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GEOMETRICAL APPROACH TO THE DYNAMICS OF A RELATIVISTIC STRING IN THE D-DIMENSIONAL MINKOWSKI SPACE

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I. INTRODUCTION

The analysis of the relativistic string model with point masses at the ends is of great interest in hadron physics, for it gives an obvious demonstration of keeping quarks in hadrons, in particular, in the nonrelativistic limit the string reduces to the growing linearly potential of the interaction between point masses which are connected by the string [1,2].

The dynamics of this model is determined by the equations of motion and nonlinear boundary conditions and just at the classical level there appear difficulties when solving these boundary conditions. For the present, only a few particular solutions [3,4] of boundary equations are found.

We have used the differential geometrical approach developed in [4] which allows one to find some solutions of the boundary value problem in the three-dimensional Minkowski space E_2^1. In the present paper this approach is applied to the string with masses in a D-dimensional space E_{D-1}^1 $(D \geq 4)$. The case D=4 (E_3^1) is picked out.

The minimal surface swept by the massive string is restricted by two world trajectories of the point masses, these curves are determined by the first, second and third curvatures K, $\varkappa_1$ and $\varkappa_2$ in the space E_3^1. It is shown that if one assumes $\varkappa_1 =$ const and $\varkappa_2 =$ const, the equations of the boundaries lead to $\varkappa_2 = 0$ for the world trajectories of point masses, and the solution of the problem in E_3^1 will coincide with the solution in E_2^1 [4] and the string world surface will be a helicoid.

This result turned out to be true in the space E_{D-1}^1 provided the curvature vectors $\nu_{\mu|1} = 0$ which means that the system of the boundary equations in the D-dimensional space reduces to system in the space with D=4.

2. GEOMETRICAL APPROACH TO THE DYNAMICS OF THE RELATIVISTIC STRING WITH MASSES AT THE ENDS

Consider the motion of the relativistic string in the D-dimensional pseudo-Euclidean space E^1_{D-1}. At the ends of the string there are point masses and the world surface S of the string will be restricted by the world trajectories of these masses. We will apply the geometrical methods not only to the surface S, but also to the restricting curves. Let x^μ, $\mu = 0,1 \ldots D-1$ be the coordinates in the Minkowski space and $u^1 = \tau$, $u^2 = \sigma$ curvilinear coordinates on the world surface S of the string which is an external of the functional of action:

$$S = -\gamma \int_{\tau_1}^{\tau_2} \int_{\sigma_1(\tau)}^{\sigma_2(\tau)} d^2u \sqrt{\det\|g_{ij}\|} - \sum_{i=1}^{2} m_i \int_{\tau_1}^{\tau_2} d\tau \sqrt{\left(x^\mu_{,1}(\tau,\sigma_i)\right)^2}, \quad (1)$$

where $x_{,i} = \frac{\partial x^\mu}{\partial u^i}$ and $g_{ij} = x^\mu_{,i} x_{\mu,j}$ are the components of the metric induced on the surface S. Constant γ has the dimension of mass squared and determines the string tension.

On the surface S one can always introduce isothermal (conformal) coordinates in terms of which the metric tensor g_{ij} is diagonal and traceless [5,6]:

$$g_{11} = -g_{22} = (x^\mu_{,1})^2 = -(x^\mu_{,2})^2 = g, \quad (2)$$
$$g_{12} = (x^\mu_{,1} \cdot x^\mu_{,2}) = 0 .$$

Variation of the functional of action (I) gives the equation of motion (D'Alembert equation)

$$X^{\mu}_{,11}(\tau,\sigma) - X^{\mu}_{,22}(\tau,\sigma) = 0 \tag{3}$$

and nonlinear boundary conditions

$$m_i \frac{d}{d\tau}\left\{\frac{X^{\mu}_{,1}(\tau,\sigma_i) + X^{\mu}_{,2}(\tau,\sigma_i)\cdot\sigma_{i,1}}{\sqrt{(X^{\mu}_{,1}(\tau,\sigma_i))^2\cdot(1-\sigma_{i,1}^2)}}\right\} = (-1)^{i+1}\gamma\left[X^{\mu}_{,2}(\tau,\sigma_i) + X^{\mu}_{,1}\cdot\sigma_{i,1}\right], \tag{4}$$

