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ON THE ENERGY-MOMENTUM TENSORS FOR FIELD THEORIES IN SPACES WITH AFFINE CONNECTION AND METRIC.

Generalized Bianchi Identities and Different Energy-Momentum Tensors

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I. INTRODUCTION

1 In classical field theories the notion of energy-momentum tensor is used for description of the evolution of a given physical system. This notion is introduced in some different ways for a given field theory and, moreover, it has been connected with notions, different in their mathematical structure and definition, such as canonical, symmetric energy-momentum tensor of Belinfante /1/ and symmetric energy-momentum tensor of Hilbert /1,2/. Some connections between all these tensors are found and used for constructing one of them by means of the others /1-3/.

2. A given physical system can be described by using a Lagrangian (scalar) density of the type /1-4/:

$$\mathcal{L} := \mathcal{L}(V^A{}_B, V^A{}_{B,i}, V^A{}_{B,i,j}) := \sqrt{-g}.L(V^A{}_B, V^A{}_{B,i}, V^A{}_{B,i,j}), \tag{0.1}$$

where

:= means "by definition", A,B,... - multiindices,

$A := i_1 \dots i_l$, $B := j_1 \dots j_m$, $l,m,\dots < N \in \mathbb{R}$, $g \neq 0$,

$i,j,k,\dots = 1,\dots,n$,

$$V^A{}_{B,i} := \frac{\partial V^A{}_B}{\partial x^i}, \qquad V^A{}_{B,i,j} := \frac{\partial^2 V^A{}_B}{\partial x^j \partial x^i} .$$

$V^A{}_B = V^A{}_B(x^k)$ are field variables, considered as components of contra- or covariant tensor fields of (finite) rank. $V^A{}_B$ can be metric ($V_B = g_{ij}$) or nonmetric ($V_B \neq g_{ij}$) field variables.

There is a possibility for describing a physical system by means of a Lagrangian density of the type /1/:

Table 1.

Methods of obtaining local conserved quantities

Method	Lagrangian density	Form of the identity $\pounds_\xi \mathcal{L} - (\mathcal{L}\xi^i)_{/i} = 0$	Local conserved quantities
1. *Method of* energy-momentum complexes	$\mathcal{L} := \mathcal{L}(V_A, V_{A,i}, V_{A,i,j})$ $V_A = g_{ij}$ $V_A \neq g_{ij}$	$A_k \xi^k + A_k{}^i \xi^k{}_{,i} + A_k{}^{ij} \xi^k{}_{,i,j} + A_k{}^{ijl} \xi^k{}_{,i,j,l} = 0$	Complexes (pseudotensors) of energy and momentum (s. 1,3,10,11,14)
2. Method of the privileged vector fields	$\mathcal{L} := \mathcal{L}(V_A, V_{A,i}, V_{A,i,j})$ $V_A = g_{ij}$ $V_A \neq g_{ij}$	$\frac{\bar{\delta}\mathcal{L}}{\bar{\delta}V_A}\pounds_\xi V_A + t^i{}_{,i} = 0,$ $\frac{\bar{\delta}\mathcal{L}}{\bar{\delta}V_A} = \frac{\partial\mathcal{L}}{\partial V_A} - \left(\frac{\partial\mathcal{L}}{\partial V_{A,i}}\right)_{,i} + \left(\frac{\partial\mathcal{L}}{\partial V_{A,i,j}}\right)_{,i,j},$ $t^i = \left[\frac{\partial\mathcal{L}}{\partial V_{A,i}} - 2\left(\frac{\partial\mathcal{L}}{\partial V_{A,i,j}}\right)_{,j}\right]\pounds_\xi V_A + \left(\frac{\partial\mathcal{L}}{\partial V_{A,i,j}}\pounds_\xi V_A\right)_{,j} - \mathcal{L}\xi^i$	Quantities in which structures the components of the vector field ξ^i with nonsimple physical interpretation together with the functions V_A and their derivatives are included (s. 14-19)

Table 1.
(continuation)

