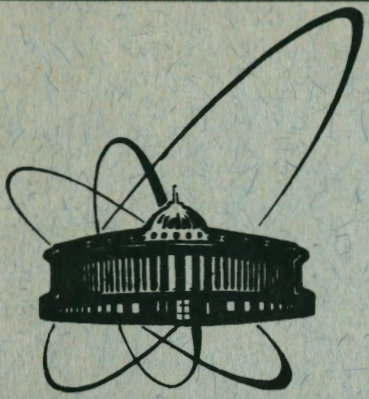

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NON-PERTURBATIVE GREEN FUNCTIONS IN QUANTUM GAUGE THEORIES

1991

1 Green functions in a quantum theory

In a quantum field theory vacuum expectations $\langle\hat{\varphi}_{a_1}(x_1)\cdots\hat{\varphi}_{a_n}(x_n)\rangle_0$ where $\hat{\varphi}_a(x)=\hat{\varphi}_a(\mathbf{x},t)=\exp(i\hat{H}t)\hat{\varphi}_a(\mathbf{x})\exp(-i\hat{H}t)$ is a field operator (a enumerates components of φ), $\hat{H}$ is a Hamiltonian determining an evolution of $\hat{\varphi}$, are usually considered. In gauge theories, i.e., when there exist first-class constraints [1], φ contains physical (gauge invariant) variables as well as unphysical ones. Nevertheless, only the first of them have a physical meaning. However, a gauge invariant description is based on a use of curvilinear coordinates which may take their values on a part of a real axis. So the corresponding to them configuration and phase spaces seem to be reduced, i.e. they differ from corresponding Euclidean spaces [2,3]. The latter leads to a modification of a Hamiltonian path integral [4-7] and quasiclassical calculations [8].

In the present work, the influence of this phenomenon on Green functions is investigated. In order to explain main points, the analysis starts from a simple quantum mechanical example (Sections 2 and 3) and then the Yang-Mills system is considered (Sec.4).

In quantum mechanics the quantity $\langle T(\hat{q}_a(t)\hat{q}_{a'}(0)\rangle_0$ where $\hat{q}_a(t)=\hat{U}_t^+\hat{q}_a(0)\hat{U}_t$, $\hat{q}_a(0)\equiv\hat{q}_a$, $\hat{U}_t=\exp(-i\hat{H}t)$ and T means a time ordering, is analogous to $\langle T(\hat{\varphi}_a(x)\hat{\varphi}_{a'}(x'))\rangle_0$ in a field theory. For example, consider a system containing Bose- and Fermi- oscillators

$$\hat{H}=\hat{b}^+\hat{b}+\hat{f}^+\hat{f} \tag{1.1}$$

where $[\hat{b},\hat{b}^+]=[\hat{f},\hat{f}^+]_+=1$. Then $\hat{q}=(\hat{b}^+ +\hat{b})/\sqrt{2}$ and we find [7]

$$D_b(t)=\langle T(\hat{q}(t)\hat{q})\rangle_0=\frac{1}{2}\theta(t)e^{-it}+\frac{1}{2}\theta(-t)e^{it}, \tag{1.2}$$

$$D_f(t)=\langle T(\hat{f}(t)\hat{f}^+)\rangle_0=\theta(t)e^{-it}. \tag{1.3}$$

It is easy to check that the functions (1.2) and (1.3) satisfy the following equations

$$\left(-\partial_t^2-1\right)D_b(t)=\left(i\partial_t-1\right)D_f(t)=i\delta(t) \tag{1.4}$$

which define the Green functions of Bose- and Fermi-oscillators. Moreover, their Fourier transforms $D(\omega)=\int_{-\infty}^{\infty}dt\exp(-i\omega t)D(t)$ have the well-known form

$$D_b(\omega)=i(\omega^2-1+i\varepsilon)^{-1},\qquad D_f(\omega)=-i(\omega+1-i\varepsilon)^{-1} \tag{1.5}$$

$\varepsilon\to 0$ and $\varepsilon>0$. So, the rules of calculations of Green functions in both quantum mechanics and quantum field theory are identical. We use this to find a modification of these rules when PCS is reduced.

2 A quantum mechanical example

Consider a two-dimensional SUSY-oscillator with a gauge group. The Lagrangian reads

$$L=\frac{1}{2}\left(\partial_t\mathbf{x}-yT\mathbf{x}\right)^2+i\psi^*\left(\partial_t-i\Gamma\right)\psi-\frac{1}{2}\mathbf{x}^2-\psi^*\psi. \tag{2.1}$$

Here a real scalar y and two-component columns $\mathbf{x}$ and ψ composed of real variables x_a and complex Grassman variables ψ_a, respectively, form dynamical degrees of freedom, $T = -i\tau_2$, $\Gamma = \tau_3$ ($\tau_{2,3}$ are Pauli matrices). Lagrangian (2.1) is invariant under gauge transformations from $SO(2)$

$$\mathbf{x} \to \exp(\omega T)\mathbf{x}, \quad \psi \to \exp(i\omega\Gamma)\psi, \quad y \to y - \dot{\omega}. \tag{2.2}$$

Let us turn now to the Hamiltonian formalism. The Hamiltonian of the system has the form

$$H = \frac{1}{2}\mathbf{p}^2 + \frac{1}{2}\mathbf{x}^2 + \psi^*\psi - y\sigma \tag{2.3}$$

where $\sigma = \mathbf{p}T\mathbf{x} + \psi^*\Gamma\psi$, $\mathbf{p}$ is a canonical moment conjugated to $\mathbf{x}$. Note that the system contains second-class constraints [1] for Grassman variables which one may eliminate changing Poisson brackets $\{,\}$ by Dirac ones $\{,\}_D$ [9]. Then the variables ψ_a^* and ψ_a seem to be canonically conjugated $\{\psi_a^*, \psi_b\}_D = \{\psi_b, \psi_a^*\}_D = -i\delta_{ab}$. The system has also two first-class constraints because of the gauge symmetry, namely, the primary one $\pi = \partial L/\partial\dot{y} = 0$ and the secondary one $\{\pi, H\} = \sigma = 0$.