$$(i = 1,2).$$

The world surface of the relativistic string may be described by the basis that is a set of orthogonal vectors at every point of the surface [6,7]. Knowing the evolution of this basis one can restore all the surface. This basis can be made up out of the vectors $X^{\mu}_{,1}$, $X^{\mu}_{,2}$ tangent to the surface and normals η^{μ}_{α}, α =3,4... D [2] of this basis on the surface S is described by derivative equations [7,8]:

$$
\begin{aligned}
X^{\mu}_{,11} &= -\frac{\varphi_{,1}}{2} X^{\mu}_{,1} - \frac{\varphi_{,2}}{2} X^{\mu}_{,2} - \sum_{\alpha=3}^{D} b_{\alpha|11}\,\eta^{\mu}_{\alpha}, \\
X^{\mu}_{,12} &= -\frac{\varphi_{,2}}{2} X^{\mu}_{,1} - \frac{\varphi_{,1}}{2} X^{\mu}_{,2} - \sum_{\alpha=3}^{D} b_{\alpha|12}\,\eta^{\mu}_{\alpha}, \\
\eta_{\alpha,i} &= -b_{\alpha|ij}\, g^{jk} X^{\mu}_{,k} - \sum_{\beta=3}^{D} \nu_{\beta\alpha|i}\,\eta^{\mu}_{\beta},
\end{aligned}
\tag{5}
$$

$$(\alpha = 3,\dots D,\ i = 1,2),$$

where $b_{\alpha|ij} = \eta^{\mu}_{\alpha}\cdot X_{\mu,ij}$ are coefficients of the second quadratic form setting the external geometry of the world surface S and $\nu_{\beta\alpha|i} = -\nu_{\alpha\beta|i}$ are the curvature vectors. The function $\varphi(\tau,\sigma)$ is defined as $e^{-\varphi} = g(\tau,\sigma)$.

Projecting (4) on any normal η^μ_α one gets

$$(1+\sigma_{i,1}^2)\cdot b_{\alpha|11}(\tau,\sigma_i) + 2\, b_{\alpha|12}(\tau,\sigma_i)\cdot\sigma_{i,1} = 0\,, \qquad (i=1,2). \tag{6}$$

Let choose the gauge of a coordinate on the surface S so as $\sigma_{i,1}=0$ and as the range of definition Ω of parameters τ and σ the rectangle: $\{\tau_1\le\tau\le\tau_2\,, 0=\sigma_1\le\sigma\le\sigma_2=\pi\}$.

Below we will see that this would does not bring any additional restriction on solving the problem (3)-(4).

Indeed, if one compares the first equation in (5) at $\sigma = \sigma_i =$ const and the boundary condition (4), equating expressions of the same basis vectors, we'll get some relations at the boundaries:

$$b_{\alpha|11}(\tau,\sigma_i) = 0 \,, \tag{7}$$

$$-\frac{\varphi_2}{2} = (-1)^{i+1}\frac{\gamma}{m_i}\, e^{-\frac{\varphi(\tau,\sigma_i)}{2}}\,, \qquad (i=1,2). \tag{8}$$

Equations (6) and (7) lead to conditions $\sigma_{i,1}=0$ since in general $b_{\alpha|12}(\tau,\sigma_i)\neq 0$.

The boundary condition (8) had been studied for the case E^1_2 [4]. With the help of solutions of these equations the world surface for some particular kinds of motion of the relativistic string with masses at the ends were found. As is generally known [2,7] a minimal surface can be described in terms of its radius vector $X^\mu(\tau,\sigma)$ or a set of variables $g_{ij}(\tau,\sigma)$, $b_{\alpha|ij}(\tau,\sigma)$, $\nu_{\alpha\beta|i}(\tau,\sigma)$ which satisfy the minimal surface conditions:

$$b_{\alpha|ij}\cdot g^{ij}=0 \quad , \tag{9}$$

Gauss equations

$$R_{ijkl} = -\sum_{\alpha=3}^{D}\left(b_{\alpha|ik}\, b_{\alpha|jl} - b_{\alpha|il}\, b_{\alpha|jk}\right), \tag{10}$$