Methods of obtaining local conserved quantities

Method	Lagrangian density	Form of the identity $\mathcal{L}_\xi \mathcal{X} - (\mathcal{X}\xi^i)_{/i} = 0$	Local conserved quantities
3. Method of the covariant derivatives of the vector fields (CDVF)	$\mathcal{X} := \mathcal{X}(g_{ij}, g_{ij,k}, g_{ij,k,l}, V_A, V_{A,i})$ $V_A \neq g_{ij}$	$B_k \xi^k + B_k{}^i \xi^k{}_{/i} + B_k{}^{ij} \xi^k{}_{/i/j} + B_k{}^{ijl} \xi^k{}_{/i/j/l} = 0$	Covariant conserved quantities (s. 2,10,13)
4. Method of Lagrangians with covariant derivatives (MLCD)	$\mathcal{X} := \mathcal{X}(g_{ij}, V_A, V_{A/i}, V_{A/i/j})$ $V_A \neq g_{ij}$	$P_k \xi^k + P_k{}^i \xi^k{}_{/i} = 0$	Covariant (tensor) conserved quantities (s. 5)

$$\mathfrak{L} := \mathfrak{L}(g_{ij}, V^A{}_B, V^A{}_{B/i}, V^A{}_{B/i/j}) :=$$
$$:= \sqrt{-g}.L(g_{ij}, V^A{}_B, V^A{}_{B/i}, V^A{}_{B/i/j}) \,, \qquad (0.2)$$

where $V^A{}_{B/i}$ is a covariant derivative of $V^A{}_B$ along the basic vector field E_i (or ∂_i).

The methods of obtaining quantities of the type of energy-momentum tensor in (pseudo)Riemannian spaces with Lagrangian density, depending on covariant tensor fields and their first (and second) partial (or covariant) derivatives are given schematically in the Table /5/.

3. In previous papers /5/ the possibilities have been considered of defining tensors of the type of an energy-momentum tensor for field theories in (pseudo)Riemannian spaces without torsion (V_n-spaces) and with torsion (U_n-spaces) /6,7/. By means of the method of Lagrangians with covariant derivatives (MLCD) the s.c. generalized covariant Bianchi type identities (GCBI) are obtained. On the ground of these identities the connections between the s.c. generalized canonical energy-momentum tensor, the symmetric energy-momentum tensor of Belinfante are investigated as well as the conditions under which the symmetric energy-momentum tensor appears to be a local conserved quantity, i.e. a quantity of the type T_{ij} obeying the condition

$$T_i{}^j{}_{/j} = 0, \text{ where } T_{ij} = g_{jk}T_i{}^k = T_{ji} \,. \qquad (0.3)$$

At the same time it was shown that the covariant Euler-Lagrange equation (CELE) for the field variables in U_n-spaces, in contrast to the case of V_n-spaces, are not sufficient conditions for the existence of the symmetric energy-momentum tensor as a local conserved quantity.

4. In this paper the possibilities for obtaining tensors of the type of energy-momentum tensor (generalized canonical, symmetric, etc.) will be considered by means of the MLCD for field

theories in spaces with affine connection Γ ($\Gamma^i_{jk} \neq \Gamma^i_{kj}$) and metric $\underline{g}$ ($g_{ij/k} :\neq 0$) (s.c. L_n-spaces). The purpose is the relations between these energy-momentum tensors to be found as well as the connections between them and the covariant Euler-Lagrange equations in L_n-spaces (s. Part II. of the article).

5. In Sec. II. the application of the MLCD will be considered in L_n-spaces and on its basis the GCBI will be obtained. In Sec.III the introduction of the notion of generalized canonical and symmetric energy-momentum tensor and the relations between them will be discussed.

6. Abbreviations, symbols and definitions

In order to have a clearer description of the results, some abbreviations, symbols and definitions, connected with the later introduced notions will be given in advance. Here it is useful to recall the paper of Thorne, Lee and Lightman /8/ where many notions, characterizing the existing gravitational theories (and not only these), are defined.

Abbreviations

CELE := covariant Euler-Lagrange equations

CEMT := canonical energy-momentum tensor

ETG := Einstein's theory of gravitation

GCBI := generalized covariant Bianchi type identity

GCEMT := generalized canonical energy-momentum tensor

LCQ := local conserved quantity

MLCD := method of Lagrangians with covariant derivatives

SEMT := symmetric energy-momentum tensor

SEMT(B) := symmetric energy-momentum tensor of Belinfante

SEMT(H) := Symmetric energy-momentum tensor of Hilbert

Symbols

A,B,C,... - multiindices

dim M - dimension of the differential manifold M

∂_k - coordinate basis vector field

$\{\partial_k\}_x$ - coordinate basis in p. $x \in T_x(L_n)$

E_k - noncoordinate basic vector field

$\{E_k\}_x$ - noncoordinate basis in p. $x \in T_x(L_n)$

$g := \det(g_{ij})$ - determinant of g_{ij}

g_{ij} - components of the metric tensor $\underline{g}$

$i,j,k,l,\ldots$ - indices

$\mathcal{L}$ - Lagrangian density

L - Lagrangian invariant (scalar function)

