сообщения
объединенного
института
ядерных
исследований
дубна

E2-91-579

I.V.Polubarinov

PHASE SPACE REPRESENTATIONS AND QUANTUM PROBABILITY THEORY

1991

1. The probability theory of quantum mechanics differs essentially from the classical probability theory[2,3,5,6], first of all by its law of addition of probabilities. The significant distinction was discovered by Bell[8-11]. The hypothesis that the correlator of components of two spins $\frac{1}{2}$ in the singlet state can be written according to the classical probability theory as an integral with respect to some hidden variables λ does contradict quantum mechanics.

Quantum and classical theories can be compared in the so-called phase space representations. Phase space representation (PSR) is a formalism, where probability densities are represented by functions of phase space variables (the term "representation" is used in sence of the Dirac theory of representations). For example, such a formalism was introduced long ago by Wigner [1]. In quantum theory PSR is not defined uniquely unlike in classical theory. Different PSR's can be introduced using a relevant completeness relation in appropriate forms. For this approach in quantum mechanics and quantum field theory see refs.[19,20] and in theories of spins $\frac{1}{2}$, $\frac{3}{2}$ and 2 see refs.[21-23]. Here we consider the case of spin 1.

2. Algebra of the spin 1 3x3 matrices is given by

$$\hat{s}_j \hat{s}_j = 2\cdot\mathbb{1}, \tag{1}$$

$$[\hat{s}_j \hat{s}_k] = i\varepsilon_{jk\ell}\hat{s}_\ell, \tag{2}$$

$$\hat{s}_i \hat{s}_j \hat{s}_k + \hat{s}_k \hat{s}_\ell \hat{s}_i = \delta_{ij}\hat{s}_k + \delta_{jk}\hat{s}_i. \tag{3}$$

The latter is the Duffin – Kemmer algebra in the 3-dimensional space.x) Using eq. (2) we can bring it into another form:

$$\{\hat{s}_i \hat{s}_j \hat{s}_k\} = 2\delta_{ij}\hat{s}_k + 2\delta_{jk}\hat{s}_i + 2\delta_{ik}\hat{s}_\ell. \tag{4}$$

Here and in what follows the braces mean the total symmetrization without division by n!, e.g.,

$$\{\hat{s}_i \hat{s}_j\} = \hat{s}_i\hat{s}_j + \hat{s}_j\hat{s}_i,$$

$$\{\hat{s}_i \hat{s}_j \hat{s}_k\} = (\hat{s}_i\hat{s}_j + \hat{s}_j\hat{s}_i)\hat{s}_k + \hat{s}_k(\hat{s}_i\hat{s}_j + \hat{s}_j\hat{s}_i) + \hat{s}_i\hat{s}_k\hat{s}_j + \hat{s}_j\hat{s}_k\hat{s}_i. \tag{5}$$

The trilinear relation (4) follows[22] immediately from the minimal annihilation polynomial $(\vec{\alpha}\hat{\vec{s}})^3 - (\vec{\alpha}\hat{\vec{s}}) = 0$ $(\vec{\alpha}^2 = 1)$.

x) For some extension of the algebra (3) see ref.[24] (Appendix, eq. (A.5).

Let us give the traces we need in what follows

$$\mathrm{tr}\,\mathbb{1}=3,\quad \mathrm{tr}\,\hat{s}_j=0,\quad \mathrm{tr}(\hat{s}_j\hat{s}_k)=2\delta_{jk},\quad \mathrm{tr}(\hat{s}_j\hat{s}_k\hat{s}_\ell)=i\varepsilon_{jk\ell},$$

$$\mathrm{tr}(\hat{s}_j\hat{s}_k\hat{s}_\ell\hat{s}_m)=\frac{1}{2}\mathrm{tr}((\hat{s}_j\hat{s}_k\hat{s}_\ell+\hat{s}_\ell\hat{s}_k\hat{s}_j)\hat{s}_m)=\delta_{jk}\delta_{\ell m}+\delta_{jm}\delta_{k\ell}. \tag{6}$$

The matrices $\mathbb{1}$, $\hat{s}_i$ and $\{\hat{s}_i\hat{s}_j\}$ form a total basis for 3x3 matrices, and the completeness relation is written in these terms as follows

$$|\mathbb{1}|\otimes\|\mathbb{1}\|+\frac{1}{2}|\hat{s}_i|\otimes\|\hat{s}_i\|+\frac{1}{4}|\{\hat{s}_i\hat{s}_j\}|\otimes\|\{\hat{s}_i\hat{s}_j\}\|=|\mathbb{1}\|\otimes\|\mathbb{1}|. \tag{7}$$

Here $|\cdots|$, $\|\cdots\|$, $|\cdots\|$, $\|\cdots|$ denote matrices with pointing out the position of matrix indices: e.g., if $|\hat{s}_i|\otimes\|\hat{s}_i\|$ means $(\hat{s}_i)_{\alpha\beta}(\hat{s}_i)_{\gamma\delta}$, then $|\mathbb{1}\|\otimes\|\mathbb{1}|$ means $(\mathbb{1})_{\alpha\delta}(\mathbb{1})_{\gamma\beta}=\delta_{\alpha\delta}\delta_{\gamma\beta}$.

3. Density matrices of interest are defined by

$$(\vec{a}\hat{\vec{s}})\hat{\rho}(m,\vec{a})=\hat{\rho}(m,\vec{a})(\vec{a}\hat{\vec{s}})=m\hat{\rho}(m,\vec{a})\quad(m=-1,0,1) \tag{8}$$

and are

$$\hat{\rho}(1,\vec{a})=|1,\vec{a}\rangle\langle 1,\vec{a}|=\frac{1}{2}(\vec{a}\hat{\vec{s}}+\mathbb{1})(\vec{a}\hat{\vec{s}}), \tag{9}$$

$$\hat{\rho}(0,\vec{a})=|0,\vec{a}\rangle\langle 0,\vec{a}|=\mathbb{1}-(\vec{a}\hat{\vec{s}})^2, \tag{10}$$

$$\hat{\rho}(-1,\vec{a})=|-1,\vec{a}\rangle\langle -1,\vec{a}|=\hat{\rho}(1,-\vec{a})=\frac{1}{2}(\vec{a}\hat{\vec{s}}-\mathbb{1})(\vec{a}\hat{\vec{s}}). \tag{11}$$

They have the properties

$$\mathrm{tr}\,\hat{\rho}(m,\vec{a})=1, \tag{12}$$

$$\sum_{m=-1,0,1}\hat{\rho}(m,\vec{a})=\mathbb{1}, \tag{13}$$

$$3\int d\mu(\vec{s})\,\hat{\rho}(m,\vec{s})=\mathbb{1}, \tag{14}$$

where $d\mu(\vec{s})=\frac{1}{2\pi}\delta(\vec{s}^2-1)\,d^3s$ is a measure on S^2. Equation (14) is the completeness relation for the usual spin states $|m,\vec{s}\rangle$, which can be treated as coherent states[x]. Sphere S^2 ($\vec{s}^2=1$ or $\vec{a}^2=1$) serves as a phase space in the theory of spins, components of $\vec{s}$ being phase space variables.

It is characteristic property of many sets of coherent states that any observable can be represented by its expectation values in the coherent states. However, it is not so for $|0,\vec{a}\rangle$. Indeed, from

$$\langle m,\vec{a}|\hat{s}_i|m,\vec{a}\rangle=\mathrm{tr}[\hat{s}_i\hat{\rho}(m,\vec{a})]=m\,a_i \tag{15}$$

x) See refs. /14-16/.

it is clear that the states $|0,\vec{a}\rangle$ lack this property: not only the "observable" zero, $\mathbf{0}$, but also the observables $\hat{s}_i$ have zero expectation values. This situation with the eigenvalue m=0 is common for all the integer spins.