A quantization is fulfilled in the usual way $\{,\} \to -i[,]$ and

$$[\hat{x}_a, \hat{p}_b] = i\delta_{ab}, \quad [\hat{\psi}_a, \hat{\psi}_b^+]_+ = \delta_{ab} \tag{2.4}$$

and operator $\hat{\sigma}$ must annihilate physical states

$$\hat{\sigma}|\Phi\rangle = 0 \tag{2.5}$$

(we may ignore y in the quantum theory since it is the Lagrangian multiplier in (2.3)).

The quantum problem may be easily solved by introducing operators $\hat{b}_a = (\hat{x}_a + i\hat{p}_a)/\sqrt{2}$ and $\hat{b}_a^+$. However, for the following generalizations to a field theory, one needs to solve Eq.(2.5) in the coordinate representation. We realize algebra (2.4) on functions $\Phi = \Phi(\mathbf{x}, \theta)$ where θ_a are complex Grassman variables, i.e. $\hat{p}_a\Phi = -i\partial/\partial x_a\Phi$, $\hat{x}_a\Phi = x_a\Phi$, $\hat{\psi}_a^+\Phi = \theta_a\Phi$ and $\hat{\psi}_a\Phi = \partial/\partial\theta_a\Phi$ (a left derivative). In this representation the scalar product reads

$$\langle\Phi_1|\Phi_2\rangle = \int_{R^2} d\mathbf{x} \int d\theta d\theta^* \exp(-\theta\theta^*)(\Phi_1(\mathbf{x}, \theta))^*\Phi_2(\mathbf{x}, \theta) \tag{2.6}$$

where $d\theta d\theta^* = \prod_a d\theta_a d\theta_a^*$ and by definition $(c\theta_1\theta_2)^* = c^*\theta_2^*\theta_1^*$. Eq.(2.5) means that physical states are gauge invariant

$$\Phi(\exp(\omega T)\mathbf{x}, \exp(-i\omega\Gamma)\theta) = \Phi(\mathbf{x}, \theta) \tag{2.7}$$

since $\hat{\sigma}$ is a generator of transformations (2.2).

In order to solve Eq.(2.5), we introduce curvilinear coordinates on a superspace $(\mathbf{x}, \theta)$ [10]

$$\mathbf{x} = \exp(\varphi T)\rho, \qquad \theta = \exp(-i\varphi\Gamma)\xi \tag{2.8}$$

where ρ is a column $(r, 0)$. Variables r and ξ_a are gauge invariant and φ is translated under gauge transformations $\varphi \to \varphi + \omega$. Therefore in the new variables $\hat{\sigma} = -i\partial_\varphi$ is the generator of translations of φ. One may directly check this rewriting $\partial/\partial x_a$ via ∂_r, ∂_φ and $\hat{\sigma}_F = \hat{\psi}^+\Gamma\hat{\psi} = \hat{\xi}^+\Gamma\hat{\xi}$ ($\hat{\xi}_a = \delta/\delta\xi_a$). The first equality in (2.8) express a passage to usual polar coordinates on a plane. However, the change of variables on the Grassman algebra contains φ. For this reason $\partial_\varphi - i\hat{\sigma}_F$ appears instead of ∂_φ in $\partial/\partial x_a$ written in the usual polar coordinates. Thus, functions $\Phi(r, \varphi, \xi) = \Phi(r, \xi)$ satisfy (2.5).

Rewriting the Laplace operator in the quantum Hamiltonian (2.3), we get the Schroedinger equation in the physical subspace $\mathcal{H}_{ph}$

$$\left(-\frac{1}{2}\partial_r^2 - \frac{1}{2r}\partial_r + \frac{1}{2r^2}\hat{\sigma}_F^2 + \frac{1}{2}r^2 + \hat{\xi}^+\hat{\xi} - 1\right)\Phi_E = E\Phi_E \tag{2.9}$$

(when passing to the quantum theory, it is necessary to order the operators $\psi^*\psi \to \hat{\psi}^+\hat{\psi} - 1 = \hat{\xi}^+\hat{\xi} - 1$).

For solving (2.9) we split $\mathcal{H}_{ph}$ into four orthogonal subspaces $\mathcal{H}^{(0)} \ni \Phi_E^{(0)}(r)$, $\mathcal{H}^{(a)} \ni \Phi_E^{(a)} = \xi_a F_E^{(a)}(r)$ and $\mathcal{H}^{(3)} \ni \Phi_E^{(3)} = \xi_1\xi_2 F_E^{(3)}(r)$. The new scalar product in $\mathcal{H}_{ph}$ induced by (2.6) reads

$$\langle\Phi_1|\Phi_2\rangle = \int_0^\infty dr r \int d\xi d\xi^* e^{-\xi\xi^*}(\Phi_1(r,\xi))^*\Phi(r,\xi) \tag{2.10}$$

(we include the constant $\int_0^{2\pi} d\varphi$ into a norm of Φ_{ph}). Since $\hat{\sigma}_F 1 = \hat{\sigma}_F\xi_1\xi_2 = 0$ and $\hat{\sigma}_F\xi_a = (\Gamma\xi)_a$ we may solve (2.9) for every $\mathcal{H}^{(\alpha)}$ ($\alpha = 0,1,2,3$) independently. These solutions normalized with respect to (2.10) and regular at $r = 0$ (see [11]) have the following form

$$\Phi_n^{(0)} = \frac{\sqrt{2}}{n!}L_n(r^2)e^{-1/2r^2}, \quad E_n^{(0)} = 2n; \tag{2.11}$$

$$\Phi_n^{(a)} = \frac{\sqrt{2}}{n!\sqrt{n+1}}r\xi_a L_n^1(r^2)e^{-1/2r^2}, \quad E_n^{(a)} = 2n+2; \tag{2.12}$$

$$\Phi_n^{(3)} = \xi_1\xi_2\Phi_n^{(0)}, \quad E_n^{(3)} = 2n+2; \tag{2.13}$$

where $n = 0,1,\ldots$. The spectrum consists of equidistant levels and each level is four-degenerated except of the vacuum which seems to be singlet. Note that the frequency of physical oscillations is doubled. It testifies that the physical phase space of the system is reduced [2-4].