Peterson – Kodazzi equations

$$\nabla_k b_{\alpha|ij} - \nabla_j b_{\alpha|ik} = -\sum_{\beta=3}^{D}\left(\nu_{\beta\alpha|k}\, b_{\beta|ij} - \nu_{\beta\alpha|j}\, b_{\beta|ik}\right), \tag{11}$$

and Ricci equations

$$\nu_{\beta\alpha|j,k} - \nu_{\beta\alpha|k,j} - \sum_{\gamma=3}^{D}\left(\nu_{\gamma\beta|j}\,\nu_{\gamma\alpha|k} - \nu_{\gamma\beta|k}\,\nu_{\gamma\alpha|j}\right) + \tag{12}$$

$$+\, g^{lm}\left(b_{\beta|lj}\, b_{\alpha|mk} - b_{\beta|lk}\, b_{\alpha|mj}\right) = 0,$$

where ∇_j is a covariant derivative with respect to the metric tensor g_{ij}:

$$\nabla_j X^{\mu}_{,i} = X^{\mu}_{,ij} - \Gamma^{k}_{ij} X^{\mu}_{,k} \quad . \tag{13}$$

Here Γ^{k}_{ij} are the Christoffel symbols for g_{ij} [7]:

$$\Gamma^{k}_{ij} = g^{kl}\,\Gamma_{l,ij} = \frac{1}{2} g^{kl}\left(g_{li,j} + g_{jl,i} - g_{ij,l}\right) \tag{14}$$

and the Riemann curvature tensor R_{ijkl} has only one essential component R_{1212}.

The minimum conditions (9) (or in other words, equality of the mean curvature in η^{μ}_{α} -direction to zero) follow from (3) and from the definition of the coefficients of the second quadratic form $b_{\alpha|ij}$. Equations (IO)-(12) are conditions of integration of the derivative formulae (5). This system will considerably get simplified, if we choose a new basis of vectors on the minimum surface so that the normal η_3^μ points into the direction $\nabla_1 X^\mu_{,2}$ and η_4^μ along $\nabla_1 X^\mu_{,1}$ [2]. This transition may be done through rotations from $SO(1,1)\times SO(D-2)$ group which don't mix the tanget vectors $X^\mu_{,i}$ (i=1,2) and normals η^μ_α $(\alpha=3,\dots D)$. Then from (5) we find

$$b_{4|12} = b_{3|11} = b_{3|22} = b_{\alpha|ij} = 0 \quad , \qquad (\alpha=5,\dots D,\ i,j=1,2), \tag{15}$$

and from Peterson - Kodazzi equations after simple transformations it follows that

$$\frac{\partial}{\partial u^{\mp}} \sum_{\alpha=3}^{D} \left(b_{\alpha|11} \pm b_{\alpha|12} \right)^2 = 0 , \tag{16}$$

where isotropic coordinates $u^{\pm} = u_1 \pm u_2 = \tau \pm \sigma$ are introduced. Integrating (16) we have

$$\sum_{\alpha=3}^{D} \left(b_{\alpha|11} \pm b_{\alpha|12} \right)^2 = A^2_{\pm}(u^{\pm}) \quad . \tag{17}$$

On the other hand, taking into account (5), we obtain

$$-\sum_{\alpha=3}^{D} \left(b_{\alpha|11} \pm b_{\alpha|12} \right)^2 = \left(\nabla_1 X^\mu_{,1} \pm \nabla_1 X^\mu_{,2} \right)^2 . \tag{18}$$

