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ABOUT SYMMETRY OF THE GRAVITATIONAL ACTION

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Many attempts were made to solve the problem of localization of the gravitational energy by introducing the nondynamical (background) object /1,2/. Usually it was a background metric (bimetric theories) and the gravitation was considered as a conventional matter field alongside with other fields /3/. The theory remained generally covariant but the dynamical invariance under the diffeomorphism group was violated. In the general case, when the background object is arbitrary, the invariance is completely violated, i.e., any residual symmetry is absent. However, if the background object permits the group of motions, the theory is invariant under this group. Usually, the background object is a metric permitting a Poincare group, and thus the energy-momentum problem seems to be solved.

In the present paper it is shown that if we want to use the Einstein equations, then, despite the action functional invariance being violated with respect to the diffeomorphism group by the background object, a new infinite - parameter invariance appears, i.e., the action invariance can be extended from the group of motions of the background object to any infinite - parameter group.

For introducing the energy - momentum density it is enough to consider the affine connection $\check\Gamma^k_{mn}$ as a background object /4,5/ (see also /6/). The difference between $\check\Gamma^k_{mn}$ and the Christoffel's symbols Γ^k_{mn} is the affine - deformation tensor $P^k_{mn} = \check\Gamma^k_{mn} - \Gamma^k_{mn}$.

We start with the Lagrangian

$$L = \sqrt{-g}\, g^{mn}(P^a_{mb}P^b_{an} - P^a_{ba}P^b_{mn}), \qquad (1)$$

where the metric tensor g_{ik} describes the gravitation field and $\check\Gamma^l_{km}$ is the nondynamical background affine connection without torsion. Being varied with respect to g_{ik}, the action

$$S = \int L\, d^4x \qquad (2)$$

leads to the variational derivative

$$\Psi^{mn} = 2\frac{\delta S}{\delta g_{mn}} = \sqrt{-g}\, g^{ma}g^{nb}(\check R_{ab} + \check R_{ba} - \check R_{ij}g^{ij}g_{ab} - 2G_{ab}), \qquad (3)$$

where $G_{ab} = R_{ab} - \frac{1}{2}R\, g_{ab}$ is the Einstein tensor; $\check R_{ik} = \check R^p_{pik}$ is the Ricci tensor, $\check R^p_{lik} = \partial_l\check\Gamma^p_{ik} - \partial_i\check\Gamma^p_{lk} + \check\Gamma^p_{ls}\check\Gamma^s_{ik} - \check\Gamma^p_{is}\check\Gamma^s_{lk}$ is the Riemann tensor for the background connection. As is clear from (3), if $\check R_{(ik)}=0$, then the equations

$$\Psi^{mn} = 0 \qquad (4)$$

coincide with the Einstein equations

$$G_{ik} = 0. \qquad (5)$$

By $\check\nabla_k$ we denote the covariant derivative with respect to $\check\Gamma^k_{mn}$. Let the following terms be defined as

$$t^k_a = \frac{\partial L}{\partial g_{mn,k}}\check\nabla_a g_{mn} - L\,\delta^k_a, \qquad (6)$$

$$\sigma^{jk}_a = \frac{\partial L}{\partial g_{mn,j}}(g_{ma}\delta^k_n + g_{na}\delta^k_m), \qquad (7)$$

where comma before an index means the partial derivative.

All introduced terms are the tensor densities of weight one. What do these terms mean see /6/ or /7/.

The action S is invariant under the Lie variations with an arbitrary vector field ξ^j:

$$\delta x^j = \varepsilon\xi^j, \qquad (8)$$

$$\delta\check\Gamma^k_{mn} = -(\check\nabla_m\check\nabla_n(\varepsilon\xi^k) + \check R^k_{amn}\,\varepsilon\xi^a), \qquad (9)$$

$$\delta g_{mn} = -(g_{ma}\check\nabla_n(\varepsilon\xi^a) + g_{na}\check\nabla_m(\varepsilon\xi^a) + \varepsilon\xi^a\check\nabla_a g_{mn}). \qquad (10)$$

Here ε is an infinitesimal parameter. This invariance is a consequence of the general covariance. But it is not the dynamic invariance because (9) is the transformation of the nondynamic object.