Functions (2.11)-(2.13) may be rewritten in the explicit gauge invariant form by using the equalities $r^2 = x^2$, $\xi_1\xi_2 = \theta_1\theta_2$, $r\xi_1 = z\theta_1$ and $r\xi_2 = z^*\theta_2$ where $z = x_1 + ix_2$ left-hand sides of which are gauge invariant (under gauge transformations $z \to \exp(i\omega)z$). Thus found functions satisfy both the Schroedinger equation with the quantum Hamiltonian (2.3) and (2.5) written in the initial "Cartesian" operators (2.4).

Let us turn now directly to a calculation of the Green function for physical variables $D_b(t) = \langle T(\hat{r}(t)\hat{r}(0))_0$ and $D_f^{ab} = \langle T(\hat{\xi}_a(t)\hat{\xi}_b(0))_0$. Simple calculations like (1.2) and (1.3) show us that their Fourier transforms read as

$$D_b(\omega) = \sum_{n=0}^{\infty}\frac{\Gamma^2(n-1/2)}{4n!^2}\qquad\frac{in}{\omega^2 - 4n^2 + i\epsilon} \tag{2.14}$$

$$D_f^{ab} = \delta^{ab}\sum_{n=0}^{\infty}\frac{\Gamma^2(n+1/2)}{4n!^2(n+1)}\qquad\frac{-i}{\omega + 2n + 2 - i\epsilon} \tag{2.15}$$

In accordance with the De Morgan theorem [12] series (2.14) and (2.15) are absolutely convergent and define analytical functions on a complex plane of ω with simple poles. In contrast with (1.5) distributions into Green functions give all eigenstates of the Hamiltonian and moreover $D_b(t)$ and $D_f^{ab}(t)$ do not satisfy Eqs.(1.4). The reason just finds shelter in that the PCS is reduced. Indeed, there should be $r > 0$ and $\varphi \in [0, 2\pi)$ in (2.8) so that (2.8) could be a change of variables, and, as a consequence, the integration region in (2.10) of the variable r turns out to be a semiaxis. The latter automatically means that the amplitudes $\langle 0|\hat{r}|\Phi_n^{(0)}\rangle$ and $\langle 0|\hat{\xi}_a|\Phi_n^{(a)}\rangle$ do not vanish at all n, but their squares, in fact, determine coefficients in series (2.14) and (2.15), respectively [1].

[1]Note that an anharmonic addition to the potential ($L \to L - V$ in (2.1)) does not influence our conclusion because V is always a function of gauge invariant objects $x^2, \theta_1\theta_2, z\theta_1$ and $z^*\theta_2$ (the Lagrangian must be gauge-invariant).

Thus, we see that a modification of Green functions in a quantum theory is conditioned by a PCS reduction. Therefore the questions when this phenomenon takes place and how it is connected with the choice of physical variables in a theory are, undoubtedly, interesting. These questions will be discussed in the next section.

3 A PCS reduction and a residual gauge symmetry in a non-invariant approach

Note first of all that states (2.11)-(2.13) are invariant under the discrete group $S = Z_2$

$$S: \quad 1)\ r \to r,\ \xi_a \to \xi_a; \qquad 2)\ r \to -r,\ \xi_a \to -\xi_a. \tag{3.1}$$

This group, in a sense, seems to be a memory (an original code) of the reduced system (in which all unphysical degrees of freedom are eliminated) about its gauge origin. Indeed, a gauge invariant state is an analytical function of gauge invariant polynomials of $\mathbf{x}$ and θ. All gauge invariant polynomials can be constructed with the help of four generating elements $\mathbf{x}^2$, $\theta_1\theta_2$, $z\theta_1$ and $z\theta_2$.

On the other hand, as is shown in the previous section, there exists a one-to-one correspondence between these generating elements and generating elements of S-invariant polynomials. So the S-invariance of states written in physical variables means their gauge invariance in the whole configuration space (see also [6,10]).

Using this symmetry, we may continue functions from $\mathcal{H}_{ph}$ to the unphysical region $r < 0$ and one may change $\int_0^\infty dr r \to \int_{-\infty}^\infty dr|r|$ in (2.8) keeping orthogonality of the basis in $\mathcal{H}_{ph}$. However states $\hat{r}\Phi_E = r\Phi_E$ and $\hat{\xi}_a^+\Phi_E = \xi_a\Phi_E$ appearing in a calculation of the Green functions (2.14) and (2.15) are not S-invariant. Therefore, if we analytically continue these states to the unphysical region, then we get obvious equalities $D_f^{ab} = D_b = 0$. It means that acting on a physical state by the operators $\hat{r}$ and $\hat{\xi}_a$ we take out this state from the physical subspace. The correct correspondence to the quantum theory found in Sec.2 appears when continuing states $\hat{r}\Phi_E$ and $\hat{\xi}_a^+\Phi_E$ to be S-invariant, i.e. as $|r|\Phi_E$ and $\epsilon(r)\xi_a\Phi_E$, respectively, where $\epsilon(r)$ is the sign function. Thus, we conclude that the reduction of PCS may be ignored in the scalar product if we change all variables in a configuration space by their S-invariant continuations

$$r \to r^Q \equiv \hat{Q}r = \sum_S \theta(\hat{s}r)\hat{s}r = |r| \tag{3.2}$$

$$\xi_a \to \xi_a^Q \equiv \hat{Q}\xi_a = \sum_S \theta(\hat{s}r)\hat{s}\xi_a = \epsilon(r)\xi_a \tag{3.3}$$

where $\theta(r)$ is the Heviside function (the function of a region of physical values for r), the summation is carried out over all elements of the group S (3.1).