Our choice of the normals η_3^μ and η_4^μ leads to $\nabla_1 X^\mu_{,1}\cdot \nabla_1 X^\mu_{,2} = 0$ and, hence, $A^2_- = A^2_+ = A^2$. Taking into consideration (15) we simplify equation (17):

$$b_{4|11}^2 + b_{3|12}^2 = A^2. \tag{19}$$

To satisfy this equation, it is necessary to put down

$$b_{4|11} = A\cos\frac{\theta}{2}, \quad b_{3|12} = A\sin\frac{\theta}{2}. \tag{20}$$

where the function $\theta = \theta(\tau,\sigma)$ defines the curvature vectors $\nu_{\alpha\beta|i}$. Indeed, from equation (11) it follows that

$$\nu_{34|1} = -\frac{\theta_{,2}}{2}, \quad \nu_{34|2} = -\frac{\theta_{,1}}{2} \tag{21}$$

$$\nu_{3\alpha|1} = \nu_{4\alpha|2}\cdot\operatorname{ctg}\frac{\theta}{2}, \quad \nu_{3\alpha|2} = \nu_{4\alpha|1}\operatorname{ctg}\frac{\theta}{2}, \quad (\alpha = 5\ldots D)$$

and Gauss equations become

$$\varphi_{,11} - \varphi_{,22} = 2A^2\cos\theta\, e^{\varphi}. \tag{22}$$

From (20) due to (7) it follows that at the boundary $(\sigma=\sigma_i)$ the function $\theta(\tau,\sigma_i) = \pi(2n_i+1)$, where n_i is an integer. The only coefficient different from zero of the second quadratic form is $b_{4|12} = A$ and the curvature vectors $\nu_{3\alpha|1} = \nu_{3\alpha|2} = 0\ (\alpha = 5,\ldots D)$. Now equations (5) at the boundaries become more simple in the new notation $\xi_1^\mu = \frac{X^\mu_{,1}(\tau,\sigma_i)}{\sqrt{g(\tau,\sigma_i)}}$;

$$\xi_2^\mu = \frac{X^\mu_{,2}(\tau,\sigma_i)}{\sqrt{g(\tau,\sigma_i)}}; \quad \varkappa_1 = \frac{A}{\sqrt{g(\tau,\sigma_i)}}; \quad \varkappa_2 = \frac{\theta_{,2}}{\sqrt{g(\tau,\sigma_i)}}; \quad K_i = (-1)^{i+1}\frac{\gamma}{m_i}; \tag{23}$$

$$\frac{d\xi_1^\mu}{ds} = K_i\xi_2^\mu, \quad \frac{d\xi_2^\mu}{ds} = K_i\xi_1^\mu - \varkappa_1\eta_3^\mu;$$

$$\frac{d\eta_3^\mu}{ds} = \varkappa_1\xi_2 - \varkappa_2\eta_4^\mu; \quad \frac{d\eta_4^\mu}{ds} = \varkappa_2\eta_3^\mu + \sum_{\beta=5}^{D}\frac{\nu_{4\beta|1}}{\sqrt{g(\tau,\sigma_i)}}\eta_\beta^\mu;$$

$$\frac{d\eta_\alpha^\mu}{dS} = \sum_{\beta=3}^{D} \frac{\nu_{\alpha\beta|1}}{\sqrt{g(\tau,\sigma_i)}} \eta_\beta^\mu \quad , \qquad (\alpha = 5,\dots D\ , i=1,2)$$

where S is a natural parameter : $dS = \sqrt{g}\, d\tau$. As could be seen from equations (23), the case D=4 is certainly on a distinct status. Further we will notice that in a particular case when $\nu_{4\alpha|1} = 0$, the system of equations (23) divides into two independent systems, one of which has a form of the system (23) for D=4.

3. THE STRING WITH MASSES IN THE FOUR-DIMENSIONAL SPACE E_3^1

Consider the system of equations (23) when D=4. It is the system of Frenet equations for the curves in the space E_3^1:

$$\frac{d\xi_1^\mu}{dS} = K_i \xi_2^\mu \quad ; \quad \frac{d\xi_2^\mu}{dS} = K_i \xi_1^\mu - \varkappa_1 \eta_3^\mu ; \qquad (24)$$

$$\frac{d\eta_3^\mu}{dS} = \varkappa_1 \xi_2^\mu - \varkappa_2 \eta_4^\mu ; \quad \frac{d\eta_4^\mu}{dS} = \varkappa_2 \eta_3^\mu .$$

- Now one can realize the geometrical meaning of the coefficients K_i, $\varkappa_1$ and $\varkappa_2$ [9] . These are respectively the first, second and third curvatures of world trajectories of point masses. Note that the first curvatures K_i (i=1,2) are constant for all D then curvatures of the mass trajectories are constant and equal $\frac{\gamma}{m_i}$.