The probability to find the spin component n along $\vec{b}$ in the state with the spin component m along $\vec{a}$ equals

$$\rho(n,\vec{b};m,\vec{a})=\mathrm{tr}[\hat{\rho}(n,\vec{b})\hat{\rho}(m,\vec{a})]. \tag{16}$$

These probabilities are given in Table 1.

Table 1. Spin 1. $4\rho(n,\vec{b};m,\vec{a})$

$m \backslash n$	1	0	-1
1	$(1+\vec{a}\vec{b})^2$	$2[1-(\vec{a}\vec{b})^2]$	$(1-\vec{a}\vec{b})^2$
0	$2[1-(\vec{a}\vec{b})^2]$	$4(\vec{a}\vec{b})^2$	$2[1-(\vec{a}\vec{b})^2]$
-1	$(1-\vec{a}\vec{b})^2$	$2[1-(\vec{a}\vec{b})^2]$	$(1+\vec{a}\vec{b})^2$

The sum of these probabilities in each row (column) equals unity

$$\sum_{n=-1,0,1}\rho(n,\vec{b};m,\vec{a})=\sum_{m=-1,0,1}\rho(n,\vec{b};m,\vec{a})=1. \tag{17}$$

4. The completeness relation (7) can be written via the above density matrices and in other relative forms as follows

$$2\cdot 3\int d\mu(\vec{s})\sum_{m=0,1}|\hat{\rho}(m,\vec{s})|\otimes\|\hat{\rho}(m,\vec{s})\|=|\mathbb{1}|\otimes\|\mathbb{1}\|+|\mathbb{1}\|\otimes\|\mathbb{1}|, \tag{18}$$

$$3\int d\mu(\vec{s})\,|\hat{\rho}(m,\vec{s})|\otimes\|\hat{X}(m,\vec{s})\|=|\mathbb{1}\|\otimes\|\mathbb{1}|\quad(m=1), \tag{19}$$

$$3\int d\mu(\vec{s})\,|\hat{Y}(\vec{s})|\otimes\|\hat{Y}(\vec{s})\|=|\mathbb{1}\|\otimes\|\mathbb{1}|. \tag{20}$$

Equations (19) and (20) can be solved for $\hat{X}(1,\vec{s})$ and $\hat{Y}(\vec{s})$ with the solutions (see Appendix B).

$$\hat{X}(1,\vec{a})=\alpha\cdot\mathbb{1}+\beta(\vec{a}\hat{\vec{s}})+\gamma(\vec{a}\hat{\vec{s}})^2, \tag{21}$$

$$\alpha=-3,\ \beta=1,\ \gamma=5, \tag{21.a}$$

$$\hat{Y}(\vec{\alpha}) = \alpha\cdot\mathbb{1} + \beta(\vec{\alpha}\hat{\vec{s}}) + \gamma(\vec{\alpha}\hat{\vec{s}})^2, \tag{22}$$

$$\alpha = \frac{1}{3}(-2\gamma \pm 1), \quad \beta = \pm\frac{1}{\sqrt{2}}, \quad \gamma = \pm\sqrt{\frac{5}{2}}. \tag{22.a}$$

Any set of the signs is acceptable. The matrices $\hat{X}(1,\vec{\alpha})$ and $\hat{Y}(\vec{\alpha})$ have the properties

$$\mathrm{tr}\,\hat{X}(1,\vec{\alpha}) = 3\alpha + 2\gamma = 1, \tag{23}$$

$$\mathrm{tr}\,\hat{Y}(\vec{\alpha}) = 3\alpha + 2\gamma = \pm 1, \tag{24}$$

$$3\int d\mu(\vec{s})\,\hat{X}(1,\vec{s}) = (3\alpha + 2\gamma)\mathbb{1} = \mathbb{1}, \tag{25}$$

$$3\int d\mu(\vec{s})\,\hat{Y}(\vec{s}) = (3\alpha + 2\gamma)\mathbb{1} = \pm\mathbb{1}. \tag{26}$$

The signs in eq. (26) correspond to those in α (22.a).

Matrices $\hat{X}(0,\vec{s})$ do not exist. From eq. (19) there follow

$$3\int d\mu(\vec{s})\hat{X}(1,\vec{s})\,\mathrm{tr}(\hat{s}_i\hat{\rho}(1,\vec{s})) \equiv 3\int d\mu(\vec{s})\hat{X}(1,\vec{s})s_i = \hat{s}_i. \tag{27}$$

However, for m=0 such an analog of eq. (19) would be false:

$$3\int d\mu(\vec{s})\hat{X}(0,\vec{s})\,\mathrm{tr}(\hat{s}_i\hat{\rho}(0,\vec{s})) = 0 \neq \hat{s}_i \tag{28}$$

(see eq. (15) and the remark after it).

5. Definition of PSR's. The completeness relations (18)-(20) suggest that we can introduce for any observable either the two-component representative

$$(F_0(\vec{s}), F_1(\vec{s})) = (\mathrm{tr}(\hat{\rho}(0,\vec{s})\hat{F}), \mathrm{tr}(\hat{\rho}(1,\vec{s})\hat{F})), \tag{29}$$

or the one-component representatives

$$F_1(\vec{s}) = \mathrm{tr}(\hat{\rho}(1,\vec{s})\hat{F}), \tag{30}$$

$$F_{1'}(\vec{s}) = \mathrm{tr}(\hat{X}(1,\vec{s})\hat{F}), \tag{31}$$

$$F_Y(\vec{s}) = \mathrm{tr}(\hat{Y}(\vec{s})\hat{F}). \tag{32}$$

There are several Y-representatives, for each set of signs in eqs. (22.a).

Restoration theorems. Due to the completeness relations (18)-(20) any observable can be restored via its representatives

$$\hat{F} = -\mathbb{1}\cdot\mathrm{tr}\,\hat{F} + 2\cdot 3\int d\mu(\vec{s})\sum_{m=0,1}\hat{\rho}(m,\vec{s})F_m(\vec{s}) = \tag{33.a}$$

$$= 3\int d\mu(\vec{s})\,\hat{X}(1,\vec{s})F_1(\vec{s}) =$$

$$= 3\int d\mu(\vec{s})\,\hat{\rho}(1,\vec{s})F_{1'}(\vec{s}) =$$

$$= 3\int d\mu(\vec{s})\,\hat{Y}(\vec{s})F_Y(\vec{s}). \tag{33.b}$$

The trace of $\hat{F}$ is expressed as follows

$$\mathrm{tr}\,\hat{F} = 3\int d\mu(\vec{s})F_m(\vec{s}) = \qquad (m=0,1)$$

$$= 3\int d\mu(\vec{s})F_{m'}(\vec{s}) = \qquad (m=1)$$

$$= 3\int d\mu(\vec{s})F_Y(\vec{s}). \tag{34}$$

For trace of the product of two operators we get

$$\mathrm{tr}(\hat{F}\hat{G}) = -\mathrm{tr}\,\hat{F}\cdot\mathrm{tr}\,\hat{G} + 2\cdot 3\sum_{m=0,1}\int d\mu(\vec{s})F_m(\vec{s})G_m(\vec{s}) \tag{35.a}$$

$$= 3\int d\mu(\vec{s})F_1(\vec{s})G_{1'}(\vec{s}) = 3\int d\mu(\vec{s})F_{1'}(\vec{s})G_1(\vec{s}) =$$

$$= 3\int d\mu(\vec{s})F_Y(\vec{s})G_Y(\vec{s}). \tag{35.b}$$

In particular, expectation values can be represented as

$$\mathrm{tr}(\hat{F}\hat{\rho}(m,\vec{\alpha})) = -\mathrm{tr}\,\hat{F} + 2\cdot 3\sum_{n=0,1}\int d\mu(\vec{s})F_n(\vec{s})\rho(n,\vec{s};m,\vec{\alpha}) = \tag{36.a}$$

$$= 3\int d\mu(\vec{s})F_{1'}(\vec{s})\rho_1(\vec{s};m,\vec{\alpha}) = 3\int d\mu(\vec{s})F_1(\vec{s})\rho_{1'}(\vec{s};m,\vec{\alpha}) =$$

$$= 3\int d\mu(\vec{s})F_Y(\vec{s})\rho_Y(\vec{s};m,\vec{\alpha}) \qquad (m=-1,0,1). \tag{36.b}$$