The variables r^Q and ξ_a^Q cannot describe elementary excitations. In fact, they are viewed as coherent states of "elementary" physical excitations described by the S-invariants r^2 and $r\xi_a$. Just this gives rise to poles in (2.14) and (2.15) from all physical eigenstates of the Hamiltonian (2.11)-(2.13). However, the physical excitations r^2 and $r\xi_a$ (their wave functions are (2.11) and (2.12) at $n = 1$, respectively) are also composed. The first of them may be considered as a simultaneous excitation of two bosons and the second is viewed as a composed state of both a boson and a fermion. Constraints just give rise to a "nonlocality" of the simplest physical excitations in the theory. Thus, the description of dynamics of the composed gauge invariant objects r^2 and $r\xi_a$ looks, in a sense, simpler than the one for r and ξ_a with a reduced PCS.

Let the first equality in (2.8) define the mapping $(r,\varphi) \subset R^2 \to \mathbf{x} \in R^2$. Consider a symmetry group of the mapping (2.8) $\tilde{S}$, i.e. such transformations of r, φ and ξ under which $\mathbf{x}$ and θ do not change. Apparently, $\tilde{S}$ consists of two subgroups $\tilde{S} = S_\varphi \times S$, $S_\varphi: \varphi \to \varphi + 2\pi n$, n is a number,

and r, ξ are not changed, under S-transformations $\varphi \to \varphi + \pi$ and r, ξ are transformed in accordance with (3.1). Points of the plane (r, φ) connected with the help of $\tilde{S}$-transformations correspond to the same point $\mathbf{x}$ of the plane R^2. In order to convert the mapping (2.8) into a change of variables, one should reduce the plane (r, φ) in the subset $R^2/\tilde{S} = R(\varphi)/S_\varphi \bigcup R(r)/S = [0, 2\pi) \bigcup [0, \infty)$. As a result the group S defined in (3.1) may be also found as a subgroup of the symmetry group of a mapping separating variables of a theory into physical and unphysical ones.

Consider the total configuration space of $\mathbf{x}$. According to (2.2) circles with centers at $\mathbf{x} = 0$ compose gauge group orbits. The gauge invariant description is fulfilled by passing to the orbit space which is described by the variable $r = (\mathbf{x})^{1/2}$. There is another way called non-invariant when only one representative of each orbit is picked out by fixing the gauge arbitrariness. Practically, it means that we fix a law of time evolution of the unphysical variable φ in (2.8), i.e. we must choose a concrete form of the Lagrangian multiplier y since $\dot{\varphi} = \{\varphi, H\} = -y$ (σ is the momentum canonically conjugated to φ). The simplest case $\varphi = 0$ corresponds to a gauge $x_2 = 0$, hence a particle moves along the first axis. However, the line $x_2 = 0$ intersects every orbits twice therefore points x_1 and $-x_1$ are gauge equivalent. They correspond to the same physical state of a particle. Gauge transformations $x_1 \to \pm x_1$ induce gauge transformations of the fermion variable $\xi_a \to \pm \xi_a$ (the rotation through the angle π). Thus, the residual gauge group appearing as a consequence of a gauge condition incompleteness coincides with S, but one should emphasize that these groups are different in nature. They are identical only for a special change of variables corresponding to both a gauge transformation law and a chosen gauge condition.

One may determine a gauge in the form of a line $\chi(x_1, x_2) = 0$. It must intersect every orbit at least once. We parametrize this line as $x_a = f(u)$ and u runs over the real axis R when a particle passes along the whole line $\chi = 0$. In this case the residual gauge symmetry, generally speaking, does not form a group (a subgroup of the gauge group as Z_2 in the previous case), but it may have a highly intricate form because the line may intersect each orbit several times and moreover a number of intersections may depend on u. Indeed, corresponding gauge transformations satisfy the following condition

$$\mathbf{x}(u) \to \exp(T\varphi_s(u))\mathbf{x}(u) = \mathbf{x}(u_s), \qquad u_s = u_s(u) \tag{3.10}$$

since the points $\mathbf{x}(u)$ and $\mathbf{x}(u_s)$ must belong to the curve $x_a = f_a(u)$. Consider three points of that type, $\mathbf{x}(u)$, $\mathbf{x}(u_s)$ and $\mathbf{x}(u_{s'})$, being on the same gauge orbit. Then with a general choice of $f(u)$ the group composition of two residual gauge transformations (3.1) $\mathbf{x}(u) \to \mathbf{x}(u_s)$ and $\mathbf{x}(u_s) \to \mathbf{x}(u_{s'})$ does not define a new residual gauge transformation, i.e. there is no point on the curve $x_a = f(u)$ at which $\mathbf{x}(u)$ could pass under this new transformation. This simple example gives a good illustration for the Yang Mills system [7] where for any chosen gauge condition the residual gauge symmetry does not also form a subgroup of a gauge group [13] (see also Sec.4).

The invariant description connected with a chosen gauge condition may be found when introducing curvilinear coordinates of the following form (instead of (2.8)) [6,7,10]

$$\mathbf{x} = \exp(\varphi\Gamma)\mathbf{f}(u), \quad \theta = \exp(-i\varphi\Gamma)\xi \tag{3.5}$$

where $\mathbf{f}$ is the column (f_1, f_2). For these variables $\dot{\sigma} = -i\partial_\varphi$, i.e. u and ξ are gauge invariant variables. Rewriting the Laplace-Beltrami operator in coordinates (3.5) and omitting in it terms with ∂_φ, we get the physical quantum Hamiltonian. In contrast with (2.9) in this case $\Phi_j = \Phi_j(u, \xi)$.

