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"HEAVEN" AND "SKY" SPACES IN QUANTUM GRAVITY

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1. Introduction

Quantization of the Einstein theory

$$\mathcal{W} = \int d^4x\sqrt{-g}\,\mathcal{L} \;\; ; \; \mathcal{L} = -\frac{1}{2\kappa}R + \mathcal{L}_M \; , \qquad (1)$$

as a gauge theory means that we consider the metric components $g_{\mu\nu}(x)$ as fields in an abstract space-time $x^\mu = (t, x^i)$ (which we shall call the "Heaven" space).

This paper is devoted to the construction of the physical observable space-time $\xi^\mu = (\tau, \xi^i)$ ("Sky" space) and dynamical invariant variables. The main differences of our construction from others [1] are the strict fulfilment of the principles of quantum theory and the choice of invariant dynamical variables.

2. The Heisenberg uncertainty principles and the choice of variables

We choose the ADM metric [1,2]

$$g_{\mu\nu} = \begin{pmatrix} \alpha^2 - \beta^2 & \beta_i \\ \beta_j & -\gamma_{ij} \end{pmatrix} \; ; \; \beta^i = \gamma^{ij}\beta_j \; ; \; \beta^2 = \beta_i \cdot \beta^i \; ; \; \sqrt{-g} = \alpha\sqrt{\gamma} \; . \qquad (2)$$

In this case the Lagrangian (1) does not depend on the time derivative of the fields α and β_i

$$\mathcal{L} = \mathcal{L}(\alpha, \beta_i, \gamma_{ij}, \partial_0\gamma_{ij}, \cdots)$$

and the corresponding momenta p_α, p_{β_i} are equal to zero. There are two ways of quantization of a theory like that:

i). to consider the equation $p_\alpha = 0, p_\beta = 0$ as constraints on the dynamical variables and to quantize all variables (α, β, γ) by the Dirac method, or

ii). to construct the minimal set of classical dynamical variables on the explicit solutions of the classical equation

$$\frac{\delta\mathcal{W}}{\delta\alpha} = 0 \; ; \frac{\delta\mathcal{W}}{\delta\beta_i} = 0 \; ,$$

and to quantize only these variables (The physical meaning of this "minimal approach" is discussed in detail in ref.s [3,4]).

In the first method we are forced to fix simultaneously the fields α, β_i and their momenta, which contradicts the Heisenberg relation in quantum theory. Therefore, we use here the second way, and solve the classical equation for fields α and β_i explicitly.

In particular, we should solve exactly the invariant Einstein equation ($\partial\mathcal{W}/\delta\beta i = 0$) :

$$R^0_i \equiv \frac{1}{2}[\frac{1}{\alpha}\pi_i^k]_{;k} - \frac{1}{2}[\frac{1}{\alpha}\pi_k^k]_{;i} = T^0_i(M)\varkappa \qquad (3)$$

where

$$\pi_{ki} = \partial_0\gamma_{ki} - \beta_{k;i} - \beta_{i;k} \; ,$$

$(;i)$ is a covariant derivative in the metric γ_{ij}. $T_i^0(M)$ are the matter energy- momentum tensor components.

We can devide the field β_k into two parts with respect to the transitiveness of a general covariant group transformation

$$\beta_i = \beta_i^{tr} + \beta_i^M \; ; \; (R_i^0 = R_i^0(\beta^{tr}) + R_i^0(\beta^M)) \; .$$

It is clear that the nontransitive part β^M is defined by the matter

$$2[\frac{1}{\alpha}\beta^{Mk}{}_{;k}]_{;i} - [\frac{1}{\alpha}(\beta^{Mk}{}_{;i} + \beta_i^{M\,;k})]_{;k} = T_i^0(M)2\kappa \; , \tag{4}$$

and β_i^{tr} is defined by the fields ($\partial_0\gamma_{ki}$) according to the equation of the type of (3) without the matter tensor

$$R_i^0(\beta^{tr}) = 0 \; .$$

The explicit solution of this equation means that instead of six fields γ_{ij} we get only three dynamical independent variables γ_{ij}^T satisfying the identity:

$$[\frac{1}{\alpha}(\gamma^T)^{ki}(\partial_0\gamma^T)_{il}]_{;k} - [\frac{1}{\alpha}(\gamma^T)^{ki}(\partial_0\gamma^T)_{ik}]_{;l} \equiv 0 \cdot \tag{5}$$