Let the background connection permit the r - parameter group of motion G_r, and let $\xi^j_{(\lambda)}$, $\lambda = 1, \ldots, r$, generate this group, i.e., the equations

$$\check\nabla_m\check\nabla_n\xi^k_{(\lambda)} + \check R^k_{amn}\xi^a_{(\lambda)} = 0 \qquad (11)$$

are satisfied. Then infinitesimal transformations of G_r are

$$\delta x^j = \varepsilon^{(\lambda)}\xi^j_{(\lambda)}, \qquad (12)$$

$$\delta g_{mn} = -(g_{ma}\check\nabla_n(\varepsilon^{(\lambda)}\xi^a_{(\lambda)}) + g_{na}\check\nabla_m(\varepsilon^{(\lambda)}\xi^a_{(\lambda)}) + \varepsilon^{(\lambda)}\xi^a_{(\lambda)}\check\nabla_a g_{mn}). \qquad (13)$$

According to the first Noether theorem the following identities take place /6,8/:

$$\partial_j J^j_{(\lambda)} = X_{mn(\lambda)}\Psi^{mn}, \qquad (14)$$

where

$$J^j_{(\lambda)} = \sigma^{jk}_a\check\nabla_k\xi^a_{(\lambda)} + t^j_a\,\xi^a_{(\lambda)}, \qquad (15)$$

are Noether currents and

$$X_{mn(\lambda)} = -\frac{1}{2}\xi^a_{(\lambda)}\check\nabla_a g_{mn} - g_{ma}\check\nabla_n\xi^a_{(\lambda)} \qquad (16)$$

are generators.

It is easy to see that

$$X_{mn(\lambda)}\Psi^{mn} = \xi^j_{(\lambda)}\nabla_a\Psi^a_j - \partial_k(\Psi^k_a\,\xi^a_{(\lambda)}), \qquad (17)$$

where ∇_a is the covariant derivative with respect to Γ^k_{mn}.

Since we want to use the Einstein equations, then

$$\xi^j_{(\lambda)}\nabla_a\Psi^a_j = 0 \qquad (18)$$

due to the Bianchi identities. As a result, (14) turns into

$$\partial_j(J^j_{(\lambda)} - \Psi^j_a\,\xi^a_{(\lambda)}) \equiv 0, \qquad (19)$$

i.e., $J^j_{(\lambda)}$ is improper /9/. It is clear that this property of $J^j_{(\lambda)}$ is closely connected with (18).

Expression (18) means that there are r identities among the equations $\Psi^{mn} = 0$. These identities can be symbolically written down as

$$\int \Psi^{mn}(x') \Lambda_{mn(\lambda)}(x'x)\, d^4x' = 0, \qquad (20)$$

where $\Lambda_{mn(\lambda)}$ are generators

$$\Lambda_{mn(\lambda)}(x',x) = - \xi_{(\lambda)(m}(x) \nabla_{n)}\delta(x'- x). \qquad (21)$$

Here $\xi_{(\lambda)m} = g_{ma} \xi^a_{(\lambda)}$; $\nabla_n\delta(x'- x)$ is a covariant derivative of the four - dimensional δ - function with respect to x'.

Let us consider the infinitesimal transformations

$$\delta_\Lambda g_{mn}(x) = \int \Lambda_{mn(\lambda)}(x,x') \delta\nu^{(\lambda)}(x')\, d^4x', \qquad (22)$$

where $\delta\nu^{(\lambda)}$ are arbitrary infinitesimal functions of coordinates vanishing at the boundary of the range of integration. Let us substitute (21) into (22) and perform the integration. Then we obtain

$$\delta_\Lambda g_{mn} = - \tfrac{1}{2} (g_{ma}\nabla_n(\delta\nu^{(\lambda)}\xi^a_{(\lambda)}) + g_{na}\nabla_m(\delta\nu^{(\lambda)}\xi^a_{(\lambda)})). \qquad (23)$$

Now we shall find the action variation

$$\delta_\Lambda S = \int \tfrac{1}{2} \Psi^{mn} \delta_\Lambda g_{mn} d^4x. \qquad (24)$$

If we substitute (22) into (24), then we get

$$\delta_\Lambda S = \int \delta\nu^{(\lambda)}(x')\, d^4x' \int \tfrac{1}{2} \Psi^{mn}(x) \Lambda_{mn(\lambda)}(x,x')\, d^4x = 0.$$

It means that the action is invariant with respect to the group generated by (22). But generators (21) are not independent, and not all of the parameters $\delta\nu^{(\lambda)}$ are essential. For generators to be independent, the system of equations

$$\int \Lambda_{mn(\lambda)}(x,x') \delta\nu^{(\lambda)}(x')\, d^4x' = 0 \qquad (25)$$

must have a single solution $\delta\nu^{(\lambda)} = 0$ for arbitrary g_{mn}.

If we substitute the definition (21) into (25) and perform integration, we obtain

$$\nabla_{(n}(\xi_{(\lambda)m)}\delta\nu^{(\lambda)}) = 0. \qquad (26)$$

It is clear , that the left - hand side of (26) up to a factor coincides with the right - hand side of (23). Consequently, the condition that all parameters in (23) are essential coincides with the condition that the solution $\delta\nu^{(\lambda)} = 0$ of the system (26) is single.