In order to define the change of variables (3.5), it is necessary to find a physical region of values for r and u. For this purpose we consider the symmetry group of mapping (3.5) $\tilde{S}$. Obviously, $S = S_\varphi \times \tilde{S}$ where $S: \varphi \to \varphi - \varphi_s(u)$, $u \to u_s(u)$, $\xi \to \exp(-i\varphi_s\Gamma)\xi$ and $\varphi_s(u)$ is determined by the condition (3.11). However, one has to define functions u_s and φ_s in more detail to show that

S is a group. Split the axis $R = \cup_\alpha R_\alpha$ so that a number of solutions of (3.4) could be fixed when $u \in R_\alpha$. Let $\{u_s\}$ be a set of solutions of (3.4) for a fixed $u \in R_\alpha$. We define an element $\hat{s}$ of $S = S_\alpha$ when $u \in R_\alpha$ as a rearrangement of points u_s. Then the transformation $u \to \hat{s}u \equiv u_s(u)$ defines a function $u_s: \ R_\alpha \to R_\alpha$. Obviously, a composition of two thus defined functions corresponds to a new element of S_α, i.e. S_α is a group. Therefore $S = \prod_\alpha S_\alpha$. Note that rotations through angles $\varphi_s(u)$ corresponding to $u_s(u)$, $u \in R_\alpha$, do not form a subgroup of the gauge group because φ_s are functions of u. Thus, physical values of u belong to $K = \cup_\alpha K_\alpha$, where K_α is a fundamental region of R_α with respect to S_α, i.e. $K_\alpha = R_\alpha/S_\alpha$. Physical values of u belong to $K = R/S$ and K defines the integration region for the scalar product in $\mathcal{H}_{ph}$ $\int d^2x = 2\pi \int_K du\mu(u)$ where $\mu = (\mathbf{f}', \mathbf{f})$ is the Jacobian.

In this approach a basis in $\mathcal{H}_{ph}$ consists, as above, of functions (2.11)-(2.13) in which one should put $r^2 \to \mathbf{f}^2(u)$, $r\xi_1 \to z_f\xi_1$ and $r\xi_2 \to z_f^*\xi_2$ where $z_f = f_1 + if_2$. This is a simple consequence of that the quantum Hamiltonian in $\mathcal{H}_{ph}$ for the variable u may be transformed to the one in (2.9) by the substitution $\partial_u = \partial r/\partial u \partial_r = r^{-1}\mu\partial_r$ and moreover $\int_K du\mu(u) = \int_0^\infty dr r$ by the definition of K (quantum theories corresponding to different $\mathbf{f}$ are unitary equivalent in our approach!). [2]

Thus, Green functions $\langle T(\hat{u}(t)\hat{u})\rangle_0$ and $\langle T(\hat{\xi}_a(t)\hat{\xi}_b^+)\rangle_0$ depend on the choice of $\mathbf{f}$ (K depends on $\mathbf{f}$). However it does not means that Green functions depend on a gauge since u and ξ are gauge invariant. In other words, one may say that Green functions depend on a parametrization of the orbit space which is determined by physical reasons, for example, by our wish to have elementary physical excitations with required quantum numbers (see also Sec.4). Because of the PCS reduction, a description of excitations corresponding to these variables may became highly complicated. Therefore it is simpler to give a classification of gauge invariant objects and to consider their Green functions at the beginning.

Indeed, physical wave functions are S-invariant $\hat{s}\Phi_E(u,\xi) \equiv \Phi_E(\hat{s}u, \hat{s}\xi) = \Phi_E(u,\xi)$ where $\hat{s}u = u_s(u)$ and $\hat{s}\xi = \exp(-i\varphi_s\Gamma)\xi \equiv \xi^s$, $\hat{s} \in S$ since they analytically depend on the gauge invariant generating elements $\mathbf{x}^2$, $z\theta_1$, $z^*\theta_2$ and $\theta_1\theta_2$. So we may continue all Φ_E to the unphysical region $R \ominus K$ in the S-invariant way. When calculating scalar products in $\mathcal{H}_{ph}$ we may change $\int_K du\mu(u) \to \sum_S \int_{K_s} du_s\mu(u_s)$, because of the S-invariance of Φ_E. The region K_s is determined by the mapping $\hat{s}: \ K \to K_s$ ($\hat{s}u \in K_s$ if $u \in K$) and $\bigcup_S K_s = R$. However, the states $\hat{u}\Phi_E = u\Phi_E$ and $\hat{\xi}_a^+\Phi_E = \xi_a\Phi_E$ arising in a calculation of Green functions are not S-invariant and must be defined in the unphysical region in the S-invariant way with the help of the operator $\hat{Q}$ by an analogy with (3.2)

$$u^Q = \sum_S \Theta_K(\hat{s}u)\hat{s}u, \qquad \xi_a^Q = \sum_S \Theta_K(\hat{s}u)\hat{s}\xi_a \tag{3.6}$$

where $\Theta_K(u) = 1,0$ if $u \in K$, $u \in R \ominus K$, respectively. It is necessary for the one-to-one correspondence with the quantum theory in the reduced PCS. The functions $u^Q(u)$ and $\xi_a^Q(u,\xi)$ contain all degrees of the elementary gauge invariant objects $\mathbf{x}^2$, $\overline{z}\theta_1$, $z^*\theta_2$ when decomposing them into a series. Hence the structure of poles of Green functions $\langle T(\hat{u}(t)\hat{u})\rangle_0$ and $\langle T(\hat{\xi}_a(t)\hat{\xi}^+_b)\rangle_0$ is preserved and like (2.14) and (2.15). Only coefficients in series (2.14), (2.15) depend on $\mathbf{f}$. On the other hand, gauge invariant functions like $\langle T(\mathbf{x}^2(t)\mathbf{x}^2)\rangle_0$ (analogously for $z\theta_1$ and $z^*\theta_2$) contain only one pole corresponding to a composed (boson + boson or boson + fermion) gauge invariant state ($\Phi_1^{(0)}$ or $\Phi_1^{(a)}$). In fact Green functions $\langle T(\hat{u}(t)\hat{u})\rangle_0$ and $\langle T(\hat{\xi}_a(t)\hat{\xi}^+_b)\rangle_0$ can be expressed via Green functions of gauge invariant objects $(\mathbf{x}^2)^n$, $(\mathbf{x}^2)^{n-1}z\theta_1$ and $(\mathbf{x}^2)^{n-1}z^*\theta_2$, $n = 1, 2, ...$

[2] Note that quantum theories corresponding to different $\mathbf{f}$ and derived by the elimination of the unphysical variable φ in classical Hamiltonian (2.3) with the help of supplementary conditions $p_\varphi = \sigma = 0$ and $\mathbf{x} = \mathbf{f}(u)$ and by the following quantization ($p_u \to -i\partial_u$) are not unitary equivalent (here p_φ and $p_u = 1/2(\mathbf{p},\mathbf{x})\partial_u \ln \mathbf{x}^2$ are momenta canonically conjugated to φ and u, respectively). The reason is simple: operations of quantization and introduction of curvilinear coordinates do not commute (in contrast with our consideration when the unphysical variable φ is eliminated after a quantization in Cartesian variables) (see also [6.7]).