L_n - space with affine connection and metric

$T_{ij}{}^{k}$ - components of the torsion tensor T ($T_{ij}{}^{k} = - T_{ji}{}^{k}$)

$T_x(L_n)$ - tangential space in p. $x \in L_n$

U_n - (pseudo)Riemannian space with torsion

V^A - components of a contravariant tensor field

V_B - components of a covariant tensor field

V_n - (pseudo)Riemannian space without torsion

Γ^{i}_{jk} - components of the affine connection Γ

$/i$ - covariant derivative with respect to the basic vector field E_i (if $E_i = \partial_i$, then $_{/i}$ means covariant derivative with respect to the coordinate x^i $(i = 1,2,\ldots,n)$)

$,j$ - partial derivative with respect to the vector field E_j (or ∂_j)

$\pounds_u v^i$ - Lie derivative of the components of the vector field v along the vector field u

Definitions

L_n-space := n-dimentional C^k-differential manifold M ($k = 2, 3, \ldots, \infty$) provided with nonsymmetric affine connection Γ and (symmetric) metric $\underline{g}$ with $g_{ij/k} :\neq 0$

$$\pounds_u v^i := v^i{}_{/j} u^j - u^i{}_{/j} v^j - T_{jk}{}^{i} u^j v^k , \quad u := u^k E_k, \quad v := v^i E_i$$

$$T_{ij}{}^{k} := -T_{ji}{}^{k} = \Gamma^{k}_{ji} - \Gamma^{k}_{ij} - C_{ij}{}^{k} \quad \text{(in noncoordinate basis)}$$
$$= \Gamma^{k}_{ji} - \Gamma^{k}_{ij} \quad \text{(in coordinate basis)}$$
$$\mathcal{L}_{E_i} E_j := [E_i, E_j] := E_i E_j - E_j E_i = C_{ij}{}^{k} E_k$$
$$v^{i}{}_{/j} := v^{i}{}_{,j} + \Gamma^{i}_{kj} . v^{k} , \quad v^{i}{}_{,j} := E_j v^{i} \ (= \partial_j v^{i})$$
$$V^{A}{}_{B/i} := E_i V^{A}{}_{B} + \Gamma_{Ci}{}^{A} V^{C}{}_{B} - \Gamma_{Bi}{}^{D} V^{A}{}_{D}$$
$$\Gamma_{Ci}{}^{A} := - S_{Cm}{}^{Aj} \Gamma_{ji}{}^{m} , \quad A := i_1 \dots i_l , \quad C := j_1 \dots j_l ,$$
$$B := k_1 \dots k_m , \quad D := l_1 \dots l_m$$
$$S_{Cm}{}^{Aj} := - \sum_{k=1}^{l} g^{j}_{j_k} . g^{i_k}_{m} . g^{i_1}_{j_1} \dots g^{i_{k-1}}_{j_{k-1}} . g^{i_{k+1}}_{j_{k+1}} \dots g^{i_l}_{j_l} .$$

Lagrangian system := A set of field variables g_{ij}, $V^{A}{}_{B}$ (with $V^{A}{}_{B/i}$, $V^{A}{}_{B/i/j}$, $V_B = g_{ij}$ or $V_B \neq g_{ij}$), characterized by means of a scalar density $\mathcal{L}$ of the type (0.2) called Lagrangian density.

Functional variation of the invariant function L with respect to $V^{A}{}_{B}$ $:= \dfrac{\delta L}{\delta V^{A}{}_{B}}$

$$\frac{\delta L}{\delta V^{A}{}_{B}} := \frac{\partial L}{\partial V^{A}{}_{B}} - \left(\frac{\partial L}{\partial V^{A}{}_{B/i}}\right)_{/i} + \left(\frac{\partial L}{\partial V^{A}{}_{B/i/j}}\right)_{/j/i} ,$$

where

$$\frac{\partial L}{\partial V^{A}{}_{B/i}} := \frac{\partial L}{\partial (V^{A}{}_{B/i})}, \quad \frac{\partial L}{\partial (V^{A}{}_{B/i/j})}$$