As examples of representatives let us give those for the density matrices

A) $\{\rho(0,\vec{s};0,\vec{\alpha}), \rho(1,\vec{s};0,\vec{\alpha})\} = \{\mathrm{tr}(\hat{\rho}(0,\vec{s})\hat{\rho}(0,\vec{\alpha})), \mathrm{tr}(\hat{\rho}(1,\vec{s})\hat{\rho}(0,\vec{\alpha}))\} =$

$$= \{(\vec{s}\vec{\alpha})^2, \tfrac{1}{2}[1-(\vec{s}\vec{\alpha})^2]\}, \tag{37}$$

$$\{\rho(0,\vec{s};1,\vec{\alpha}), \rho(1,\vec{s};1,\vec{\alpha})\} = \{\mathrm{tr}(\hat{\rho}(0,\vec{s})\hat{\rho}(1,\vec{\alpha})), \mathrm{tr}(\hat{\rho}(1,m)\hat{\rho}(1,\vec{\alpha}))\} =$$

$$= \{\tfrac{1}{2}[1-(\vec{s}\vec{\alpha})^2], \tfrac{1}{4}(1+\vec{s}\vec{\alpha})^2\}, \tag{38}$$

B) $\rho_1(\vec{s};0,\vec{\alpha}) = tr(\hat{\rho}(1,\vec{s})\hat{\rho}(0,\vec{\alpha})) = \frac{1}{2}[1-(\vec{s}\vec{\alpha})^2]$, (39)

$$\rho_1(\vec{s};1,\vec{\alpha}) = tr(\hat{\rho}(1,\vec{s})\hat{\rho}(1,\vec{\alpha})) = \frac{1}{4}(1+\vec{s}\vec{\alpha})^2, \quad (40)$$

C) $\rho_{1'}(\vec{s};0,\vec{\alpha}) = tr(\hat{X}(1,\vec{s})\hat{\rho}(0,\vec{\alpha})) = 2-5(\vec{s}\vec{\alpha})^2$, (41)

$$\rho_{1'}(\vec{s};1,\vec{\alpha}) = tr(\hat{X}(1,\vec{s})\hat{\rho}(1,\vec{\alpha})) = \frac{1}{2}[-1+2(\vec{s}\vec{\alpha})+5(\vec{s}\vec{\alpha})^2] =$$

$$= \frac{1}{3}[1+3P_1(\vec{s}\vec{\alpha})+5P_2(\vec{s}\vec{\alpha})]. \quad (42)$$

The last function can be replaced by

$$\rho_{1''}(\vec{s};1,\vec{\alpha}) = \frac{1}{3}\delta_{S^2}(\vec{s},\vec{\alpha}) = \frac{1}{3}\lim_{\eta\to 1-}\frac{1-\eta^2}{[1-2\eta(\vec{s}\vec{\alpha})+\eta^2]^{3/2}} =$$

$$= \frac{1}{3}\lim_{\eta\to 1-}\Big[1+\sum_{\ell=1}^{\infty}\eta^{\ell}(2\ell+1)P_\ell(\vec{s}\vec{\alpha})\Big] = \frac{1}{3}\Big[1+\sum_{\ell=1}^{\infty}(2\ell+1)P_\ell(\vec{s}\vec{\alpha})\Big], \quad (43)$$

since ρ are always integrated with polynomials in $\vec{s}$ of degree $n \leq 2$.

D) $\rho_Y(\vec{s};0,\vec{\alpha}) = tr(\hat{Y}(\vec{s})\hat{\rho}(0,\vec{\alpha})) = \alpha+\gamma-\gamma(\vec{s}\vec{\alpha})^2$, (44)

$$\rho_Y(\vec{s};1,\vec{\alpha}) = tr(\hat{Y}(\vec{s})\hat{\rho}(1,\vec{\alpha})) = \alpha+\frac{1}{2}\gamma+\beta(\vec{s}\vec{\alpha})+\frac{1}{2}\gamma(\vec{s}\vec{\alpha})^2. \quad (45)$$

In eqs. (42) and (43) $P_\ell(x)$ are the Legendre polynomials, and $\delta_{S^2}(\vec{s},\vec{\alpha})$ is the δ-function on the sphere S^2 (the Poisson kernel). In the case D) α, β, γ are given by eqs. (22.a).

The above densities are normalized as follows

$$3\int d\mu(\vec{s})\,\rho(n,\vec{s};m,\vec{\alpha}) = 1 \qquad (n=0,1), \quad (46)$$

$$3\int d\mu(\vec{s})\,\rho_i(\vec{s};m,\vec{\alpha}) = 1, \quad (47)$$

where i takes the values 1, $1'$ (or $1''$), Y.

Some of the densities introduced are not positive definite, like the Wigner densities in the general case [1,4]. Negative probabilities were discussed by Feynman in his last papers [17,18].

The representatives of the spin 1 components $\hat{F} = \hat{s}_j$ are given by

$$\{tr(\hat{\rho}(0,\vec{s})\hat{s}_j),\ tr(\hat{\rho}(1,\vec{s})\hat{s}_j)\} = \{0,\ s_j\}, \quad (48.a)$$

$$F_1(\vec{s}) \equiv (\hat{s}_j)_1 \equiv tr(\hat{\rho}(1,\vec{s})\hat{s}_j) = s_j, \quad (48.b)$$

$$F_{1'}(\vec{s}) \equiv (\hat{s}_j)_{1'} \equiv tr(\hat{X}(1,\vec{s})\hat{s}_j) = 2\beta s_j = 2s_j, \quad (48.c)$$

$$F_Y(\vec{s}) \equiv (\hat{s}_j)_Y \equiv tr(\hat{Y}(\vec{s})\hat{s}_j) = 2\beta s_j = \pm\sqrt{2}\,s_j, \quad (48.d)$$

$$(\hat{s}_j)_1(\hat{s}_j)_{1'} = (\hat{s}_j)_Y(\hat{s}_j)_Y = s(s+1) = 2. \quad (49)$$

In eq. (48.d) the signs correspond to those in β (22.a).

In PSR's the expectation values of the spin component $(\vec{b}\hat{\vec{s}})$ can be represented as follows

$$\underset{=(\vec{b}\vec{\alpha}),}{tr((\vec{b}\hat{\vec{s}})\hat{\rho}(m,\vec{\alpha}))} = 3w_i\int d\mu(\vec{s})(\vec{b}\vec{s})\rho_i(\vec{s};m,\vec{\alpha}), \quad (50)$$

where i takes the values $1, 1'$ (or $1''$), Y;

$$w_1 = 2,\ w_{1'} = w_{1''} = 1,\quad w_Y = 2\beta = \pm\sqrt{s(s+1)} = \pm\sqrt{2}$$

$$w_1 w_{1'} = s(s+1) = 2,\quad w_Y^2 = s(s+1) = 2. \quad (51)$$

Although the expectation value, represented in the form (50) with $i = 1''$, conforms with the definition of the classical probability theory, we cannot guarantee for all densities to be positive definite in this PSR.