Thus we may conclude that the main reason of a modification of Green functions for physical degrees of freedom is the following. After an elimination of all unphysical degrees of freedom with the help of a supplementary condition $\chi = 0$, there exists a group S acting in PCS. Physical states annihilated by a quantum version of constraints must be S-invariant: $\hat{s}\Phi = \Phi \in \mathcal{H}_{ph}$ (a consequence of their gauge invariance). However, a state with a certain value of a chosen physical variable (i.e. an eigenstate of $\hat{u}$ or $\hat{\xi}_a^+$ in our case) cannot belong to $\mathcal{H}_{ph}$ since operators $\hat{s} \in S$ do not commute with the operator of this variable (with $\hat{u}$ or $\hat{\xi}_a^+$: $\hat{s}\hat{u} = u_s(\hat{u})\hat{s}$ or $\hat{s}\hat{\xi}_a^+ = \xi_a^s(\hat{u}, \hat{\xi}^+)\hat{s}$). Hence, they cannot have common eigenstates except trivial, equal to zero. Just based on this point of view we turn now directly to the Yang-Mills system.

4 Green functions in the Yang-Mills theory and Singer theorem

After the consideration of the quantum mechanical example, we may state that, first, physical variables in a gauge theory may have a reduced configuration space (i.e. it differs from an Euclidean space); second, this reduction takes place if after an elimination of all unphysical variables with the help of constraints and chosen gauge conditions, there exists a residual gauge symmetry which cannot decrease the number of physical degrees of freedom but it can decrease their configuration space identifying some points in it; third, the PCS reduction leads to a modification of Green functions and moreover an excitation of a physical degree of freedom with a reduced PCS must be viewed as a coherent excitation of all gauge invariant states of a system. It turns out that all these points take place for the Yang-Mills system with fermions.

The quantum Yang-Mills theory is defined by the functional Schroedinger equation [14,15]

$$\left[\frac{1}{2}\langle\hat{\mathcal{E}}_k, \hat{\mathcal{E}}_k\rangle + \langle B_k, B_k\rangle + i\langle\bar{\psi}, \gamma_k\nabla_k\psi\rangle\right]\Phi_E = E\Phi_E \tag{4.1}$$

and the quantum equations of constraints

$$\hat{\sigma}\Phi_E = (\partial_k\hat{\mathcal{E}}_k + g[A_k, \hat{\mathcal{E}}_k] + gJ_0)\Phi_E = 0 \tag{4.2}$$

Here $\hat{\mathcal{E}}_k = -i\delta/\delta A_k$ $(k = 1,2,3)$, A_k are Yang-Mills potentials being elements of a Lie algebra X of a semisimple compact gauge group G, B_k is the colour magnetic field, ∇_k is the covariant derivative, $\bar{\psi} = \psi^+\gamma_0$, γ_μ are Dirac matrices ($\mu = 0, k$), the brackets $(,)$ denote the integration over $\mathbf{x} \in R^3$ and a corresponding scalar product in a space of components $\mathcal{E}_K$, B_k and ψ; J_0 is the null component of the colour 4-current J_μ of fermions. States Φ_E viewed as functionals of $A_k(\mathbf{x})$ and of a Grassman field $\theta(\mathbf{x})$ (in this functional representation, operators $\psi(\mathbf{x})$ and $\psi^+(\mathbf{x})$ act on Φ_E as the left functional derivative $\partial/\partial\theta(\mathbf{x})$ and the left multiplication on $\theta(\mathbf{x})$, respectively) are normalized as follows

$$\int_{[\mathbf{A}]} D\mathbf{A} \int D\theta D\theta^* \exp(-(\theta,\theta^*))\Phi_E^*\Phi_{E'} = \delta_{EE'} \tag{4.3}$$

where $D\mathbf{A} = \prod_\mathbf{x}(\prod_{comp} d\mathbf{A}(\mathbf{x}))$, $D\theta D\theta^* = \prod_\mathbf{x}(\prod_{comp} d\theta(\mathbf{x})\theta^*(\mathbf{x}))$, $\prod_{comp}$ denotes a product over all components of A_k or θ. The integration region $[\mathbf{A}]$ in (4.3) is chosen so that the integration over each component of $A_k(\mathbf{x})$ ($\mathbf{x}$ is fixed) is carried out over a real axis, i.e. each degree of freedom is assumed to be Cartesian. We also assume that Yang-Mills fields tend to zero at spatial infinity ($|\mathbf{x}| \to 0$) in the measure (4.4), i.e. they are defined on the 3-sphere ($\mathbf{x} \in S^3$) [16].

Eq.(4.2) means that Φ_E are invariant with respect to gauge transformations [14,15]

$$A_k \to \Omega A_k \Omega^{-1} + g^{-1}\Omega\partial_k\Omega^{-1}, \qquad \theta \to \theta\Omega^{-1} \tag{4.4}$$

generated by σ where $\Omega = \Omega(\mathbf{x}) \in G$. To solve it, we introduce new curvilinear functional variables [7]

$$A_k = W\tilde{A}_k W^{-1} + g^{-1}W\partial_k W^{-1}, \qquad \theta = \xi W^{-1} \tag{4.5}$$

where $\tilde{A}_k = A_k[\alpha]$ so that N $(N = \dim X)$ identities $F[\tilde{\mathbf{A}}[\alpha]] \equiv 0$ are fulfilled, i.e. new variables are $\xi, \omega = \omega[\mathbf{A}]$, where $\delta\omega = W^{-1}\delta W$, and $2N$ components of fields $\alpha = \alpha[\mathbf{A}]$. Under gauge transformations ξ and α are not changed, but ω is translated. Hence, $\Phi_E[\omega, \alpha, \xi] = \Phi_E[\alpha, \xi] \in \mathcal{H}_{ph}$, i.e. the variables ω, $\tilde{A}_k$ and α are defined by analogy with $\varphi, \mathbf{f}$ and u from Sec.3, respectively.