Covariant Euler-Lagrange equations for the fields $V^{A}{}_{B}$

$$\frac{\delta L}{\delta V^{A}{}_{B}} := 0 .$$

II. ENERGY-MOMENTUM TENSORS FROM LAGRANGIANS WITH COVARIANT DERIVATIVES

1. The MLCD consists of three essential steps /5/ needed to obtain the s.c. generalized covariant Bianchi type identities (GCBI) by means of which the corresponding energy-momenta are defined:

a) Representation of the Lie variation of a Lagrangian density along an arbitrary vector field by means of the Lie derivative of the components of the tensor fields (and their first and second covariant derivatives) along this vector field. In this way an identity for L will be obtained.

b) Representation of the Lie derivatives of the components of the tensor fields (and their first and second covariant derivatives) by means of their covariant derivatives only and the components of the torsion tensor using the commutation relation between the Lie derivatives and the covariant derivatives and the connections between these derivatives. Writing down the identity for L in a form, dependent on the components of the vector field and its first covariant derivatives.

c) Obtaining the GCBI from the identity for L under the condition of arbitrariness of the vector field.

2. Let us now take up all three steps of the MLCD in the case of L_n-space to find the corresponding GCBI.

a) The Lie derivative of the Lagrangian density $\mathfrak{L}$ of the type

$$\mathfrak{L} = \sqrt{-g}.L(g_{ij}, V^A{}_B, V^A{}_{B/i}, V^A{}_{B/i/j}) , \quad \det(g_{ij}) \neq 0, \quad -g > 0, \tag{2.1}$$

along an arbitrary vector field ξ can be written in the following form:

$$\pounds_\xi \mathfrak{L} = \sqrt{-g}\{\xi L + \frac{1}{2}.L.\bar{g}[\pounds_\xi g]\} = $$
$$= \sqrt{-g}[L_{/k}\xi^k + \frac{1}{2}.L.g^{ij}\pounds_\xi g_{ij}] . \tag{2.2}$$

If one assumes that the functional variation $\delta\mathfrak{L}$ of $\mathfrak{L}$

$$\delta\mathfrak{L} = \frac{\partial\mathfrak{L}}{\partial V^A{}_B}.\delta V^A{}_B + \frac{\partial\mathfrak{L}}{\partial V^A{}_{B/i}}.\delta(V^A{}_{B/i}) + \frac{\partial\mathfrak{L}}{\partial V^A{}_{B/i/j}}.\delta(V^A{}_{B/i/j}) + $$
$$+ \frac{\partial\mathfrak{L}}{\partial g_{ij}}.\delta g_{ij} \tag{2.3}$$

is connected with the variation of the field variables $V^A{}_B$ (and their first and second covariant derivatives) and g_{ij} along the vector field ξ, i.e. $\delta := \pounds_\xi$, then the s.c. Lie variation $\pounds_\xi\mathfrak{L}$ of $\mathfrak{L}$, equal to the Lie derivative of $\mathfrak{L}$ along

ξ, can be given in the form:

$$\pounds_\xi \mathcal{L} = \sqrt{-g}(L_{/i}\xi^i + \tfrac{1}{2}.L.g^{ij}\pounds_\xi g_{ij}) =$$

$$= \frac{\partial \mathcal{L}}{\partial V^A{}_B}.\pounds_\xi V^A{}_B + \frac{\partial \mathcal{L}}{\partial V^A{}_{B/i}}.\pounds_\xi (V^A{}_{B/i}) + \frac{\partial \mathcal{L}}{\partial V^A{}_{B/i/j}}.\pounds_\xi (V^A{}_{B/i/j}) +$$

$$+ \frac{\partial \mathcal{L}}{\partial g_{ij}}.\pounds_\xi g_{ij} \ . \qquad (2.4)$$

From the last identity for $\mathcal{L}$ it follows the identity for the scalar function (invariant) L:

$$\frac{\partial L}{\partial V^A{}_B}.\pounds_\xi V^A{}_B + \frac{\partial L}{\partial V^A{}_{B/i}}.\pounds_\xi (V^A{}_{B/i}) + \frac{\partial L}{\partial V^A{}_{B/i/j}}.\pounds_\xi (V^A{}_{B/i/j}) +$$

$$+ \frac{\partial L}{\partial g_{ij}}.\pounds_\xi g_{ij} - L_{/i}\xi^i = 0 \ . \qquad (2.5)$$