Let us give the representatives for $\hat{F} = (\vec{x}\hat{\vec{s}})^2$:

$$\{F_0(\vec{s}), F_1(\vec{s})\} = \{\vec{x}^2-(\vec{s}\vec{x})^2,\ \tfrac{1}{2}[\vec{x}^2+(\vec{s}\vec{x})^2]\}, \quad (52.a)$$

$$F_1(\vec{s}) = \tfrac{1}{2}[\vec{x}^2+(\vec{s}\vec{x})^2], \quad (52.b)$$

$$F_{1'}(\vec{s}) = -\vec{x}^2+5(\vec{s}\vec{x})^2, \quad (52.c)$$

$$F_Y(\vec{s}) = (2\alpha+\gamma)\vec{x}^2+\gamma(\vec{s}\vec{x})^2 \quad (52.d)$$

($\vec{x}$ - an arbitrary 3-vector); for $\hat{F} = \{\hat{s}_j\hat{s}_k\}$:

$$\{F_0(\vec{s}), F_1(\vec{s})\} = \{2(\delta_{jk}-s_js_k),\ \delta_{jk}+s_js_k\}, \quad (53.a)$$

$$F_1(\vec{s}) = \delta_{jk}+s_js_k, \quad (53.b)$$

$$F_{1'}(\vec{s}) = 2(-\delta_{jk}+5s_js_k), \quad (53.c)$$

$$F_Y(\vec{s}) = 2(2\alpha+\gamma)\delta_{jk}+2\gamma s_js_k; \quad (53.d)$$

for the most general operator $\hat{F} = a\mathbf{1}+b_j\hat{s}_j+c_{jk}\{\hat{s}_j\hat{s}_k\}$:

$$\{F_0(\vec{s}), F_1(\vec{s})\} = \{a+2c_{jk}(\delta_{jk}-s_js_k),\ a+b_js_j+c_{jk}(\delta_{jk}+s_js_k)\}, \quad (54.a)$$

$$F_1(\vec{s}) = a+b_js_j+c_{jk}(\delta_{jk}+s_js_k), \quad (54.b)$$

$$F_{1'}(\vec{s}) = a+2b_js_j+2c_{jk}(-\delta_{jk}+5s_js_k), \quad (54.c)$$

$$F_Y(\vec{s}) = a+2\beta b_js_j+2c_{jk}[(2\alpha+\gamma)\delta_{jk}+\gamma s_js_k] \quad (54.d)$$

and at last for the irreducible tensor $\hat{F} = \{\hat{s}_j \hat{s}_k\} - \frac{4}{3}\delta_{jk}\mathbb{1}$:

$$\{F_0(\vec{s}), F_1(\vec{s})\} = \{-2(s_j s_k - \tfrac{1}{3}\delta_{jk}),\ s_j s_k - \tfrac{1}{3}\delta_{jk}\}, \tag{55.a}$$

$$F_1(\vec{s}) = s_j s_k - \tfrac{1}{3}\delta_{jk}, \tag{55.b}$$

$$F_{1'}(\vec{s}) = 10(s_j s_k - \tfrac{1}{3}\delta_{jk}), \tag{55.c}$$

$$F_Y(\vec{s}) = 2\gamma(s_j s_k - \tfrac{1}{3}\delta_{jk}). \tag{55.d}$$

The expectation values of the latter operator can be represented in terms of PSR's as follows

$$\mathrm{tr}\left[(\{\hat{s}_j \hat{s}_k\} - \tfrac{4}{3}\delta_{jk}\mathbb{1})\hat{\rho}(m,\vec{\alpha})\right] =$$

$$= 2\cdot 3\int d\mu(\vec{s})(s_j s_k - \tfrac{1}{3}\delta_{jk})[-2\rho(0,\vec{s};m,\vec{\alpha}) + \rho(1,\vec{s};m,\vec{\alpha})] =$$

$$= 3W_i\int d\mu(\vec{s})(s_j s_k - \tfrac{1}{3}\delta_{jk})\rho_i(\vec{s};m,\vec{\alpha}), \tag{56}$$

where $W_1 = 10$, $W_{1'} = W_{1''} = 1$, $W_Y = 2\gamma$.

Note that the representatives (53.b)-(53.d) of the squares of the spin matrices $\hat{s}_j^2$ (no summation) satisfy

$$(\hat{s}_1^2)_i + (\hat{s}_2^2)_i + (\hat{s}_3^2)_i = 2 \qquad i = 1, 1', Y \tag{57}$$

in accord with eq. (1). For example, in the case (53.b)

$$\tfrac{1}{2}(s_1^2+1) + \tfrac{1}{2}(s_2^2+1) + \tfrac{1}{2}(s_3^2+1) = 2. \tag{58}$$

If $\mathrm{tr}\hat{Y} = 3\alpha + 2\gamma = -1$ in the case (53.d), one must divide by this normalization constant.

The quantum probabilities $\rho(n,\vec{\beta};m,\vec{\alpha})$ can be expressed in terms of PSR's as follows

$$\rho(n,\vec{\beta};m,\vec{\alpha}) = -1 +$$

$$+ 2\cdot 3\int d\mu(\vec{s})[\rho(n,\vec{\beta};0,\vec{s})\rho(0,\vec{s};m,\vec{\alpha}) + \rho(n,\vec{\beta};1,\vec{s})\rho(1,\vec{s};m,\vec{\alpha})] =$$

$$= 3\int d\mu(\vec{s})\rho_{1'}(\vec{s};n,\vec{\beta})\rho_1(\vec{s};m,\vec{\alpha}) = 3\int d\mu(\vec{s})\rho_1(\vec{s};n,\vec{\beta})\rho_{1'}(\vec{s};m,\vec{\alpha}) =$$

$$= 3\int d\mu(\vec{s})\rho_Y(\vec{s};n,\vec{\beta})\rho_Y(\vec{s};m,\vec{\alpha}), \tag{59.a}$$

$$\rho(1,\vec{\beta};m,\vec{\alpha}) = 3\int d\mu(\vec{s})\rho_{1''}(\vec{s};n,\vec{\beta})\rho_1(\vec{s};m,\vec{\alpha}), \tag{59.b}$$

$$\rho(n,\vec{\beta};1,\vec{\alpha}) = 3\int d\mu(\vec{s})\rho_1(\vec{s};n,\vec{\beta})\rho_{1''}(\vec{s};1,\vec{\alpha}). \tag{59.c}$$

All these expressions have no classical construction: some include not positive definite densities, others lack a symmetry between initial and final states (compare with the equation

$$\delta_{S^2}(\vec{\beta},\vec{\alpha}) = \int d\mu(\vec{s})\,\delta_{S^2}(\vec{\beta},\vec{s})\,\delta_{S^2}(\vec{s},\vec{\alpha}) \tag{60}$$

of the classical nature).

The representative of the product of two observables $\hat{F}$ and $\hat{G}$ can be expressed via their representatives as follows

$$F(\vec{s}) * G(\vec{s}) = \mathrm{tr}(\hat{Z}(\vec{s})\hat{F}\hat{G}) =$$

$$= 3^2\int d\mu(\vec{s}')\int d\mu(\vec{s}'')K(\vec{s};\vec{s}',\vec{s}'')F(\vec{s}')G(\vec{s}''), \tag{61}$$

where the kernel K is

$$K(\vec{s};\vec{s}',\vec{s}'') = \mathrm{tr}(\hat{Z}(\vec{s})\hat{Z}'(\vec{s}')\hat{Z}''(\vec{s}'')). \tag{62}$$

The matrices $\hat{Z}$, $\hat{Z}'$, and $\hat{Z}''$ are those entering eqs. (33.b), for example, $\hat{Z}(\vec{s}) = \hat{\rho}(1,\vec{s})$, $\hat{Z}'(\vec{s}) = \hat{Z}''(\vec{s}) = \hat{X}(1,\vec{s})$.