(Like in QED the explicit solution of the same equation

$$\partial_i\partial_0 A_i - \partial_i^2 A_0 = 0 \;\; ; \;\; (A_0 = \frac{1}{\partial^2}\partial_i\partial_0 A_i)$$

leads to the gauge-invariant transversal variables

$$A_i^T = (\delta_{ij} - \partial_i\frac{1}{\partial^2}\partial_j)A_j$$

satisfying the identity $\partial_i A^T \equiv 0$).

For purposes of the definition of an observable "Sky" space and the proof of the Newton law we separate also the variable $\sqrt{\gamma}$ by the definition of a new metric h_{kl}

$$\gamma_{kl}^T = a^2(x^\mu)h_{kl}(x^\mu) \;\; ; \;\; \sqrt{\gamma} = a^3(x^\mu) \; , \tag{6}$$

with the condition

$$det\;(h) \; = 1 \;\; ; \;\; (h^{ik}\delta h_{ik} = 0) \; . \tag{7}$$

Just in this case the constraint (5) has a very simple form

$$p^k_{\;l;\,k} = 4 \;\; \partial_l(\frac{\dot{a}}{a}) \; , \tag{8}$$

where

$$p_i^k = h^{kl}\ p_{il}\ ;\ p_{kl} = \frac{1}{\alpha}\partial_0 h_{kl} = \frac{\partial}{\partial\tau}h_{kl} \equiv \dot{h}_{kl} \tag{9}$$

and $d\tau = \alpha dx^0$ is the "Sky"-time differential.

The Einstein equations

$$R_0^0 - \frac{1}{2}R = \kappa T_0^0(M)\ ;\ R_0^0 = \frac{\kappa}{2}(T_0^0(M) - T_k^k(M))$$

in terms of these variables have the forms

$$\frac{1}{2}[\mathcal{K} + {}^3\mathcal{R}] = \kappa T_0^0(M) \tag{10}$$

$$\mathcal{K} + \Sigma = \frac{\kappa}{2}(T_0^0(M) - T_k^k(M)) \tag{11}$$

where ${}^3\mathcal{R}$ is the curvature for the metric $\gamma_{ij} = a^2 h_{ij}$

$${}^3\mathcal{R} = \mathcal{R}_a - \mathcal{R}_h\ . \tag{12}$$

(The explicit expression of ${}^3\mathcal{R}(\gamma)$ is given in ref. [5], see also Appendix, where we give also the complete expressions for $\mathcal{K}, \Sigma$).

$\mathcal{K}$ is the kinetic part

$$\mathcal{K} = 6(\frac{\dot{a}}{a})^2 - \frac{1}{4}(p_l^k p_k^l)\ . \tag{13}$$

Σ becomes the total derivative under the sign of the integral ($\int d^4x\sqrt{-g}$)

$$\Sigma = \frac{1}{\alpha}\alpha_{;k}^{;k} - \frac{1}{a^3}\frac{\partial^2}{\partial\tau^2}(a^3)\ . \tag{14}$$

We introduce also the coordinates of the "Sky" space

$$\alpha dx^0 = d\tau\ ;\ a dx^i = d\xi^i\ ;\ \xi^\mu = (\tau, \xi^i)\ ;\ \int d^4x\sqrt{-g} = \int d^4\xi\ , \tag{15}$$

where we can make the synchronization of our watches [6] and define the procedure of measuring the intervals of physical time and space

$$(\tau_1 - \tau_2)^2\ ;\ (\xi_1 - \xi_2)^2\ .$$

So, the explicit solution of equation for β_i (3) leads to the natural separation of variables

$$\mathcal{W}(\alpha, \beta, \gamma, M)|_{\beta=\beta^{tr}(\gamma)+\beta^M} = \mathcal{W}(\alpha, a, h_{ij}, M)\ , \tag{16}$$

and to the definition of the physical "Sky"- space- time (15), in the terms of which the action (1) depends on only the variables

$$\log(a) = \sqrt{\kappa}\mu\ ,\ h_{ij} \tag{17}$$

(or more correctly, on its derivatives $\dot{\mu}, \partial_{\xi^i}\mu$).