The variation of the field variables $V^A{}_B$ and the metric tensor components g_{ij} along the vector field ξ are given by means of their Lie derivatives along ξ, and the variation of the nonsymmetric affine connections (and with it also the components of the torsion tensor) is given implicitly by the Lie derivative of the first and second covariant derivatives of $V^A{}_B$.

b) By means of the Lie derivatives of the tensor fields g_{ij}, $V^A{}_B$ and of their first and second covariant derivatives, the identity (2.5) can be obtained in the form:

$$(P_i + P_k{}^j.T_{ij}{}^k)\xi^i + P_i{}^j\xi^i{}_{/j} = 0 \ , \qquad (2.6)$$

where the following expressions for the Lie derivatives are used:

$$\pounds_\xi V^A{}_B = [\, V^A{}_{B/i} + (S_{Ck}{}^{Aj}V^C{}_B - S_{Bk}{}^{Dj}V^A{}_D)T_{ij}{}^k]\xi^i +$$

$$+ (S_{Ci}{}^{Aj}V^C{}_B - S_{Bi}{}^{Dj}V^A{}_D)\xi^i{}_{/j} \ , \qquad (2.7)$$

$$\pounds_\xi (V^A{}_{B/i}) = [\, V^A{}_{B/i/j} + (S_{Cl}{}^{Ak}V^C{}_{B/i} - S_{Bl}{}^{Dk}V^A{}_{D/i})T_{jk}{}^l +$$

$$+ V^A{}_{B/k}T_{ji}{}^k]\xi^j + (S_{Cj}{}^{Ak}V^C{}_{B/i} - S_{Bj}{}^{Dk}V^A{}_{D/i})\xi^j{}_{/k} +$$

$$+ V^A{}_{B/j}\xi^j{}_{/i} \ , \qquad (2.8)$$

$$\pounds_\xi (V^A{}_{B/i/j}) = [\, V^A{}_{B/i/j/k} + (S_{Cl}{}^{Am}V^C{}_{B/i/j} - S_{Bl}{}^{Dm}V^A{}_{D/i/j})T_{km}{}^l +$$

$$+ V^A{}_{B/l/j}T_{ki}{}^l + V^A{}_{B/i/l}T_{kj}{}^l]\xi^k +$$

$$+ (S_{Ck}{}^{Al}V^C{}_{B/i/j} - S_{Bk}{}^{Dl}V^A{}_{D/i/j})\mathcal{E}^k{}_{/l} +$$

$$+ V^A{}_{B/k/j}\mathcal{E}^k{}_{/i} + V^A{}_{B/i/k}\mathcal{E}^k{}_{/j} \quad . \tag{2.9}$$

c) The components of the vector field $\mathcal{E} = \mathcal{E}^i E_i$ and their first covariant derivatives $\mathcal{E}^i{}_{/j}$ can be considered as arbitrary, independent functions of the coordinates in L_n-space (i.e. in a coordinate system /9/, in which in p. $x \in L_n$ $\Gamma^i_{jk} = 0$, $\mathcal{E}^i{}_{/j_{/x}} = \mathcal{E}^i{}_{,j}$). If the identity (2.6) is fulfilled for arbitrary $\mathcal{E}^i$ and $\mathcal{E}^i{}_{,j}$, then the components of the tensors, sitting as coefficients before $\mathcal{E}^i$ and $\mathcal{E}^i{}_{/j}$ have to be identically equal to zero, i.e.

$$P_i + P_k{}^j T_{ij}{}^k = 0 \ , \qquad P_i{}^j = 0 \ . \tag{2.10}$$

From the second identity in (2.10) it follows that the first condition would have the form $P_i = 0$. In this way the generalized covariant Bianchi type identities (GCBI) for the invariant L can be written in the form:

$$P_i = 0 \ , \tag{2.11}$$

$$P_i{}^j = 0 \ . \tag{2.12}$$

III. GENERALIZED COVARIANT BIANCHI TYPE IDENTITIES AND ENERGY-MOMENTUM TENSORS

1. The identity (2.12) can be rewritten in the following form:

$$P_i{}^j = t_i{}^j - K_i{}^j - V_i{}^{kj}{}_k + 2.\frac{\partial L}{\partial g_{jk}}.g_{ik} + g^j_i.L - Q_i{}^j = 0 \ , \tag{3.1}$$