The search of the general solutions to equations (24) is very complicated, that's why one can find only a few particular solutions. It is known that in the space E^1_2 there is a solution for the case when the torsions ˣ are constant [4].

4. THE CONSTANT CURVATURES IN THE SPACE D = 4

When $\varkappa_1$ and $\varkappa_2$ are constant, the system (24) reduces to the linear equation for ξ_1^μ [9]:

$$\frac{d^4\xi_1^\mu}{ds^4} + (\varkappa_1^2+\varkappa_2^2-K_i^2)\frac{d^2\xi_1^\mu}{ds^2} - K_i^2\varkappa_2^2\xi_1^\mu = 0 \qquad (25)$$

that has the characteristic equation

$$\lambda^4 + (\varkappa_1^2+\varkappa_2^2-K_i^2)\lambda^2 - K_i^2\varkappa_2^2 = 0 \qquad (26)$$

with roots:

$$\lambda^2 = -\frac{\varkappa_1^2+\varkappa_2^2-K_i^2}{2} \pm \frac{1}{2}\sqrt{(\varkappa_1^2+\varkappa_2^2-K_i^2)+4K_i^2\varkappa_2^2} \; . \qquad (27)$$

There are four kinds of solution of (25) depending on the sign of the radical and relations between K_i, $\varkappa_1$ and $\varkappa_2$, though equalities $(\xi_1^\mu)^2 = 1$, $(\xi_2^\mu)^2 = -1$ distinguish two solutions:

$$\xi_1^\mu = A^\mu\cos\alpha s + B^\mu\sin\alpha s + C^\mu s + D^\mu, \quad \alpha = \sqrt{\varkappa_1^2-K_i^2} \qquad (28)$$

when $\varkappa_2 = 0$, $\varkappa_1^2 > K_i^2$ and

$$\xi_1^\mu = A^\mu e^{-\bar\alpha s} + B^\mu e^{\bar\alpha s} + C^\mu\cos\bar\alpha s + D^\mu\sin\bar\alpha s, \quad \bar\alpha = \sqrt{\varkappa_2|K_i|} \qquad (29)$$

ˣ The first and second curvatures are called the curvature and the torsion in the space E^1_2.

when $\varkappa_1^2+\varkappa_2^2=K_i^2$, $\varkappa_2\neq 0$.

Here $A^\mu, B^\mu, D^\mu, C^\mu$ some constant vectors determined by normalization of vectors ξ_1^μ and ξ_2^μ .

These solutions concern the trajectories of the point masses, but we are interested in the world surface of the string. For this purpose use can be made of the solution of the equations for string (3) in terms of isotropic vectors.

On the plane tangent to the world surface of the string one can always transform the vectors $X^\mu_{,1}$ and $X^\mu_{,2}$ into isotropic ones $\Psi'^\mu_\pm(u^\pm)$ (the prime means the derivative of the function Ψ with respect to its argument). Further, we will use the general solution of the string equation (3):

$$X^\mu(\tau,\sigma)=\frac{\Psi^\mu_+(u^+)+\Psi^\mu_-(u^-)}{2} \qquad (30)$$

By definition of ξ_1^μ and ξ_2^μ it follows that

$$\Psi'^\mu_\pm(u^\pm)=(\xi_1^\mu\pm\xi_2^\mu)\sqrt{g}\;,\quad (\Psi'^\mu_\pm)^2=0 \qquad (31)$$