In order to find the Schroedinger equation in $\mathcal{H}_{ph}$, one should calculate the corresponding functional Laplace-Beltrami operator for variables (4.5) omitting functional derivatives of ω in it (like (2.9)). However, we shall not consider this problem (see some details in [7] where a Hamiltonian path integral is discussed for systems with a reduced PCS).

As was shown above, different F in (4.5) correspond to different choices of gauge invariant, physical variables. For example, one may put $F[\tilde{\mathbf{A}}] = \partial_k\tilde{A}_k = 0$, i.e. $\tilde{A}_k[\alpha] = \alpha_k$ where α_k is a transverse field of gluons (note that α_k is gauge invariant in this approach!). Then ξ is a corresponding gauge invariant quark field. So Green functions in this case describe a propagation of excitations with quantum numbers of gluons and quarks (compare with the analogous functions considered in Sec.3).

May a PCS reduction take place in this theory? The answer seems to be positive for any choice of F if Yang-Mills fields tend to zero at the spatial infinity. Therefore Green functions must be modified in the theory.

First of all, mapping (4.5) $X \otimes [\alpha] \to [\mathbf{A}]$ exists (the functional space $[\alpha]$ is defined like $[\mathbf{A}]$, for example, when $F = \partial_k\tilde{A}_k = 0$, $[\alpha]$ coincides with the functional space of all transversal fields tending to zero when $|\mathbf{x}| \to 0$) if any $A_k \in [\mathbf{A}]$ can be transformed to the form $F[A_k] = 0$ with the help of a non-singular gauge transformation preserving a topology of A_k [16,17]. Let it be true. Consider the symmetry group $\tilde{S}$ of mapping (4.5). It has the form $\tilde{S} = S_G \times S$ where S_G consists of transformations of ω not changing W, i.e. they are translations of ω through periods of the G-group manifold. Thus, permissible values of ω belong to the $X/S_G = G$-group manifold (compare with $S_\varphi : \varphi \to \varphi + 2\pi n$, $n \in Z$ in (2.8)). The group S contains the following transformations

$$W \to WW_s^{-1}, \quad \tilde{A}_k \to \hat{s}\tilde{A}_k = W_s\tilde{A}_k W_s^{-1} + g^{-1}W_s\partial_k W_s^{-1}, \quad \xi \to \hat{s}\xi = \xi W_s^{-1} \tag{4.6}$$

where $W_s \in G$ and $F[\tilde{\mathbf{A}}] = F[\hat{s}\tilde{\mathbf{A}}] = 0$, hence $\hat{s}\tilde{A}_k[\alpha] = A_k[\alpha_s]$, $\alpha_s = \alpha_s[\alpha] \equiv \hat{s}\alpha$ (there is a complete analogy with the analysis of change of variables (3.5)). Here W_s must be non-singular group elements, i.e. $\hat{s}$ preserves a topology of $\tilde{A}_k$ (a choice of a gauge condition $F = 0$, generally speaking, fixes an instanton number [16]). Thus, the group S is not trivial if the equation

$$F[\hat{s}\mathbf{A}] = 0 \qquad \text{where} \qquad F[\mathbf{A}] = 0 \tag{4.7}$$

has non-trivial solutions with respect to $\hat{s}$. It means that the equation $F[\mathbf{A}] = 0$ has some solutions connected with each other by gauge transformations. Therefore the problem reduces to the question of the existence of a global condition in $[\mathbf{A}]$ fixing gauge arbitrariness without ambiguities [18]. There is the Singer theorem [19,16] forbidding the existence of such a condition for the Yang-Mills fields defined on the 3-sphere (or, which is equivalent, tending to zero at spatial infinity [16]). Hence the group S is always non-trivial and PCS of field α must be reduced $[\alpha] \to K = [\alpha]/S$. So, in the scalar product (4.4) for states from $\mathcal{H}_{ph}$, one should make the change $\int_{[\mathbf{A}]} D\mathbf{A} \to \int_K D\alpha\mu[\alpha]$ where $\mu[\alpha]$ is the Jacobian [7].

Physical states must be S-invariant $\hat{s}\Phi[\alpha] = \Phi[\alpha_s] = \Phi[\alpha] \in \mathcal{H}_{ph}$. Indeed, let a state $\tilde{\Phi}[\mathbf{A}]$ satisfy Eq.(4.2) then it is invariant under gauge transformations (4.4). Hence we have the following chain of equalities $\tilde{\Phi}[\mathbf{A}] = \tilde{\Phi}[\tilde{\mathbf{A}}[\alpha]] = \tilde{\Phi}[\hat{s}\tilde{\mathbf{A}}[\alpha]] = \tilde{\Phi}[\tilde{\mathbf{A}}[\alpha_s]] = \Phi[\alpha_s] = \Phi[\alpha]$ where $\tilde{\Phi}[\tilde{\mathbf{A}}[\alpha]] = \Phi[\alpha]$. Thus, the S-invariance of physical states is a "memory" of the reduced system (i.e. all unphysical variables are eliminated) about its gauge origin.

In the functional representation the operators $\hat{\alpha}$ and $\hat{\xi}^+$ act as the multiplication $\hat{\alpha}\Phi = \alpha\Phi$ and $\hat{\xi}^+\Phi = \xi\Phi$. However, the action of these operators on physical states takes them out of $\mathcal{H}_{ph}$ since $\hat{\alpha}$ and $\hat{\xi}^+$ do not commute with $\hat{s} \in S$: $\hat{s}\hat{\alpha} = \alpha_s[\hat{\alpha}]\hat{s}$ and $\hat{s}\hat{\xi}^+ = \hat{\xi}^+ W_s^{-1}[\hat{\alpha}]\hat{s}$. Hence there are no common eigenstates of $\hat{s}$ and $\hat{\alpha}$, $\hat{\xi}^+$, i.e. eigenstates are absent in $\mathcal{H}_{ph}$. So we may conclude that quantum Green functions must be modified.