where

$$t_i{}^j = [\frac{\partial L}{\partial V^A{}_{B/j}} - (\frac{\partial L}{\partial V^A{}_{B/k/j}} + \frac{\partial L}{\partial V^A{}_{B/j/k}})_{/k}]V^A{}_{B/i} +$$

$$+ (\frac{\partial L}{\partial V^A{}_{B/k/j}}.V^A{}_{B/i})_{/k} - g^j_i.L \ , \tag{3.2}$$

$$K_i{}^j = (S_{Cm}{}^{An}V^C{}_B - S_{Bm}{}^{Dn}V^A{}_D)\frac{\partial L}{\partial V^A{}_{B/k/j}}.R^m{}_{nik} +$$

$$+ \frac{\partial L}{\partial V^A{}_{B/k/j}}.V^A{}_{B/l}T_{ik}{}^l \ , \tag{3.3}$$

$$V_i{}^{kj}{}_l = Q_i{}^{kj}{}_l - P_i{}^{kj}{}_l = (Q_i{}^{kj} - P_i{}^{kj})_{/l} \ , \tag{3.4}$$

$$Q_i{}^{kj}{}_l = S_{Bi}{}^{Dj}\Big[\frac{\partial L}{\partial V^A{}_{B/k}}.V^A{}_D + \Big(\frac{\partial L}{\partial V^A{}_{B/k/m}} + \frac{\partial L}{\partial V^A{}_{B/m/k}}\Big)V^A{}_{D/m} - $$

$$- \Big(\frac{\partial L}{\partial V^A{}_{B/k/m}}.V^A{}_D\Big)_{/m}\Big]_{/l} = Q_i{}^{kj}{}_{/l} , \qquad (3.5)$$

$$P_i{}^{kj}{}_l = S_{Ci}{}^{Bj}\Big[\frac{\partial L}{\partial V^A{}_{E/k}}.V^C{}_B + \Big(\frac{\partial L}{\partial V^A{}_{B/k/m}} + \frac{\partial L}{\partial V^A{}_{B/m/k}}\Big)V^C{}_{B/m} - $$

$$- \Big(\frac{\partial L}{\partial V^A{}_{B/k/m}}.V^C{}_E\Big)_{/m}\Big]_{/l} + \Big(\frac{\partial L}{\partial V^A{}_{B/j/k}}.V^A{}_{B/i}\Big)_{/l} = P_i{}^{kj}{}_{/l}, \qquad (3.6)$$

$$V_i{}^{kj}{}_k = V_i{}^{kj}{}_{/k} = g^l_k.V_j{}^{kj}{}_{/l} , \qquad (3.7)$$

$$Q_i{}^j = (S_{Bi}{}^{Dj}V^A{}_D - S_{Ci}{}^{Aj}V^C{}_B)\frac{\delta L}{\delta V^A{}_B} . \qquad (3.8)$$

After representation of $V_i{}^{kj}{}_l$ by means of combination of a symmetric and antisymmetric part in k and j, i.e.

$$V_i{}^{kj}{}_l = {}_sV_i{}^{kj}{}_l + {}_aV_i{}^{kj}{}_l , \qquad (3.9)$$

$${}_sV_i{}^{kj}{}_l = \frac{1}{2}(V_i{}^{kj}{}_l + V_i{}^{jk}{}_l) = {}_sV_i{}^{jk}{}_l := V_i{}^{(kj)}{}_l , \qquad (3.10)$$

$${}_aV_i{}^{kj}{}_l = \frac{1}{2}(V_i{}^{kj}{}_l - V_i{}^{jk}{}_l) = - {}_aV_i{}^{jk}{}_l := V_i{}^{[kj]}{}_l , \qquad (3.11)$$

the components $P_i{}^j$ will have the form:

$$P_i{}^j = \Theta_i{}^j - {}_sT_i{}^j - Q_i{}^j , \qquad (3.12)$$

where

$$\Theta_i{}^j = t_i{}^j - K_i{}^j - W_i{}^{jk}{}_k , \qquad (3.13)$$

$$W_i{}^{jk}{}_k = g^l_k W_i{}^{jk}{}_l = g^l_k g_{im} W^{mjk}{}_l, \qquad (3.14)$$

$$W^{mjk}{}_l = {}_sV_n{}^{jm}{}_l g^{nk} - {}_sV_n{}^{km}{}_l g^{nj} - {}_aV_n{}^{jk}{}_l g^{nm} , \qquad (3.15)$$