6. Left and right representatives. Besides the above nonoperator representatives in some cases operator representatives can be introduced

$$\mathrm{tr}(\hat{Z}(\vec{s})\hat{F}\hat{G}) = F^{\ell}\,\mathrm{tr}(\hat{Z}(\vec{s})\hat{G}) = F^{\ell}G(\vec{s}) \tag{63.a}$$

$$= G^{r}\,\mathrm{tr}(\hat{Z}(\vec{s})\hat{F}) = G^{r}F(\vec{s}) \tag{63.b}$$

(cf. refs. /19-20,23/). The left and right operator representatives F^l and G^r are partial differential operators acting on $\vec{s}$ (on functions of $\vec{s}$). Note that

$$\mathrm{tr}(\hat{Z}(\vec{s})\hat{F}) = F^{\ell}_{r}\,\mathrm{tr}(\hat{Z}(\vec{s})) = F^{\ell}_{r}\cdot 1 = F(\vec{s}) \qquad (\mathrm{tr}\,\hat{Z}(\vec{s}) = 1). \tag{64}$$

Equations (63.a) and (63.b) supply us (if F^l and F^r exist) with two more expressions for the nonoperator representative of the product of two operators in addition to eq. (61).

The left representatives are multiplied in the same order as original operators, while the right ones in the inverse order:

$$\mathrm{tr}(\hat{Z}(\vec{s})\hat{F}_1\hat{F}_2\hat{G}) = F_1^{\ell}F_2^{\ell}\,\mathrm{tr}(\hat{Z}(\vec{s})\hat{G}), \tag{65}$$

$$\mathrm{tr}(\hat{Z}(\vec{s})\hat{G}\hat{F}_1\hat{F}_2) = F_2^{r}F_1^{r}\,\mathrm{tr}(\hat{Z}(\vec{s})\hat{G}). \tag{66}$$

The left representatives commute with the right ones

$$[F^{\ell}, G^{r}] = 0. \tag{67}$$

as a general rule in all the associative theories.

The operator representatives can be introduced in the representation with $\hat{Z}(\vec{s}) = \hat{\rho}(1,\vec{s})$ (spin $s = 1$, $m = 1$)

$$\mathrm{tr}(\hat{\rho}(1,\vec{s})\hat{s}_j\hat{F}) = s_j^{\ell}\,\mathrm{tr}(\hat{\rho}(1,\vec{s})\hat{F}), \tag{68}$$

$$\mathrm{tr}(\hat{\rho}(1,\vec{s})\hat{F}\hat{s}_j) = s_j^{r}\,\mathrm{tr}(\hat{\rho}(1,\vec{s})\hat{F}) \tag{69}$$

and are given by

$$s_j^{\ell}_{r} = s_j + \tfrac{1}{2}s_k\left(s_k\frac{\partial}{\partial s_j} - s_j\frac{\partial}{\partial s_k}\right) \mp \tfrac{i}{2}\varepsilon_{jk\ell}s_k\frac{\partial}{\partial s_\ell} \tag{70}$$

(for derivation see ref.[23]). The operator representatives must satisfy all the relations of the spin algebra (1) - (3) at least in application to the polynomials in $\vec{s}$ of degree $n \leq 2$. However the relations (1) and (2) are fulfilled exactly

$$\hat{s}_j^{\substack{\ell \\ r}} \hat{s}_j^{\substack{\ell \\ r}} = 2 \tag{71}$$

$$[s_j^{\ell} s_k^{\ell}] = i\varepsilon_{jk\ell} s_{\ell}^{\ell}, \quad [s_j^r s_k^r] = -i\varepsilon_{jk\ell} s_{\ell}^r, \quad [s_j^{\ell}, s_k^r] = 0. \tag{72}$$

The representatives F^l and F^r of any operator $\hat{F}$ may be obtained explicitly by replacing in $\hat{F}$ the spin matrices $\hat{s}_j$ by their left and right representatives, respectively, taking into account the above rule of order of factors.

For equations of motion in terms of PSR's see refs.[21,23].

7. <u>The singlet state of two spins 1 in PSR's.</u> The singlet state is defined as follows

$$(\hat{s}_j^a + \hat{s}_j^b)|\text{singlet}\rangle = 0 \qquad (j = 1, 2, 3), \tag{73}$$

$$(\hat{s}_j^a + \hat{s}_j^b)\hat{\rho}^{\text{singlet}} = \hat{\rho}^{\text{singlet}}(\hat{s}_j^a + \hat{s}_j^b) = 0, \tag{74}$$

$$|\text{singlet}\rangle = \frac{1}{\sqrt{3}}\left[u^a(1)\otimes u^b(-1) - u^a(0)\otimes u^b(0) + u^a(-1)\otimes u^b(1)\right], \tag{75}$$

$$\hat{\rho}^{\text{singlet}} = |\text{singlet}\rangle\langle\text{singlet}| = \frac{1}{3}\left[-\mathbb{1}^a \otimes \mathbb{1}^b + (s_i^a \otimes s_i^b)^2\right]. \tag{76}$$

In fact, the singlet state is independent of any quantization axis, and the last expression for $\hat{\rho}^{\text{singlet}}$ demonstrates this fact manifestly (in spite of the state vector in form (75) implies the use of some or other quantization axis).

The probabilities to find definite components of two spins 1 in the singlet state are expressed via the one-spin probabilities

$$\begin{aligned} \rho(m,\vec{a};n,\vec{b}|\text{singlet}) &= \underset{a}{tr}\,\underset{b}{tr}\left[\hat{\rho}^a(m,\vec{a})\hat{\rho}^b(n,\vec{b})\hat{\rho}^{\text{singlet}}\right] = \\ &= \hat{\rho}^a_{\alpha'\alpha}(m,\vec{a})\,\hat{\rho}^b_{\beta'\beta}(n,\vec{b})\,\hat{\rho}^{\text{singlet}}_{\alpha\beta,\alpha'\beta'} = \\ &= \frac{1}{3}\rho(m,\vec{a};-n,\vec{b}) = \frac{1}{3}\rho(m,\vec{a};n,-\vec{b}) = \\ &= \frac{1}{3}\rho(-m,\vec{a};n,\vec{b}) = \frac{1}{3}\rho(m,-\vec{a};n,\vec{b}) \end{aligned} \tag{77}$$

(see, e.g., refs.[12,22]).

$$\sum_{m,n=-1,0,1} \rho(m,\vec{a};n,\vec{b}|\text{singlet}) = 1, \tag{78}$$

$$3^2\int d\mu(\vec{s}^a)\int d\mu(\vec{s}^b)\,\rho(m,\vec{s}^a;n,\vec{s}^b|\text{singlet}) = 1. \tag{79}$$

The probabilities $\rho(m,\vec{a};n,\vec{b}|\text{singlet})$ are given explicitly in Table 2.