Owing to

$$\xi_2^\mu=\frac{1}{K_i}\frac{d\xi_1^\mu}{ds} \qquad (32)$$

from equation (28) one finds $\Psi'^\mu_\pm(\tau\pm\sigma_i)$; then by integration $\Psi^\mu_\pm(\tau\pm\sigma)$ and, hence, $X^\mu(\tau,\sigma)$. Since two boundaries give two solutions, it is necessary to make them consistent, i.e., to compare at $\sigma=\sigma_i$ (i=1,2). In this case the solutions are

$$X^\mu(\tau,\sigma) = \left(\frac{A_i^\mu}{\alpha}\sin\alpha\sqrt{g}\tau - \frac{B_i^\mu}{\alpha}\cos\alpha\sqrt{g}\tau\right)\times$$
$$\times\left(\cos\alpha\sqrt{g}(\sigma-\sigma_i) + \frac{\alpha}{K_i}\sin\alpha\sqrt{g}(\sigma-\sigma_i)\right) +$$
$$+ g C_i^\mu \frac{\tau^2+(\sigma-\sigma_i)^2}{2} + \sqrt{g}\tau\left(D_i^\mu + \frac{C_i^\mu}{K_i}\right) + F_i, \tag{33}$$
$$(i=1,2)$$

where $A_i^\mu, B_i^\mu, C_i^\mu, D_i^\mu, F_i^\mu$ are some vectors. Without loss of generality we put $m_i = m$, i.e. $K_i = (-1)^{i+1} K, (i=1,2)$. Then the consistency between solutions (33) is possible provided that

$$\operatorname{tg}\omega\pi = \frac{2\omega K\sqrt{g}}{\omega^2 - (K\sqrt{g})^2}, \quad \omega = \alpha\sqrt{g}, \tag{34}$$

and also $A_i^\mu = A^\mu$, $B_i^\mu = B^\mu$, $C_i^\mu = 0$, $D_i^\mu = D^\mu$, $F_i^\mu = F^\mu$, $(i=1,2)$. The condition (34) holds when the coefficients of the same degrees of the parameter τ are compared, the conditions normalization $\xi_1^2 = 1$ and $\xi_2^2 = -1$ lead to the following conditions on the vectors:

$$A^2 = B^2 = -\frac{K^2}{2\alpha^2}, \quad D^2 = 1 + \frac{K^2}{2\alpha^2}, \quad D\cdot A = B\cdot A = DB = 0. \tag{35}$$

So, the world surface of the relativistic string has the form:

$$X^\mu(\tau,\sigma) = \left(\frac{A^\mu}{\alpha}\sin\alpha\sqrt{g}\tau - \frac{B^\mu}{\alpha}\cos\alpha\sqrt{g}\tau\right)\times$$
$$\times\left(\cos\alpha\sqrt{g}\sigma - \frac{\alpha}{K}\cdot\sin\alpha\sqrt{g}\sigma\right) + \sqrt{g}\tau D^\mu + F^\mu, \tag{36}$$
$$\alpha = \sqrt{\varkappa_1^2 - K^2},$$

where F^μ is an arbitrary constant vector.

The condition (34) coincides with the frequency equation derived in [10] for string motions when the parameter τ is the proper time of massive points at the ends of the string. This means that

$$X_{,1}^{2}(\tau,0) = X_{,2}^{2}(\tau,\pi) = g = m^{-2} \quad . \tag{37}$$

In the gauge $t \sim \tau$ (t — proper time) for different choices of the vectors $A^{\mu}, B^{\mu}, C^{\mu}, F^{\mu}$ we get a set of two--dimensional surfaces which, after appropriate transformations of global coordinates, become helicoids embedded into the four--dimensional Minkowski space.

In the case (29) the solution satisfying one boundary contradicts the other. So, for constant curvatures, there exists only one kind of solution (36), $\varkappa_2$ being zero. It means that $\theta_{,2}(\tau,\sigma_i) = 0$. As $\theta(\tau,\sigma_i)$ = const and using the Ricci equation (12) for $\alpha = 3$, $\beta = 4$

$$\theta_{,11} - \theta_{,22} = 2 A^{2} e^{\varphi} \sin\theta \quad , \tag{38}$$

one can show that $\frac{\partial^{n}\theta(\tau,\sigma_i)}{\partial u_i^{m} \partial u_j^{n-m}}$, where i, j = 1,2 and n,m are arbitrary (n > m). Indeed, from $\theta_{,1}(\tau,\sigma_i) = 0$, $\theta(\tau,\sigma_i) = \pi \cdot (2n_i + 1)$ and (38) it follows that $\theta_{,22}(\tau,\sigma_i) = 0$. Then by derivation of (38) at $\sigma = \sigma_i$ and by induction one proves the above statement. Under the assumption that the function $\theta(\tau,\sigma)$ is infinitely differentiable in the extended range $\Omega^{*} = \{ \tau_1 \le \tau \le \tau_2 \ , \ -\varepsilon \le \sigma \le \pi + \varepsilon \ , \ 0 < \varepsilon < 1 \}$, we find that $\theta(\tau,\sigma)$ is constant and equals $\theta = \pi(2n+1)$ in Ω^{*} , i.e. at both the boundaries it has the same values.