$${}_sT_i{}^j = \mathcal{T}_i{}^j - 2.\frac{\partial L}{\partial g_{jk}}.g_{ik} - g^j_i.L , \qquad (3.16)$$

$$\mathcal{T}_i{}^j = g_{ik}\mathcal{T}^{kj} , \quad \mathcal{T}^{kj} = \mathcal{T}^{jk} = g^l_m \mathcal{T}^{kjm}{}_l = \mathcal{T}^{kjl}{}_l , \qquad (3.17)$$

$$\mathcal{T}^{ijk}{}_l = {}_sV_m{}^{ik}{}_l g^{mj} + {}_sV_m{}^{jk}{}_l g^{mi} - {}_sV_m{}^{ij}{}_l g^{mk} . \qquad (3.18)$$

2. The identity (2.11), having the form

$$P_i = \frac{\partial L}{\partial V^A{}_B}.V^A{}_{B/i} + \frac{\partial L}{\partial V^A{}_{B/j}}.V^A{}_{B/j/i} + \frac{\partial L}{\partial V^A{}_{B/j/k}}.V^A{}_{B/j/k/i} + $$

$$+ \frac{\partial L}{\partial g_{jk}}.g_{jk/i} - L_{/i} = 0 , \qquad (3.19)$$

can after a sequence of transformations be written as

$$P_i = \frac{\delta L}{\delta V^A{}_B} . V^A{}_{B/i} + W_i + \Theta_i{}^j{}_{/j} = F_i + \Theta_i{}^j{}_{/j} = 0 \, , \tag{3.20}$$

where

$$F_i = \frac{\delta L}{\delta V^A{}_B} . V^A{}_{B/i} + W_i \quad , \tag{3.21}$$

$$W_i = S_i - S_k{}^j T_{ij}{}^k + \frac{\partial L}{\partial g_{jk}} . g_{jk/i} \quad , \tag{3.22}$$

$$\begin{aligned} S_i = {} & W_i{}^{jk}{}_{k/i} - W_i{}^{jk}{}_{/k/i} - \\ & - \tfrac{1}{2}[W_i{}^{jk}{}_{/l} T_{jk}{}^l + W_l{}^{jk}(T_{<ij}{}^l{}_{/k>} + T_{<ij}{}^m T_{mk>}{}^l) + \\ & + W_i{}^{jk}(T_{jk}{}^l{}_{/l} + T_{jk} - T_{kj} + T_l T_{jk}{}^l - R^m{}_{mjk})] + \\ & + \mathcal{T}^{njk} g_{nl} R^l{}_{jik} \, , \end{aligned} \tag{3.23}$$

$$W_i{}^{jk} = g_{il} W^{ljk} = - W_i{}^{kj} \, , \tag{3.24}$$

$$W^{ljk} = - W^{lkj} = {}_sV_m{}^{jl} g^{mk} - {}_sV_m{}^{kl} g^{mj} + {}_aV_m{}^{kj} g^{ml} \, , \tag{3.24a}$$

$$V_m{}^{ij} = {}_sV_m{}^{ij} + {}_aV_m{}^{ij} \, , \quad {}_sV_m{}^{ij} = V_m{}^{(ij)} \, , \quad {}_aV_m{}^{ij} = V_m{}^{[ij]} , \tag{3.25}$$

$$V_i{}^{jk} = W_i{}^{jk} + \mathcal{T}_i{}^{jk} \, , \quad \mathcal{T}_i{}^{jk} = g_{il} \mathcal{T}^{ljk} \, , \tag{3.26}$$

$$\mathcal{T}^{ljk} = \mathcal{T}^{jlk} = {}_sV_m{}^{jk} g^{ml} + {}_sV_m{}^{lk} g^{mj} - {}_sV_m{}^{jl} g^{mk} \, , \tag{3.27}$$

$$T_{<ij}{}^l{}_{/k>} = T_{ij}{}^l{}_{/k} + T_{ki}{}^l{}_{/j} + T_{jk}{}^l{}_{/i} \, , \quad T_{jk} = g^m_l . T_{mj}{}^l{}_{/k} \, ,$$

$$T_i = g^j_k . T_{ij}{}^k = T_{ik}{}^k \, ,$$

$$S_k{}^j = \Big[\frac{\partial L}{\partial V^A{}_{B/j}} - \Big(\frac{\partial L}{\partial V^A{}_{B/j/l}}\Big)_{/l}\Big] V^A{}_{B/k} + \frac{\partial L}{\partial V^A{}_{B/l/j}} . V^A{}_{B/l/k} \, . \tag{3.28}$$