Using the S-invariance of physical states we may widen the integration region in the scalar product from K to $[\alpha]$ changing all fields α and ξ by $\alpha^Q \equiv \hat{Q}\alpha = \sum_S \Theta_K[\alpha_s]\alpha_s$ and $\xi^Q \equiv \hat{Q}\xi = \sum_S \Theta_K[\alpha_s]\xi W_s^{-1}$ where $W_s = W_s[\alpha]$ and the functional $\Theta_K[\alpha] = 1, 0$ if $\alpha \in K$, $[\alpha] \ominus K$, respectively (compare with $\Theta_K(u)$ in Sec.3). The operator $\hat{Q}$ defines physical fields α and ξ in the unphysical region $[\alpha] \ominus K$ in the S-invariant way. Note that for physical states $\Phi[\hat{Q}\alpha, \hat{Q}\xi] = \Phi[\alpha, \xi]$ because of their S-invariance and the equality $\sum_S \Theta_K[\alpha_s] = 1$ since $K = [\alpha]/S$. So, instead of Green functions of field α and ξ with the reduced PCS (i.e. defined in K) , we may consider Green functions of fields α^Q and ξ^Q in the total PCS $[\alpha]$. Thus, Green functions turn out to be modified. The analysis in the framework of the path integral approach leads to the same result [7]. The reason is that the evolution operator $\hat{U}_t^{ph} = \exp(-i\hat{H}_{ph}t)$ has an unusual form because of the PCS reduction, namely, $\hat{U}_t^{ph} = \hat{U}_t^{sfJ}\hat{Q}$ where $\hat{U}_t^{sfJ}$ is defined by the standard Hamiltonian path integral in a gauge $F = 0$.

Since operators of constraints $\hat{\sigma}$ composing a basis of X commute with the Hamiltonian in (4.1), every solution of (4.1) may be represented in the form

$$\Psi_E[\mathbf{A}, \theta] = \sum_{(n)} \Psi_E^{(n)}[\alpha, \xi] Y_{(n)}[\omega] \tag{4.8}$$

where $Y_{(n)}[\omega]$ are eigenstates of the Casimir operators of the algebra of the operators $\hat{\sigma}$, and (n) denotes a set of their eigenvalues. The physical subspace is composed of states $\Psi_E^{(0)} \equiv \Phi_E$ $(Y_{(0)} = const)$ which must be S-invariant. Thus, there exists a one-to-one correspondence between gauge invariant states $\Phi[\mathbf{A}]$ in the total configuration space $[\mathbf{A}]$ and S-invariant states in $[\alpha]$. So, when calculating Green functions of the fields α^Q and ξ^Q, we may decompose them into a power series of simplest gauge invariant (or S-invariant) objects, for example,

$$Tr\ P\exp g\int_{\mathbf{x}} A_k dz_k = Tr\ P\exp g\int_{\mathbf{x}} \tilde{A}_k[\alpha] dz_k\ ;$$

$$\theta(\mathbf{x})P\exp\left(-g\int_{\mathbf{y}}^{\mathbf{x}} A_k dz_k\right)\theta^+(\mathbf{y}) = \xi(\mathbf{x})P\exp\left(-g\int_{\mathbf{y}}^{\mathbf{x}} \tilde{A}_k[\alpha] dz_k\right)\xi^+(\mathbf{y}) \tag{4.9}$$

etc. Green functions of objects like (4.9) may have simple poles in the momentum space if the dynamics of the system allows as to consider them as bound states [19] of the origin fields.

What are qualitative peculiarities of a modification of Green functions? Based on the analysis of sections 2 and 3 one may expect that total Green functions of the fields α^Q and ξ^Q in $[\alpha]$ (or α and ξ in K) contain sums over all poles corresponding to physical excitations of the system. Let now the fields α and ξ be fields describing transversal gluons and quarks, respectively $(\tilde{A}_k = \alpha_k, \quad \partial_k\alpha_k \equiv 0)$. Then the propagators for α_k and ξ cannot have poles corresponding to a propagation of quanta of fields α_k and ξ because there are no S-invariant objects linear in α and ξ (or gauge invariant objects linear in A_k and θ). Thus, the "kinematic" confinement of gluons and quarks exists because of the PCS reduction for the corresponding dynamical field variables.

5 Conclusion

What happens in a perturbative theory? In this case field fluctuations of A_k are small (quasiclassics in the neighbourhood of the solution $A_k = 0$) and one should consider the change of variables (4.5) at $W \approx 1 + g\omega$ since $g \to 0$. Locally a gauge condition $F = 0$ may always be chosen so that the group S could be trivial [3], i.e. $S = 1$. Hence $\alpha^Q \approx \alpha$ and $\xi^Q \approx \xi$ and Green functions must have the well-known form. Thus, the modification of Green functions takes place only in a non-perturbative region when field fluctuations become large (the group S is not trivial). However, in our opinion (and as it follows from the above consideration), the physical more consistent explanation why the perturbative theory works should be based on an investigation of objects like (4.9) (i.e. strings).

To make stricter investigations of the modified Green functions for the fields α and ξ in a non-perturbative region, an explicit form of the functional α^Q and ξ^Q should be considered. However, it is impossible to realize it practically since the group S has a highly complicated form for covariant gauges [13]. Therefore it seems to us that it is impossible to hope that there exists a practically fit non-perturbative quantum theory of non-Abelian gauge fields using covariant gauge conditions. In other words, physical variables separated by such gauges are not natural to describe physical excitations being coherent states of the initial fields of a non-Abelian gauge theory (for example, strings).

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Received by Publishing Department

on February 7, 1991.

[3] As is shown in [13], a local gauge condition always exists in the Yang-Mills theory when fields belong to a definite functional class.