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# RELATIVITY THEORY IN (2+2)-SPACE

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## 1. INTRODUCTION

Recently the problems involving a conventional character of the concepts of simultaneity and distance have been widely discussed. In particular, the so-called generalized Lorentz transformations* have been considered that apparently take into account the indicated fact by introducing simultaneity parameter $\epsilon$ or space parameter $\epsilon_1$ [6,7] and both quantities in the general case[8]. For $\epsilon = 1/2(\epsilon_1 = 1/2)$ these transformations transit to ordinary Lorentz ones whereas for $\epsilon \neq 1/2$ "time anisotropy"** is allowed which leads to the anisotropy of propagation velocities of physical signals.

Remind that in defining the concepts of simultaneity and distance, the existence of the elements of conventional agreement is associated with the fact that time sending ($t_1$) and arrival ($t_2$) of a reflected light signal are quantities directly measured by the radar method by means of which the indicated concepts are practically introduced. We get rid of the conventional agreement when passing from coordinates t and z*** to the times of sending and return of the light signal. Mathematically this simply means the replacement of variables.

Unhabitualness of the introduced "time description" is the pay for this transition. The advantage of the new description lies in direct measuring and other corresponding kinematical and dynamical quantities. In this case the mathematical language of relativity theory is simplified.

However, in the general case of four-dimensional space such a simple picture already takes no place. On the contrary, the transition to directly measured times makes mathematical apparatus complicated. In addition, the formulation of relativity in the new variables**** has the same (equivalent) right to exist and in a series of cases it may be more convenient. Spe-

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*See [1-4] and also [5] where references to other papers on this subject can be found.

**Space anisotropy for $\epsilon_1 \neq 1/2$.

***In the frame of (1+1)-space.

****Taking into account a similar replacement for another pair of coordinates.

cial Lorentz transformations and quantum mechanics of particles having spin 1/2 are the examples of this.

## 2. TRANSITION TO QUANTITIES DIRECTLY MEASURED

In the above radar experiment the coordinates

$$t = \frac{t_1 + t_2}{2}, \quad z = \frac{t_2 - t_1}{2} \tag{1}$$

are assigned to time t of the reflected light signal and distance z to the point of reflection.

To transit to directly measured quantities $t_1$ and $t_2$, denoted below by $x^1$ and $x^2$, replace the variables*

$$x^1 \equiv t_1 = t - z, \quad x^2 \equiv t_2 = t + z. \tag{2}$$

For the interval squared we find

$$d\tau^2 = dt - dz = dx^1 dx^2 = g_{ik} dx^i dx^k, \tag{3}$$

i.e.

$$g_{11} = g_{22} = 0 \quad \text{and} \quad g_{12} = g_{21} = 1/2.$$

According to (2), for components of covariant velocity we get

$$u^1 = \frac{dt}{d\tau} - \frac{dz}{d\tau}, \quad u^2 = \frac{dt}{d\tau} + \frac{dz}{d\tau} \tag{4}$$

or

$$u^1 = \frac{dx^1}{d\tau} = \sqrt{\frac{dx^1}{dx^2}}, \quad u^2 = \frac{dx^2}{d\tau} = \sqrt{\frac{dx^2}{dx^1}} = (u^1)^{-1}. \tag{4'}$$

Taking (4) and (4') into account, instead of energy E and momentum p for a body with mass m we have

$$p^1 = E - p = mu^1, \quad p^2 = E + p = mu^2 \tag{5}$$

whence

$$p^1 p^2 = (m)^2. \tag{5'}$$

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*Some time ago these variables were considered by Kotel'nikov /9/ and Dirac /10/, see also /11/.

It is important to note that $u^1$ and $u^2$, as well as $x^1$ and $x^2$, are directly observed quantities. In fact, from an experiment on measuring Doppler frequency $\omega$ we find

$$u^1 = \omega/\omega^\circ \quad \text{or} \quad u^2 = \omega/\omega^\circ. \tag{6}$$

Knowing m, we can find $p^1$ and $p^2$.