The components W_i of the covariant vector field W depend on the components of the torsion tensor and of the covariant derivatives of the metric tensor in such a way, that in the case of $g_{ij/k} = 0$, $T_{ij}{}^k = 0$ they are identical with those defined in V_n-spaces /5/. The quantities F_i, $\Theta_i{}^j$, ${}_sT_i{}^j$ and $Q_i{}^j$ can therefore have the same notations in L_n-spaces as in the case of V_n- and U_n-spaces.

Now the GCBI can be written as follows:

$$F_i + \Theta_i{}^j{}_{/j} = 0 \tag{3.29}$$

$$\Theta_i{}^j - {}_sT_i{}^j = Q_i{}^j \, . \tag{3.30}$$

3. On the basis of the obtained GCBI and in analogy with the tensors of the type of energy-momentum tensor, introduced in V_n-and U_n-spaces, the tensors of the same type can be defined also in L_n-spaces. In accordance with the structure of $\theta_i{}^j$, ${}_sT_i{}^j$, $t_i{}^j$ and $Q_i{}^j$, the following notations can be used:

- $t_i{}^j$ - canonical energy-momentum tensor for a Lagrangian system,
- $\theta_i{}^j$ - generalized canonical energy-momentum tensor for a Lagrangian system,
- ${}_sT_i{}^j$ - symmetric energy-momentum tensor (of Belinfante)
- $Q_i{}^j$ - variational energy-momentum tensor (of Euler-Lagrange).

The variational energy-momentum tensor vanished if the covariant Euler-Lagrange equations (CELE) are fulfilled. It means that this tensor appears as typical of the Lagrangian system only when the CELE are broken down under the interaction of other, except for $V^A{}_B$ and g_{ij}, fields which are not *included in the Lagrangian system, i.e.*

$$\frac{\delta L}{\delta V^A{}_B} = 0 : \qquad Q_i{}^j = 0 ,$$

$$\frac{\delta L}{\delta V^A{}_B} \neq 0 : \qquad Q_i{}^j \neq 0 \text{ or } Q_i{}^j = 0 . \qquad (3.31)$$

4. By means of GCBI and in analogy with similar propositions for the cases in V_n- and U_n-spaces some propositions can be proved, expressing the connections between the different types of energy-momentum tensors.

Proposition 1. *The necessary and sufficient conditions for the* equality of the generalized canonical energy-momentum tensor (GCEMT) and the symmetric energy-momentum tensor (SEMT(B)), i.e. the necessary and sufficient condition for

$$\theta_i{}^j = {}_sT_i{}^j \qquad (3.32)$$

are the conditions

$$\zeta_i{}^j = (S_{Bi}{}^{Dj}V^A{}_D - S_{Ci}{}^{Aj}V^C{}_E).\frac{\delta L}{\delta V^A{}_B} = 0 . \qquad (3.33)$$

Proof: a) Necessity: follows from conditions (3.32) and identity (3.30).

b) Sufficiency: follows from conditions (3.33) and identity (3.30).

<u>Proposition 1.1.</u> If $V^A{}_B = \varphi$ is a scalar field, then the condition $\theta_i{}^j = {}_sT_i{}^j$ is always valid.
Proof: For scalar fields $S_{Di}{}^{Dj}V^A{}_B = S_{Ci}{}^{Aj}V^C{}_E = 0$ (rank $V^A{}_B = 0$) and therefore the condition $\zeta_i{}^j = 0$ is always fulfilled. From identity (3.30) follows (3.32). Hence, the variational energy-momentum tensor for Lagrangian systems, constructed only of scalar fields, is identically equal to zero regardless of the fulfilment of the CELE.

IV. CONCLUSION

In the present paper the possibilities are considered for obtaining energy-momentum tensors for theories in spaces with affine connection and metric. By means of the MLCD the s.c. generalized covariant Bianchi type identities are found. On the ground of these identities, the connections between the energy-momentum tensors (canonical, symmetric and variational) are investigated and possibilities for the existence of the symmetric energy-momentum tensor as a local conserved quantity can be easily found.

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