Let us consider an arbitrary point M within the range of integration. Let the orbit of the point M, i.e., the multitude of the points of the area which can be transferred to the point M by the transformations of the group G_r, be denoted by the term Q_M. Let among the vector fields $\xi_{(\lambda)}$ there be exactly m fields which are zero fields in M. It can be assumed without loss of generality that the zero fields are $\xi_{(\rho)}$, $\rho = 1, \ldots , m$. It means that $\xi_{(\rho)}$ form the Lie algebra of the stability subgroup of the point M. In differential geometry the stability subgroup is more often called the group of isotropy of M. Let us denote this group by the symbol H_M.

So, the vector fields $\xi_{(\gamma)}$, $\gamma = m+1, \ldots ,r$, are not equal to zero in M. It should be remarked that these fields do not generally

form the Lie algebra. Let us prove that in any neighborhood of the point M they form a set of basis fields of the orbit Q_M.

Indeed, according to the Frobenious theorem /10/ the integral curves of the fields $\xi_{(\lambda)}$ compose a family of the submanifolds of the initial manifold, because they form the Lie algebra. Each of the points of the initial manifold belongs to one of these submanifolds which are orbits of these points. A linear envelope spanned over $\xi_{(\lambda)}$ at the point M is a tangent space for the Q_M. Let it be denoted by T_M. Since $\xi_{(\rho)}|_M = 0$, then T_M coincides with the linear envelope of $\xi_{(\gamma)}|_M$. Then, Q_M is homogeneous under action of the G_r by definition. Therefore, Q_M is isomorphic to G_r/H_M which is a factor space of the group of motion to the group of isotropy. Hence, $dim\ Q_M = dim\ G_r - dim\ H_M = r - m \equiv p$. Consequently, the dimension of the linear envelope $\xi_{(\gamma)}|_M$ is equal to the number of the vectors $\xi_{(\gamma)}$, therefore, these vectors in M form a basis set of T_M.

Vector fields $\xi_{(\gamma)}$ are assumed to be differentiable, therefore there is any neighborhood U_M of the point M in which these fields remain linearly independent, and because their integral curves completely belong to Q_M, then in U_M the vector fields $\xi_{(\gamma)}$ form a basis set of Q_M.

Let us consider a vector field $\eta^m = \xi^m_{(\gamma)}\delta\nu^{(\gamma)}$. In U_m an arbitrary vector field tangent to Q_M can be decomposed over the fields $\xi_{(\gamma)}$ with any variable coefficients. Consequently, in a neighborhood of M any a priori given tangent to Q_M vector field can be obtained from η^m by a suitable choice of $\delta\nu^{(\gamma)}$. It means, the generator of an arbitrary diffeomorphism of Q_M at the point M has the form $\xi_{(\gamma)}\delta\nu^{(\gamma)}$.

Now we return to (26). It has been shown that in the neighborhood of M an arbitrary, tangent to Q_M, vector field can be decomposed over the fields $\xi_{(\gamma)}$. The field $\xi_{(\lambda)}\delta\nu^{(\lambda)}$ for arbitrary $\delta\nu^{(\lambda)}$ is tangent to Q_M since all $\xi_{(\lambda)}$ are tangent to Q_M. Therefore, for any $\delta\nu^{(\lambda)}$, $\delta\mathring{\nu}^{(\gamma)}$ can be picked out such that in some neighborhood of M, $\xi_{(\lambda)}\delta\nu^{(\lambda)} = \xi_{(\gamma)}\delta\mathring{\nu}^{(\gamma)}$. Then (26) transforms to the form

$$\nabla_{(n}(\xi_{(\gamma)m)}\delta\mathring{\nu}^{(\gamma)}) = 0. \qquad (27)$$

But (27) is the Killing equation for the covector field $\mathring{\eta}_m = \xi_{(\gamma)m}\delta\mathring{\nu}^{(\gamma)}$. As an arbitrary metric tensor has no Killing vectors, then the only solution of (27) is $\mathring{\eta} = 0$, and since $\xi_{(\gamma)m}$ are linearly independent, we obtain $\delta\mathring{\nu}^{(\gamma)} = 0$.

Summarizing we conclude that the group generated by the infinitesimal transformations (23) has $p = dim\ G_r - dim\ H_M$ essential parameters depending on coordinates. It is the group of the metric transformations corresponding to arbitrary diffeomorphisms of the orbits. The diffeomorphism of the orbits is such a diffeomorphism of the whole manifold that integral curves of the generating vector fields do not leave the orbits.

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