Taking (2) and (5) into account, for invariant action S we have

$$S = Et - pz = \frac{1}{2}(p^1x^2 + p^2x^1). \tag{7}$$

In the frame of our approach let us introduce quantity (v) which corresponds to usual velocity of motion $\beta$. The quantity introduced describes the change of one coordinate relative to another and is expressed through $\beta$

$$v = \frac{dx^2}{dx^1} = \frac{1+\beta}{1-\beta}. \tag{8}$$

It is evident that v can vary from $1(\beta = 0)$ to $\infty(\beta = 1)$, i.e. the velocity of light is an infinite value as for limiting transition to classics. If the sign of $\beta$ is variable, i.e. the direction of motion reverses, instead of (8) we have

$$v' = \frac{1-\beta}{1+\beta} = v^{-1}. \tag{8'}$$

Thus, in the opposite direction the velocity varies from 1 to 0. In its physical sense v is equal to the ratio

$$v = \frac{t^b + t^\ell}{t^b - t^\ell}, \tag{9}$$

where $t^\ell$ ($t^b$) is the time during which light (a material body) travels some space cut back and forth ($t^\ell = t_2 - t_1$).

## 3. TRANSFORMATIONS FOR COORDINATES. THE "ADDITION" THEOREM FOR VELOCITIES

Using (2) and (8), special Lorentz transformations for the coordinates are written as

$$x^{1'} = v^{1/2}x^1, \quad x^{2'} = v^{-1/2}x^2 \tag{10}$$

or

$$x^1 = ux^{1'}, \quad x^2 = u^{-1}x^2, \tag{10'}$$

where $u = u^1$.

The corresponding transformations for $p^1$ and $p^2$ take a similar form. Taking the known substitution $\beta =$ th$\alpha$ into account, eq. (10) can be presented as

$$x^{1'} = e^{\alpha}x^1, \quad x^{2'} = e^{-\alpha}x^2. \tag{10''}$$

Combining (10') with their particular (U) Lorentz transformations

$$x^{1'} = Ux^{1''}, \quad x^{2'} = U^{-1}x^{2''}, \tag{11}$$

we are led to the "addition" theorem for velocities

$$w = uU \quad \text{(multiplication rule)} \tag{12}$$

with w the velocity of the resulting transformation which takes the place of the two initial ones.

## 4. TIME DECRIPTION OF EVENTS IN (1+2)-SPACE

In a more general case of (1+2)-space the coordinates of reflection of a radar signal are expressed, e.g., as

$$t = \frac{1}{2}(t_{21} + t_1), \quad r = \frac{1}{2}(t_{21} - t_1), \quad r + x + z = t_{22} - t_1. \tag{13}$$

Here the first index denotes as before the sending or arrival of the signal;and the second one,its ordinal number. The reflected signal is devided into two. The latter goes in the direction of axes x and z to the starting point. The expressions for x and z calculated using eqs. (13) are extremely cumbersome. Therefore already in this case the transition to the times directly observed makes mathematical apparatus greatly involved. Measuring the coordinates z (instead of r), we have

$$z = \frac{1}{2}(t_2 - t_{12}), \quad r + x + z = t_2 - t_{11}, \quad t = \frac{t_2^2 + t_{12}^2 - 2t_{11}^2}{2(t_2 + t_{12} - 2t_{11})}. \tag{14}$$

Already from the expression for t one can see that the transition to the time coordinates is hardly reasonable.

Thus, only in the particular case ($t_{22} = t_{21}$ and $t_{12} = t_{11}$, i.e. when the point of reflection lies on axis z) the considered replacement of the variables is related to the transition to the quantities observed. As far as the special Lorentz transformations is concerned, in this case they are simple in form (10) as before.

On the other hand, as any space coordinate can be expressed through time, the common use of one time and three space coordinates cannot be considered preferable to another description (equivalent to it). The only criterion appears to be only simplicity and convenience of this or that approach.

## 5. TRANSITION TO (2+2)-SPACE

Rewriting formulae (2) in the form

$$x^1 = -z + ix_4, \quad x^2 = z + ix_4, \tag{2'}$$

where $x_4 = -it$, introduce

$$x^3 = -x + iy, \quad x^4 = x + iy \tag{15}$$

by analogy to (2'). In this case for the interval squared we find

$$d\tau^2 = dx^1 dx^2 + dx^3 dx^4 = g_{ik} dx^i dx^k, \tag{16}$$

i.e. the matric tensor $g_{ik}$ takes the form

$$2g_{ik} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{17}$$

Further on we shall conditionally say that (2+2)-space is defined by the quadratic form (16).