The equation for α (10) $\delta W/\delta\alpha = 0$ in terms of the "Sky"- space turns out into the constraint for the dynamical fields (μ, h_{ij}, M). Equation (11) is the difference of equations for α and a $(\delta W/\delta\alpha - \delta W/\delta a) = 0$. The Newton law $\Delta\phi = (\kappa/2)M\delta(\mathbf{x})$ is the joint solution of equations for α and a in the approximation $a = 1 - \phi$, $\alpha = 1 + \phi$, $h_{ij} = \delta_{ij}$, $T_0^0(M) = M\delta(\mathbf{x})$.

Let us consider here the nontrivial vacuum ($T_\nu^\mu = 0$, $h_{ij} = \delta_{ij}$) solutions of equations (10), (11), which have the form in "Sky"-space coordinates:

$$\frac{1}{\kappa}[3(\frac{\dot{a}}{a})^2 - \frac{4}{3}\frac{1}{a^{3/2}}\partial^2_{\xi^i}(a^{3/2})] = 0 \ , \tag{18}$$

$$-3\frac{\ddot{a}}{a} + \frac{1}{\alpha}[\partial^2_{\xi^i}\alpha + 2\partial_{\xi^i}\alpha(\partial_{\xi^i}(\log(a)))] = 0 \ . \tag{19}$$

The Laplace operator $\partial^2_{\xi^i}$ has a set of eigen- functions

$$\partial^2_{\xi^i} f_n = H_n^2 f_n(H_n\xi) \ ; \ f_n(0) = 1 \ , \tag{20}$$

characterized by the "quantum number", H_n and unit vector $\mathbf{n}$, $(\mathbf{n}^2 = 1)$ (or an orbital momentum l and its projectors m), for example

$$f_n(H_n\xi) = e^{H_n(\mathbf{n}\xi)} \ . \tag{21}$$

We can represent the general solution of eq. (18) in the form

$$a(\tau, \xi) = e^{H_n\tau} a_n^0(\xi) \ ; \ a_n^0 = [f_n(\frac{3}{2}H_n\xi)]^{2/3} \tag{22}$$

Eq. (19) reduces to an equation of the Schrödinger type

$$(\partial_\xi^2 + V(\xi))\psi = \frac{9}{2}H_n^2\psi \ ; \ \psi = (a_n^0\alpha(\xi)) \ ; \ V = \frac{1}{2}(\partial_{\xi^i}\log a_n^0)^2 \ . \tag{23}$$

In particular, for (21) we get two solutions

$$\alpha_1 = e^{H_n(\xi\mathbf{n})} \ ; \ \alpha_2 = e^{-3H_n(\xi\mathbf{n})} \ . \tag{24}$$

If H_R is equal to zero, we have only the "Minkowski" space solution $\alpha = a = 1$ (There is an opinion that this space is unstable [7]).

3. The correspondence and observability principles and the quantum state of the Universe

The correspondence principle dictates to quantize field a in the "Heaven" space (otherwise we cannot get the Newton law as the classical approximation of quantum equation for field $a(\xi)$).

However, we should keep in our mind that the quantum equations have physical meaning only in the "Sky"- space, according to the observability principle.

At first, we consider the possibility of the quantization of the space- scale field $a(x)$ in the vacuum approximation (18),(19). Taking into account eq.(17) we get the action

$$\mathcal{W} = \int d^4x \alpha a^3[-\frac{1}{2\alpha^2}6(\partial_0\mu)^2 + \frac{1}{2\kappa}\mathcal{R}_a] = \int d^4x \mathcal{L}_H \ , \tag{25}$$

which leads to the "Heaven" momentum, and to the Hamiltonian

$$\Pi_{(\mu)} = \frac{\partial \mathcal{L}}{\partial(\partial_0\mu)} = -\alpha a^3 6(\partial_0\mu) \tag{26}$$

$$\mathcal{H} = \partial_0\mu \cdot \Pi_{(\mu)} - \mathcal{L}_H = \frac{\alpha}{2}[-\frac{1}{6}a^{-3}\Pi^2_{(\mu)} - \frac{a^3}{\kappa}\mathcal{R}_a] \ . \tag{27}$$