Table 2. Spin 1. $12\rho(m,\vec{a};n,\vec{b}|\text{singlet})$

m \ n	1	0	-1
1	$(1-\vec{a}\vec{b})^2$	$2[1-(\vec{a}\vec{b})^2]$	$(1+\vec{a}\vec{b})^2$
0	$2[1-(\vec{a}\vec{b})^2]$	$4(\vec{a}\vec{b})^2$	$2[1-(\vec{a}\vec{b})^2]$
-1	$(1+\vec{a}\vec{b})^2$	$2[1-(\vec{a}\vec{b})^2]$	$(1-\vec{a}\vec{b})^2$

The singlet state can be represented in PSR's by the 4-component representative

$$\rho(m,\vec{a};n,\vec{b}|\text{singlet}), \quad m,n = 0,1 \tag{80.a}$$

or by any of the following 1-component representatives

$$\rho_1(\vec{a},\vec{b}|\text{singlet}) = \rho(1,\vec{a};1,\vec{b}|\text{singlet}) = \frac{1}{12}(1-\vec{a}\vec{b})^2, \tag{80.b}$$

$$\begin{aligned}\rho_{1'}(\vec{a},\vec{b}|\text{singlet}) &= \underset{a}{tr}\,\underset{b}{tr}\left[\hat{X}^a(1,\vec{a})\hat{X}^b(1,\vec{b})\hat{\rho}^{\text{singlet}}\right] = \\ &= \frac{1}{3}\left[-8 - 2(\vec{a}\vec{b}) + 25(\vec{a}\vec{b})^2\right],\end{aligned} \tag{80.c}$$

$$\begin{aligned}\rho_Y(\vec{a},\vec{b}|\text{singlet}) &= \underset{a}{tr}\,\underset{b}{tr}\left[\hat{Y}^a(\vec{a})\hat{Y}^b(\vec{b})\hat{\rho}^{\text{singlet}}\right] = \\ &= \frac{1}{6}\left[-1 - 2(\vec{a}\vec{b}) + 5(\vec{a}\vec{b})^2\right] = \frac{1}{3^2}\left[1 - 3P_1(\vec{a}\vec{b}) + 5P_2(\vec{a}\vec{b})\right].\end{aligned} \tag{80.d}$$

This function can be replaced by

$$\rho_{Y'}(\vec{a},\vec{b}|\text{singlet}) = \frac{1}{3^2}\delta_{S^2}(\vec{a},-\vec{b}), \tag{80.d'}$$

since it is always integrated only with polynomials in $\vec{s}$ of degree $n \leq 2$. Note that function ρ_Y is independent of the choice of Y (i.e., of the signs in eqs. (22.a) for α, β, γ). All the densities are normalized as follows

$$3^2\int d\mu(\vec{s}^a)\int d\mu(\vec{s}^b)\,\rho_i(\vec{s}^a,\vec{s}^b|\text{singlet}) = 1. \tag{81}$$

Now the correlator of components of two spins 1 in the singlet state can be written in PSR's as

$$c(\vec{a},\vec{b}) = \underset{a}{\mathrm{tr}}\,\underset{b}{\mathrm{tr}}\left[(\vec{a}\hat{\vec{s}}^{a})(\vec{b}\hat{\vec{s}}^{b})\hat{\rho}^{\,singlet}\right] =$$
$$= 3^2 w_i^2 \int d\mu(\vec{s}^{a}) \int d\mu(\vec{s}^{b}) (\vec{a}\vec{s}^{a})(\vec{b}\vec{s}^{b})\, \rho_i(\vec{s}^{a},\vec{s}^{b}|singlet) =$$
$$= -\frac{2}{3}(\vec{a}\vec{b}), \tag{82}$$

where $i = 1, 1', Y, Y'$, w_i are given by eqs. (50), $w_{Y'} = w_Y$, $w_{Y'}^2 = w_Y^2 = s(s+1) = 2$. In all the representations correlator (82) resembles its classical counterpart assumed by Bell, however, with essential distinctions. In representations C) and D) the densities are not positive definite. In cases A), B) and D) with $\rho_{Y'}$ instead of ρ_Y the densities are positive, but extra numerical factors exclude reducing to classics.

The Bell type derivation using expressions (82) with the positive densities leads us to the following quantum analogs of the Bell inequality x)

$$|c(\vec{a},\vec{b}) - c(\vec{a},\vec{b}')| + |c(\vec{a}',\vec{b}) + c(\vec{a}',\vec{b}')| \leqslant 2\cdot w_i^2 = 2\cdot\begin{cases} 4 & i=1 \\ 2 & i=Y' \end{cases} \tag{83}$$

which of course are not contradictory. Among these estimations (83) the last is the best. It can be easily obtained for any spin s as follows. We can represent $c(\vec{a},\vec{b})$ in the form

$$c(\vec{a},\vec{b}) = -\frac{1}{3}s(s+1)(\vec{a}\vec{b}) =$$
$$= s(s+1)\int d\mu(\vec{s}^{a}) \int d\mu(\vec{s}^{b}) (\vec{a}\vec{s}^{a})(\vec{b}\vec{s}^{b})\, \delta_{S^2}(\vec{s}^{a}, -\vec{s}^{b}). \tag{84}$$

This expression leads to the inequality

$$|c(\vec{a},\vec{b}) - c(\vec{a},\vec{b}')| + |c(\vec{a}',\vec{b}') + c(\vec{a}',\vec{b})| \leqslant 2\cdot s(s+1) \tag{85}$$

for any spin s. From eq. (85) it follows that

$$|\vec{a}\vec{b} - \vec{a}\vec{b}'| + |\vec{a}'\vec{b}' + \vec{a}'\vec{b}| \leqslant 2\cdot 3 \tag{86}$$

with no dependence on the spin. Note that $\delta_{S^2}(\vec{s}^{a}, -\vec{s}^{b})$ may be treated as a classical counterpart of the singlet state. It is the density $\delta_{S^2}(\vec{s}^{a},\vec{s}_0)\,\delta_{S^2}(\vec{s}^{b},-\vec{s}_0)$ for two "spins", integrated over all the initial directions $\vec{s}_0$:

$$\delta_{S^2}(\vec{s}^{a}, -\vec{s}^{b}) = \int d\mu(\vec{s}_0)\, \delta_{S^2}(\vec{s}^{a},\vec{s}_0)\, \delta_{S^2}(\vec{s}^{b},-\vec{s}_0). \tag{87}$$

However, the quantum correlator $c(\vec{a},\vec{b})$ contains the extra factor $s(s+1)$.

x) Cirel'son[13] proposed another way to obtain quantum generalizations of the Bell inequality.

8. PSR's for other spins are constructed similarly, starting with the completeness relations[21-23]

$$u_0|\mathbb{1}|\otimes\|\mathbb{1}\| + u_1|\hat{s}_i|\otimes\|\hat{s}_i\| + u_2|\{\hat{s}_i\hat{s}_j\}|\otimes\|\{\hat{s}_i\hat{s}_j\}\| +$$
$$+ u_3|\{\hat{s}_i\hat{s}_j\hat{s}_k\}|\otimes\|\{\hat{s}_i\hat{s}_j\hat{s}_k\}\| + u_4|\{\hat{s}_i\hat{s}_j\hat{s}_k\hat{s}_l\}|\otimes\|\{\hat{s}_i\hat{s}_j\hat{s}_k\hat{s}_l\}\| + \dots =$$
$$= |\mathbb{1}\|\otimes\|\mathbb{1}|, \tag{88}$$

where

	u_0	u_1	u_2	u_3	u_4	u_5	...
$s=\frac{1}{2}$	$\frac{1}{2}$	2	0	0	0	0	...
$s=1$	-1	$\frac{1}{2}$	$\frac{1}{4}$	0	0	0	...
$s=\frac{3}{2}$	$-\frac{17}{32}$	$-\frac{293}{2^3\cdot 3^3}$	$\frac{1}{2^3\cdot 3}$	$\frac{1}{2\cdot 3^4}$	0	0	...
$s=2$	$\frac{5}{2\cdot 3}$	$-\frac{47}{2^2\cdot 3^3}$	$-\frac{127}{2^5\cdot 3^3}$	$\frac{1}{2^4\cdot 3^4}$	$\frac{1}{2^8\cdot 3^4}$	0	...