Geometrically, for example, in the plane $x^3 x^4$ of Cartesian coordinates $x^3$ and $x^4$ the square of a distance is equal to the area of a rectangle with sides $x^3$ and $x^4$. In so doing, a set of points corresponding to constant distance "a" is given by equilateral hyperbola $x^4 = (a)^2/x^3$. Now the rotation through angle $\alpha$ in the plane xy evidently corresponds to the transformations

$$x^{3'} = e^{i\alpha} x^3, \quad x^{4'} = e^{-i\alpha} x^4$$

which are similar to (10′′).

The scalar product of two vectors X and Y is defined by the expression

$$XY = \frac{1}{2}(X^1Y^2 + X^2Y^1 + X^3Y^4 + X^4Y^3) = \frac{1}{2} X^i Y_i . \tag{18}$$

According to (4′), let us introduce $u^3$ and $u^4$. Components $u^i$ are expressed through components of usual velocity $\beta_\alpha$:

$$\begin{aligned} &u^1 + u^2 = 2\gamma, \quad u^2 - u^1 = 2\beta_z\gamma, \\ &u^4 - u^3 = 2\beta_x\gamma, \quad u^3 + u^4 = 2i\beta_y\gamma, \end{aligned} \tag{19}$$

where $\gamma = (1 - \beta_\alpha^2)^{-1/2}$, and they satisfy the relation

$$u^1u^2 + u^3u^4 = 1. \tag{20}$$

Taking (19) into account, for general Lorentz transformations we get

$$\begin{aligned} x^1 &= [(\tilde{u}^1)^2 x^{1'} + u^3u^4x^{2'} - \tilde{u}^1(u^4x^{3'} + u^3x^{4'})]\tilde{\gamma}, \\ x^2 &= [u^3u^4x^{1'} + (\tilde{u}^2)^2 x^{2'} - \tilde{u}^2(u^4x^{3'} + u^3x^{4'})]\tilde{\gamma}, \\ x^3 &= [\tilde{u}^1u^3x^{1'} + \tilde{u}^2u^3x^{2'} + \tilde{u}^1\tilde{u}^2x^{3'} - (u^3)^2x^{4'}]\tilde{\gamma}, \\ x^4 &= [\tilde{u}^1u^4x^{1'} + \tilde{u}^2u^4x^{2'} - (u^4)^2x^{3'} + \tilde{u}^1\tilde{u}^2x^{4'}]\tilde{\gamma} \end{aligned} \tag{21}$$

where $\tilde{u}^{1,2} = u^{1,2} \pm 1$, $\tilde{\gamma} = (\tilde{u}^1 + \tilde{u}^2)^{-1}$.

Using (10′), special relativistic formulae of transformation for components of a second rank antisymmetric tensor $F^{ik}$ take the simple form

$$\begin{aligned} &F^{12} = F^{12'}, \ F^{13} = uF^{13'}, \ F^{14} = uF^{14'}, \\ &F^{34} = F^{34'}, \ F^{23} = u^{-1}F^{23'}, \ F^{24} = u^{-1}F^{24'}. \end{aligned} \tag{22}$$

For electromagnetic field $F^{ik}$ is expressed through intensities $\vec{E}$ and $\vec{H}$ of electric and magnetic fields by means of the equalities

$$F^{12} = E_z, \ F^{13} = -E_z + H_y + i(E_y + H_x),$$

$$F^{14} = E_x - H_y + i(E_y + H_x), \quad F^{34} = iH_z, \tag{23}$$

$$F^{23} = -E_x - H_y + i(E_y + H_x), \quad F^{24} = E_x + H_y + i(E_y + H_x).$$

The form of Maxwell's equations in the new variables is undoubtedly unchangeable.

## 6. DIRAC EQUATION IN (2+2)-SPACE

The variables being considered can be assumed "natural" for the Dirac equation. Thus, in these variables each of four equations has only three terms instead of five. If the Dirac equation is presented by matrices $\gamma^i$ in the common form

$$(i\gamma^i \frac{\partial}{\partial x^i} - m)\psi = 0, \tag{24}$$

we have

$$\gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$\gamma^3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad \gamma^4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{25}$$

In this case $\psi$ is expressed through the former wave function $\Psi$ by the equalities

$$\psi_{\substack{1\\2}} = \Psi_1 \mp \Psi_3, \quad \psi_{\substack{3\\4}} = \Psi_2 \mp \Psi_4. \tag{26}$$

Note that new matrices $\gamma^i$ have only two elements different from zero. As before, they satisfy the following communication relations

$$\gamma^i\gamma^k + \gamma^k\gamma^i = 2g^{ik}I. \tag{27}$$

For bilinear scalar form we have

$$s = \psi^*(\gamma^1 + \gamma^2)\psi = \bar{\psi}\psi. \tag{28}$$

where $\bar{\psi}$ is the function "relativistically adjoint" of $\psi$.