The variation with respect to the field α gives the constraint (9) in the form

$$\frac{1}{6}\mathcal{P}^2_{(\mu)} + \frac{1}{\kappa}\mathcal{R}_a = 0 \ , \tag{28}$$

where $\mathcal{P}_{(\mu)}$ is the physical momentum

$$\mathcal{P}_{(\mu)} = a^{-3}\Pi_{(\mu)} \ . \tag{29}$$

The quantization means the commutation relations

$$i\ [\Pi_{(\mu)}(x_0, \mathbf{x}), \mu(y_0, \mathbf{y})] = \delta(\mathbf{x} - \mathbf{y}) \ , \tag{30}$$

or

$$i\ [\mathcal{P}_{(\mu)}(x_0, \xi(\mathbf{x})), \mu(y_0, \xi(\mathbf{y}))] = \frac{1}{a^3(\xi)}\delta(\mathbf{x} - \mathbf{y}) \equiv \delta(\xi(\mathbf{x}) - \xi(\mathbf{y})) \ , \tag{31}$$

where ξ^i is the operators of the "Sky"- coordinates.

The question arises: Is there decomposition of the operator $\mu(x_0, \mathbf{x})$ with respect to the class functions (20):

$$\mu(x_0, \mathbf{x}) = \sum_{H_n, n} \mu(H_n\tau)\log f_n(H_n\xi) \ , \tag{32}$$

on each of which the operator of the curvature is constant

$$\frac{1}{2} < n|\mathcal{R}_a(\hat{a})|a >= \frac{1}{2}\mathcal{R}_a(a^0_n(\xi)) = -3H^2_n \ , \tag{33}$$

and equation (28) reproduces the classical results

$$< n|\mathcal{P}^2_{(\mu)}|n > \equiv \mathcal{P}^2_{(\mu)n} \equiv \frac{36H^2_n}{\kappa} \ ? \tag{34}$$

We do not know at present a consistent mathematical realization of such quantization and prefer to consider the space- scale field $a(x)$ as the dependent variable like $\alpha(x)$ (up to the vacuum solutions (22) and (23)).

In this case eqs.(8), (10) and (11) can be considered as the constraints which define the local deviation of the metric fields (a, α) induced by the matter and graviton energy-momentum tensors

$$\alpha(\xi) = \alpha_n(\xi) \cdot \alpha^{Local}[M, h] \ ; a(\xi) = a_n(\xi) \cdot a^{Local}[M, h] \ . \tag{35}$$

From the point of view of the observability principle it is useful to distinguish between the energy- momentum tensor of matter and of gravitons h_{ij} in the "Sky"-space and equation (10) for the "Sky"- metric, which can play the role of the "hidden matter" for the compensation of the Hubble kinetic term in eq.(10)

$$\rho_{critical} = \frac{3H^2}{\kappa} \ .$$

If the observable density of matter now is much less than the critical one $\rho_{critical}$ there is a possibility to calculate the local deviation of metrics by the perturbation theory with respect to the Newton constant κ

$$a^{Local} = 1 - \kappa\Phi_a + o(\kappa^2) \ , \alpha^{Local} = 1 + \kappa\Phi_\alpha + o(\kappa^2) \ .$$

These local deviations lead to the Newton "potential" gravitational interactions of the matter fields and of gravitons (like the current- current potential interaction in the "minimal" QED [3]).

We cannot neglect the local deviations in eq.(10), otherwise, eq.(10) means that the sum of the positive energies of matter fields and of gravitons is equal zero, i.e. these fields are also absent in the theory [4].

4. Conclusion

There is a lot of way to quantize the Einstein theory. We have shown here that the principle of quantization leave only one way of the construction of both the classical and the quantum theories. This construction contains the vacuum excitations of metric (playing the role of hidden mass) which can explain the "inflation scenario" and the Hubble scale as the parameter of the boundary condition of the creation of the "Sky"-space.

It is possible to explain the creation of the matter Universe due to the singularities of the vacuum metric according to next Phrases of the Bible?