The completeness relations (88) can be converted into the following forms:

$$(2s+1)\int d\mu(\vec{s}) \sum_{m\geqslant 0} v_m |\hat{\rho}(m,\vec{s})|\otimes\|\hat{\rho}(m,\vec{s})\| = |\mathbb{1}|\otimes\|\mathbb{1}\| + |\mathbb{1}\|\otimes\|\mathbb{1}|, \tag{89}$$

where

$s=\frac{1}{2}$ $\quad v_{\frac{1}{2}} = 3.$

$s=1$ $\quad v_0 = v_1 = 2.$

$s=\frac{3}{2}$ $\quad v_{\frac{1}{2}} = \frac{3\cdot 5}{4}, \quad v_{\frac{3}{2}} = \frac{5}{4}.$

$s=2$ $\quad v_0 = 2, \quad v_1 = \frac{10}{3}, \quad v_2 = \frac{2}{3}$

and

$$(2s+1)\int d\mu(\vec{s})|\hat{\rho}(m,\vec{s})|\otimes\|\hat{X}(m,\vec{s})\| = |\mathbb{1}\|\otimes\|\mathbb{1}| \qquad (m\neq 0), \tag{90}$$

$$(2s+1)\int d\mu(\vec{s})|\hat{Y}(\vec{s})|\otimes\|\hat{Y}(\vec{s})\| = |\mathbb{1}\|\otimes\|\mathbb{1}|. \tag{91}$$

The matrices $\hat{X}(m,\vec{a})$ and $\hat{Y}(\vec{a})$ can be found in the form

$$\alpha\mathbb{1} + \beta(\vec{a}\hat{\vec{s}}) + \gamma(\vec{a}\hat{\vec{s}})^2 + \delta(\vec{a}\hat{\vec{s}})^3 + \gamma(\vec{a}\hat{\vec{s}})^4 + \dots \tag{92}$$

with real coefficients $\alpha, \beta, \gamma, \delta, \dots$ like above for spin 1 (eqs. (21) and (22)). For spins $\frac{1}{2}$, $\frac{3}{2}$ and 2 see refs.[21-23].

Now we expose briefly PSR's for spin $\frac{1}{2}$ in the notation adopted. We start with the completeness relation in the forms

$$3\cdot 2\int d\mu(\vec{s})\,|\hat{\rho}(\tfrac{1}{2},\vec{s})|\otimes\|\hat{\rho}(\tfrac{1}{2},\vec{s})\| = |\mathbb{1}|\otimes\|\mathbb{1}\| + |\mathbb{1}\|\otimes\|\mathbb{1}|, \tag{93}$$

$$2\int d\mu(\vec{s})\,|\hat{\rho}(\tfrac{1}{2},\vec{s})|\otimes\|\hat{X}(\tfrac{1}{2},\vec{s})\| = |\mathbb{1}\|\otimes\|\mathbb{1}|, \tag{94}$$

$$2\int d\mu(\vec{s})\,|\hat{Y}(\vec{s})|\otimes\|\hat{Y}(\vec{s})\| = |\mathbb{1}\|\otimes\|\mathbb{1}|, \tag{95}$$

where

$$\hat{\rho}(\tfrac{1}{2},\vec{a}) = |\tfrac{1}{2},\vec{a}\rangle\langle\tfrac{1}{2},\vec{a}| = \tfrac{1}{2}(\mathbb{1}+\vec{a}\vec{\sigma}), \tag{96}$$

$$\hat{X}(\tfrac{1}{2},\vec{a}) = \tfrac{1}{2}(\mathbb{1}+3\vec{a}\vec{\sigma}) = \hat{\rho}(\tfrac{1}{2},3\vec{a}), \tag{97}$$

$$\hat{Y}(\vec{a}) = \tfrac{1}{2}(\mathbb{1}\pm\sqrt{3}\,\vec{a}\vec{\sigma}) = \hat{\rho}(\tfrac{1}{2},\pm\sqrt{3}\,\vec{a}) \tag{98}$$

(σ_j are the Pauli σ-matrices),

$$\mathrm{tr}\,\hat{\rho}(\tfrac{1}{2},\vec{a}) = \mathrm{tr}\,\hat{X}(\tfrac{1}{2},\vec{a}) = \mathrm{tr}\,\hat{Y}(\vec{a}) = 1, \tag{99}$$

$$2\int d\mu(\vec{s})\,\hat{\rho}(\tfrac{1}{2},\vec{s}) = 2\int d\mu(\vec{s})\,\hat{X}(\tfrac{1}{2},\vec{s}) = 2\int d\mu(\vec{s})\,\hat{Y}(\vec{s}) = \mathbb{1}. \tag{100}$$

Nonoperator representatives are defined as follows

$$F_{\frac{1}{2}}(\vec{s}) = \mathrm{tr}\big(\hat{\rho}(\tfrac{1}{2},\vec{s})\hat{F}\big), \tag{101.a}$$

$$F_{\frac{1}{2}'}(\vec{s}) = \mathrm{tr}\big(\hat{X}(\tfrac{1}{2},\vec{s})\hat{F}\big), \tag{101.b}$$

$$F_{Y}(\vec{s}) = \mathrm{tr}\big(\hat{Y}(\vec{s})\hat{F}\big). \tag{101.c}$$

For example, the representatives of the density matrix $\hat{\rho}(\frac{1}{2},\vec{a})$ are given by

$$\rho_{\frac{1}{2}}(\vec{s};\tfrac{1}{2},\vec{a}) = \mathrm{tr}\big(\hat{\rho}(\tfrac{1}{2},\vec{s})\hat{\rho}(\tfrac{1}{2},\vec{a})\big) = \tfrac{1}{2}(1+\vec{s}\vec{a}), \tag{102.a}$$

$$\rho_{\frac{1}{2}'}(\vec{s};\tfrac{1}{2},\vec{a}) = \mathrm{tr}\big(\hat{X}(\tfrac{1}{2},\vec{s})\hat{\rho}(\tfrac{1}{2},\vec{a})\big) = \tfrac{1}{2}(1+3\vec{s}\vec{a}), \tag{102.b}$$

$$\rho_{Y}(\vec{s};\tfrac{1}{2},\vec{a}) = \mathrm{tr}\big(\hat{Y}(\vec{s})\hat{\rho}(\tfrac{1}{2},\vec{a})\big) = \tfrac{1}{2}(1\pm\sqrt{3}\,\vec{s}\vec{a}). \tag{102.c}$$

The densities (102.b) and (102.c) are not positive definite, unlike the density (102.a). However the density (102.b) can be replaced by the positive one

$$\rho_{\frac{1}{2}''}(\vec{s};\tfrac{1}{2},\vec{a}) = \tfrac{1}{2}\delta_{S^2}(\vec{s},\vec{a}), \tag{102.b'}$$

when integrating with polynomials in $\vec{s}$ of degree $n\leq 1$. All observables are represented by such polynomials.