Using (2) and (15), introduce the 4-vector of probability current density for particles with spin 1/2. We obtain

$$
\begin{aligned}
j^1 &= j^t - j^z = \psi_2^* \psi_2 + \psi_3^* \psi_3 = \bar{\psi} \gamma^1 \psi, \\
j^2 &= j^t + j^z = \psi_1^* \psi_1 + \psi_4^* \psi_4 = \bar{\psi} \gamma^2 \psi, \ldots
\end{aligned}
\tag{29}
$$

i.e. as before

$$
j^k = \bar{\psi} \gamma^k \psi \tag{29'}
$$

although each expression of (29) has only two terms instead of four.

As for special relativistic transformations for components of bispinor wave function $\psi$, they are of the following simple form:

$$
\psi_{1,4} = u^{-1/2} \psi'_{1,4}, \quad \psi_{2,3} = u^{1/2} \psi'_{2,3}. \tag{30}
$$

Hence, using (29), it is easy to get the formulae of transformation for 4-vector components

$$
j^1 = u j^{1'}, \; j^2 = u^{-1} j^{2'}, \; j^{3,4} = j^{3,4'}. \tag{31}
$$

It should be also noted that spinors are now related to 4-vectors by the following simple formulae

$$
\begin{aligned}
p_{1\dot{1}} &= p^{2\dot{2}} = p_1, \quad p_{2\dot{2}} = p^{1\dot{1}} = p_2, \\
p_{2\dot{1}} &= -p^{12} = p_3, \quad -p_{1\dot{2}} = p^{21} = p_4.
\end{aligned}
\tag{32}
$$

Taking into account that in spinor representation

$$
\eta = \begin{pmatrix} \psi_1 \\ \psi_3 \end{pmatrix}, \quad \xi = \begin{pmatrix} \psi_2 \\ \psi_4 \end{pmatrix}, \tag{33}
$$

where two-component quantities

$$
\eta = \begin{pmatrix} \eta^{\dot{1}} \\ \eta^{\dot{2}} \end{pmatrix} \quad \text{and} \quad \xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \tag{34}
$$

are dotted contravariant and nondotted covariant spinors, let us write the Dirac equation in the form

$$\hat{p}^{\dot{\alpha}\beta}\xi_{\beta} = m\eta^{\dot{\alpha}}, \quad \hat{p}_{\alpha\dot{\beta}}\eta^{\dot{\beta}} = m\xi_{\alpha}. \tag{35}$$

Here, using (32), operational spinors $\hat{p}^{\dot{\alpha}\beta}$ and $\hat{p}_{\alpha\dot{\beta}}$ correspond to operational vector $\hat{p}_j = i\partial_i$ .

On the other hand, with the help of $\hat{p}_j$ eqs.(35) can be written as

$$\hat{p}_i\sigma^i\xi = m\eta, \quad \hat{p}_i\sigma_i\eta = m\xi. \tag{36}$$

Using (2) and (15), e.g., "components of contravariant matrix 4-vector" are expressed through Pauli's matrices

$$2\sigma^1 = I - \sigma_z, \quad 2\sigma^2 = I + \sigma_z, \quad 2\sigma^3 = -\sigma_x + i\sigma_y, \quad 2\sigma^4 = \sigma_x + i\sigma_y, \tag{37}$$

where I is the unit matrix which takes the form of known projection operators

$$\sigma^1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad \sigma^4 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{38}$$

## 7. CONCLUSION

As previously noted, in the frame of (1+1)-space not only the time but also the space coordinate is commonly introduced using conventional agreements. The transition to quantities directly measured in experiments (the time of sending and arrival of a radar signal) makes it possible to get rid of the above conventions. As a result, (special) Lorentz transformations, the "addition" theorem for velocities and so on are simplified.

In the general case, however, the transition to times directly observed complicates the mathematical apparatus of the theory. In addition, the considered formulation of special relativity in noneuclidean space is equivalent to the conventional one. It appears more convenient to describe spinor quantities, e.g. the Dirac equation.

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