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Appendix

We get the following components of the curvature $R = R^0_0 + R^k_k$ in the metric (2) with taking into account eqs.(3) and (5)

$$R^0_0 = \mathcal{K} + \Sigma \ ; \ R^k_k = -{}^3\mathcal{R} + \Sigma \ , \tag{36}$$

where ${}^3\mathcal{R} = \mathcal{R}_a - \mathcal{R}_h$,

$$\mathcal{R}_a = -4[(\frac{\partial_k\partial_l a}{a^3}) - \frac{1}{2}(\frac{\partial_k a}{a^2})(\frac{\partial_l a}{a^2}) + (\frac{\partial_k a}{a^3})\partial_l]h^{kl} \ , \qquad (37)$$

$$\mathcal{R}_h = \frac{1}{a^2}[\partial_k\partial_l h^{kl} + \frac{1}{4}(\partial_i h_{mn})(\partial_j h_{ab})h^{ma}(h^{ij}h^{nb} - 2h^{bi}h^{nj})] \ , \qquad (38)$$

$$\Sigma = \frac{1}{\alpha}\frac{1}{a^3}\partial_k[a h^{kl}\partial_l \alpha] - \frac{1}{a^3}\partial_\tau(\partial_\tau - \underline{\nabla}_k J^k) + \frac{1}{a^3}\nabla_k[(3\frac{\dot{a}}{a} - \underline{\nabla}_l J^l)J^k] \ , \qquad (39)$$

$$\mathcal{K} = 6(\frac{\dot{a}}{a})^2 - \frac{1}{4}(p_l^k p_k{}^l) - 4(\frac{\dot{a}}{a})(\underline{\nabla}_k J^k) + (\underline{\nabla}_k J^k)^2 - - \frac{1}{2}[(\underline{\nabla}^k J_l)(\underline{\nabla}_k J^l) + (\underline{\nabla}^k J_l)(\underline{\nabla}^l J_k)] \ , \qquad (40)$$

$$J_l = (\Pi^{-1})_l^k T_k^0 \ ; \ \Pi_k^l = 2\nabla^l\underline{\nabla}_k - \nabla_k\underline{\nabla}^l - \nabla_j\underline{\nabla}^j\delta_k^l \ , \qquad (41)$$

$$\nabla_k J^l = \partial_k J^l + \Gamma_{ki}^l J^i \ , \ \underline{\nabla}_k J^l = \frac{1}{\alpha}\nabla_k[\alpha J^l] \ , \qquad (42)$$

$$\Gamma_{ki}^l = \frac{1}{2}\gamma^{lj}(\partial_k\gamma_{ij} + \partial_i\gamma_{jk} - \partial_j\gamma_{ki}) = = \frac{1}{2}h^{lj}(\partial_k h_{ij} + \partial_i h_{jk} - \partial_j h_{ki}) + + (\delta_k^l\partial_i + \delta_i^l\partial_k - h^{ln}h_{ki}\partial_n)\log a \ . \qquad (43)$$

It is easy to be convinced that these expressions in the "Sky" space depend only on $\mu(\xi)$ (17).

For example, the Lagrangian for a scalar field has the form

$$\frac{1}{2}(g^{\mu\nu}\partial_\mu\partial_\nu + \frac{R}{6}\phi^2) = \frac{1}{2}[\dot{\phi}^2 - h^{ij}\partial_{\xi^i}\phi\partial_{\xi^j}\phi + \frac{\phi^2}{6}(\mathcal{K} + 2\Sigma - {}^3\mathcal{R})] \ . \qquad (44)$$

Expressions (37), (39) and (40) in the vacuum $h_{ij} = \delta_{ij}; T_\nu^\mu = 0$ can be represented in terms of the "Sky" coordinates

$$\mathcal{R}_a = -4[\frac{\partial_\xi^2 a}{a} + \frac{1}{2}(\frac{\partial_{\xi^i} a}{a})^2] \equiv -\frac{8}{3}\frac{1}{a^{3/2}}\partial_{\xi^i}^2[a^{3/2}] \ , \qquad (45)$$

$$\mathcal{K} + \Sigma = -3\frac{\ddot{a}}{a} + \frac{1}{\alpha}[(\partial_\xi^2\alpha) + 2(\partial_{\xi^i}\alpha)(\partial_{\xi^i}\log a)] \ . \qquad (46)$$

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