infty d\sigma\,\sigma^2 e^{-\omega\sigma} f^{-3/2}(\sigma)\right]^{-1}.$$

Inserting it in (4.5) we have

$$\tilde{\tilde{V}}(\vec r) = \vec P^2/2M = \frac{\vec P^2}{2m}\Big\{2\left[1+\frac{a\omega^{3/2}}{6\sqrt{2\pi}\mu_1^{3/2}}\int_0^\infty d\sigma\,\sigma^2 e^{-\omega\sigma} f^{-3/2}(\sigma)\right]^{-1} -$$

$$-\left[1+\frac{a\omega^{3/2}V^3}{3\sqrt{\pi}}\int_0^\infty d\sigma\,\sigma^2 e^{-\omega\sigma}F^{-3/2}(\sigma)\right]\times$$

$$\times\left[1+\frac{a\omega^{3/2}}{6\sqrt{2\pi}\,\mu_1^{3/2}}\int_0^\infty d\sigma\,\sigma^2 e^{-\omega\sigma} f^{-3/2}(\sigma)\right]^{-2}\Big\}.$$

It is clear that the Feynman expression

$$\frac{M}{m} = 1 + \frac{a\omega^{3/2}V^3}{3\sqrt{\pi}}\int_0^\infty d\sigma\,\sigma^2 e^{-\omega\sigma}F^{-3/2}(\sigma) \tag{4.7}$$

is obtained when

$$f(\sigma) = F(\sigma)/2\mu_1 V^2. \tag{4.8}$$

For the relation (4.8) eqs. (4.5), (4.6) take the form

$$\tilde{\tilde{V}}(\vec r) = i\vec P\vec v(\vec r)/\sqrt{2m}, \tag{4.9}$$

where $\vec v(\vec r)$ obeys the equation

$$\vec v(\vec r)\left[1+\frac{a\omega^{3/2}V^3}{3\sqrt{\pi}}\int_0^\infty d\sigma\,\sigma^2 e^{-\omega\sigma}F^{-3/2}(\sigma)\right] = \tag{4.10}$$

$$= -i\frac{\vec P}{\sqrt{2m}} - \frac{a\omega^{3/2}}{2\sqrt{2}\,\pi^2\mu_1^{3/2}}\int d\sigma\,\sigma^2 e^{-\omega\sigma}\int\frac{d\vec k}{\vec k}\vec k(\vec k\vec v(\vec r))\cos\vec k\vec r\cdot\exp\left[-\frac{\vec k^2}{2\mu_1V^2}F(\sigma)\right].$$

In the case of weak coupling, as has been discussed above, the mean value of r is of the order $1/\alpha$, i.e., the term with cosine in (4.10) is exponentially small and for the mass we obtain eq. (4.7) resulting in the following expression

$$M = m[1 + \alpha/6 + O(\alpha^2)],$$

which is well known from the theory of free polarons.

In the case of strong coupling $F(\sigma) \approx V$ and eq. (4.10) is of the form

$$\vec v(\vec r)\frac{32\alpha^4}{81\pi^2} = -i\sqrt{\frac{2}{m}}\vec P - \frac{32\alpha^4}{27\pi^{7/2}}\int\frac{d\vec k}{\vec k^2}\vec k(\vec k\vec v)e^{-\vec k^2}\cos(\vec k\vec r)\sqrt{2\mu_1 V}, \quad (4.11)$$

where V is given by the relation (3.4a).

The solution of eq. (4.11) is searched in the form

$$\vec v(\vec r) = -\frac{i}{\sqrt{2m}}\frac{81\pi^2}{16\alpha^4}[\vec P W_1(r\sqrt{2\mu_1 V}) + \frac{\vec r(\vec r\vec P)}{\vec r}W_2(r\sqrt{2\mu_1 V})]. \quad (4.12)$$

Inserting eq. (4.12) into (4.11) we obtain the functions W_1 and W_2

$$W_1(r) = [1 + \frac{3r}{\sqrt{\pi}}\int_0^1 d\eta\,\eta^2\sqrt{1-\eta^2}\,e^{-r^2\eta^2/4}]^{-1},$$

$$W_2(r) = -W_1(r) + [1 + 3\int_0^1 d\eta\,\eta^2(1 - r^2\eta^2/2)e^{-r^2\eta^2/4}]^{-1}.$$

Then the potential (4.9) takes the form

$$\tilde{\tilde V}(\vec r) = \frac{\vec P^2}{2m}\frac{81\pi^2}{16\alpha^4}[W_1(r\sqrt{2\mu_1 V}) + \frac{(\vec r\vec P)^2}{\vec r^2\vec P^2}W_2(r\sqrt{2\mu_1 V})]. \quad (4.13)$$

Treating the potential (4.13) as a small perturbation we find the first-order correction to the ground state energy (recall that the wave function of the ground state is given by (3.8)):

$$\delta E = \frac{\vec P^2}{2M} = \int d\vec r\frac{|\chi_0(r)|^2}{r^2}\tilde{\tilde V}(\vec r),$$

and obtain the expression for the effective mass

$$\frac{m}{M} = \frac{81\pi^2}{16\alpha^4}\frac{2}{3B(3/2;\lambda-2)}\int_0^\infty dr\frac{\mathrm{sh}^2 r}{\mathrm{ch}^{2(\lambda-1)}r}[3W_1(2xr) + W_2(2xr)],$$

where parameters x and λ are given by (3.12) and (3.13), resp.

The numerical integration produces the following value of M:

$$M = m\frac{16\alpha^4}{81\pi^2}1.161 = 232(\alpha/10)^4 m.$$

As we see, for strong coupling the effective interaction between polarons results in the increase of the mass of the system by 16% as compared to the mass of free polarons. Note that the functional integration method gives underestimated value for the mass therefore the effect of the mass increase can be even larger.

In conclusion we would like to stress that the technique we have developed allows, in principle, the energy and effective mass of the system to be calculated for another dependence of A_k and ω_k on momentum, as well as with the account of the direct interaction of particles. All the required formulae are presented.

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