Equation (22) in this case coincides with the Gauss equation for the world surface of the relativistic string in the space E_2^1 [2,4] . If the vectors $\Psi_{\pm}'^{\mu}(u^{\pm})$ (30)-(31) are represented as an expansion over the constant basis $\{a^{\mu}, b_1^{\mu}, b_2^{\mu}, c^{\mu}\}$ [2] made for E_2^1 [4]:

$$\Psi_+'(u^+) = \frac{A}{\sqrt{f_1'^2+f_2'^2}} \left(a^{\mu} + b_1^{\mu} f_1 + b_2^{\mu} f_2 + c^{\mu} \frac{f_1^2+f_2^2}{2}\right), \tag{39}$$

$$\Psi_-'(u^-) = \frac{A}{\sqrt{g_1'^2+g_2'^2}} \left(a^{\mu} + b_1^{\mu} f_1 + b_2^{\mu} f_2 + c^{\mu} \frac{f_1^2+f_2^2}{2}\right),$$

where $f_1(u^+)$, $f_2(u^+)$, $g_1(u^-)$, $g(u^-)$ are some functions and the basis $\{a^{\mu}, b_1^{\mu}, b_2^{\mu}, c^{\mu}\}$ is defined by equalities:

$$a^2 = c^2 = 0 , \quad a \cdot c = 1 , \quad b_k \cdot b_j = -\delta_{kj} , \tag{40}$$
$$a \cdot b_k = c \cdot b_k = 0 , \quad (k, j = 1,2) ,$$

then in terms of new variables we have

$$g(\tau, \sigma) = \frac{A^2}{4} \frac{(f_1 - g_1)^2 + (f_2 - g_2)^2}{\sqrt{(f_1'^2 + f_2'^2)(g_1'^2 + g_2'^2)}} \tag{41}$$

and (20) gives the expression for the function $\theta(\tau,\sigma)$:

$$\cos\theta = \frac{1}{\sqrt{(f_1'^2+f_2'^2)(g_1'^2+g_2'^2)}} \Big(f_1' g_1' + f_2' g_2' - \tag{42}$$
$$- 2 \frac{[f_1'(f_1-g_1) + f_2'(f_2-g_2)] \cdot [g_1'(f_1-g_1) + g_2'(f_2-g_2)]}{(f_1-g_1)^2 + (f_2-g_2)^2} \Big) .$$

The condition $\theta(\tau,\sigma_i) = \pi(2n+1)$ can be realized when $f_1 = g_1 = 0$ or $f_2 = g_2 = 0$. This immediately leads to the three-dimensional case studied in detail in [4].

It is to be noticed that the condition $\theta_{,2}(\tau,\sigma_i) = 0$ in the space E^1_{D-1} is not so "artificial" as the condition $\gamma_{4\alpha|1}(\tau,\sigma_i) = 0$. The Ricci equation (12) at β = 4 together with conditions (21) at $\sigma = \sigma_i$ result in the equalities

$$\theta_{,2}(\tau,\sigma_i) \cdot \gamma_{4\alpha|2}(\tau,\sigma_i) = 0, \quad (\alpha = 5,\dots D). \tag{43}$$

It means that we choose the asymptotic coordinate system [7].

5. Conclusion

In the given geometrical approach to solving the problem of a relativistic massive string the values of the curvatures $\varkappa_1(\tau)$, $\varkappa_2(\tau)$ of the world trajectories of point masses are very important. The full analysis of the boundary equations (5) is complicated. In a particular case of constant $\varkappa_1$ and $\varkappa_2$ the solutions (3)-(4) are obtained.

The search for other configurations of the string world surface is undoubtedly of great importance.

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