The representatives of the spin components $\hat{F} = \hat{s}_j = \frac{1}{2}\sigma_j$ are

$$F_{\frac{1}{2}}(\vec{s}) \equiv (\hat{s}_j)_{\frac{1}{2}} \equiv \mathrm{tr}\big(\hat{\rho}(\tfrac{1}{2},\vec{s})\hat{s}_j\big) = \tfrac{1}{2}s_j, \tag{103.a}$$

$$F_{\frac{1}{2}'}(\vec{s}) \equiv (\hat{s}_j)_{\frac{1}{2}'} \equiv \mathrm{tr}\big(\hat{X}(\tfrac{1}{2},\vec{s})\hat{s}_j\big) = \tfrac{3}{2}s_j, \tag{103.b}$$

$$F_{Y}(\vec{s}) \equiv (\hat{s}_j)_{Y} \equiv \mathrm{tr}\big(\hat{Y}(\vec{s})\hat{s}_j\big) = \pm\tfrac{\sqrt{3}}{2}s_j, \tag{103.c}$$

$$(\hat{s}_j)_{\frac{1}{2}}(\hat{s}_j)_{\frac{1}{2}'} = (\hat{s}_j)_{Y}(\hat{s}_j)_{Y} = s(s+1) = \tfrac{3}{4}.$$

Restoration theorems are written as

$$\hat{F} = -\mathbb{1}\cdot\mathrm{tr}\,\hat{F} + 3\cdot 2\cdot\int d\mu(\vec{s})\,\hat{\rho}(\tfrac{1}{2},\vec{s})F_{\frac{1}{2}}(\vec{s}) =$$

$$= 2\int d\mu(\vec{s})\,\hat{X}(\tfrac{1}{2},\vec{s})F_{\frac{1}{2}}(\vec{s}) = 2\int d\mu(\vec{s})\,\hat{\rho}(\tfrac{1}{2},\vec{s})F_{\frac{1}{2}'}(\vec{s}) =$$

$$= 2\int d\mu(\vec{s})\,\hat{Y}(\vec{s})F_{Y}(\vec{s}). \tag{104}$$

In terms of PSR's

$$\mathrm{tr}\,\hat{F} = 2\int d\mu(\vec{s})F_{\frac{1}{2}}(\vec{s}) = 2\int d\mu(\vec{s})F_{\frac{1}{2}'}(\vec{s}) = 2\int d\mu(\vec{s})F_{Y}(\vec{s}), \tag{105}$$

$$\mathrm{tr}(\hat{F}\hat{G}) = -\mathrm{tr}\,\hat{F}\cdot\mathrm{tr}\,\hat{G} + 3\cdot 2\int d\mu(\vec{s})F_{\frac{1}{2}}(\vec{s})G_{\frac{1}{2}}(\vec{s})$$

$$= 2\int d\mu(\vec{s})F_{\frac{1}{2}}(\vec{s})G_{\frac{1}{2}'}(\vec{s}) = 2\int d\mu(\vec{s})F_{\frac{1}{2}'}(\vec{s})G_{\frac{1}{2}}(\vec{s})$$

$$= 2\int d\mu(\vec{s})F_{Y}(\vec{s})G_{Y}(\vec{s}), \tag{106}$$

$$\mathrm{tr}\big(\hat{F}\hat{\rho}(m,\vec{a})\big) = -\mathrm{tr}\,\hat{F} + 3\cdot 2\int d\mu(\vec{s})F_{\frac{1}{2}}(\vec{s})\rho_{\frac{1}{2}}(\vec{s};m,\vec{a}) =$$

$$= 2\int d\mu(\vec{s})F_{\frac{1}{2}'}(\vec{s})\rho_{\frac{1}{2}}(\vec{s};m,\vec{a}) = 2\int d\mu(\vec{s})F_{\frac{1}{2}}(\vec{s})\rho_{\frac{1}{2}'}(\vec{s};m,\vec{a}) =$$

$$= 2\int d\mu(\vec{s})F_{Y}(\vec{s})\rho_{Y}(\vec{s};m,\vec{a}), \tag{107}$$

$$\mathrm{tr}\big(\hat{s}_j\hat{\rho}(m,\vec{a})\big) = 2w_i\int d\mu(\vec{s})\,s_j\,\rho_i(\vec{s};m,\vec{a}). \tag{108}$$

Here $i = \frac{1}{2}, \frac{1}{2}', \frac{1}{2}'', Y$; $w_{\frac{1}{2}} = \frac{3}{2}$, $w_{\frac{1}{2}'} = w_{\frac{1}{2}''} = \frac{1}{2}$, $w_Y = \sqrt{s(s+1)} = \frac{\sqrt{3}}{2}$.

In PSR's the quantum probability $\rho(n,\vec{b};m,\vec{a}) = \mathrm{tr}(\hat{\rho}(n,\vec{b})\hat{\rho}(m,\vec{a})) = \frac{1}{2}(1+4mn\vec{a}\vec{b})$ can be calculated via initial and final densities as follows

$$\rho(n,\vec{b};m,\vec{a}) = -1 + 3\cdot 2\int d\mu(\vec{s})\,\rho_{\frac{1}{2}}(\vec{s};n,\vec{b})\rho_{\frac{1}{2}}(\vec{s};m,\vec{a}) =$$

$$= 2\int d\mu(\vec{s})\,\rho_{\frac{1}{2}'}(\vec{s};n,\vec{b})\rho_{\frac{1}{2}}(\vec{s};m,\vec{a}) = 2\int d\mu(\vec{s})\,\rho_{\frac{1}{2}}(\vec{s};n,\vec{b})\rho_{\frac{1}{2}'}(\vec{s};m,\vec{a}) =$$

$$= 2\int d\mu(\vec{s})\,\rho_{\frac{1}{2}''}(\vec{s};n,\vec{b})\rho_{\frac{1}{2}}(\vec{s};m,\vec{a}) = 2\int d\mu(\vec{s})\,\rho_{\frac{1}{2}}(\vec{s};n,\vec{b})\rho_{\frac{1}{2}''}(\vec{s};m,\vec{a}) =$$

$$= 2\int d\mu(\vec{s})\,\rho_{Y}(\vec{s};n,\vec{b})\rho_{Y}(\vec{s};m,\vec{a}). \tag{109}$$

The singlet state of two spins $\frac{1}{2}$

$$\hat{\rho}^{\,singlet}_{\alpha\beta,\alpha'\beta'} = |singlet\rangle_{\alpha\beta}\langle singlet|_{\alpha'\beta'} = \frac{1}{4}\left[\delta_{\alpha\alpha'}\delta_{\beta\beta'} - (\sigma^a_j)_{\alpha\alpha'}(\sigma^b_j)_{\beta\beta'}\right] \quad (110)$$

can be represented by the functions

$$\rho_{\frac{1}{2}}(\vec{a},\vec{b}\,|singlet) = \underset{a}{tr}\,\underset{b}{tr}\left[\hat{\rho}^a(\tfrac{1}{2},\vec{a})\hat{\rho}^b(\tfrac{1}{2},\vec{b})\hat{\rho}^{\,singlet}\right] =$$

$$= \hat{\rho}^a_{\alpha'\alpha}(\tfrac{1}{2},\vec{a})\hat{\rho}^b_{\beta'\beta}(\tfrac{1}{2},\vec{b})\hat{\rho}^{\,singlet}_{\alpha\beta,\alpha'\beta'} = \frac{1}{4}(1-\vec{a}\vec{b}), \quad (111.a)$$

$$\rho_{\frac{1}{2}'}(\vec{a},\vec{b}\,|singlet) = \underset{a}{tr}\,\underset{b}{tr}\left[\hat{X}^a(\tfrac{1}{2},\vec{a})\hat{X}^b(\tfrac{1}{2},\vec{b})\hat{\rho}^{\,singlet}\right] = \frac{1}{4}(1-9\vec{a}\vec{b}), \quad (111.b)$$

$$\rho_{Y}(\vec{a},\vec{b}\,|singlet) = \underset{a}{tr}\,\underset{b}{tr}\left[\hat{Y}^a(\vec{a})\hat{Y}^b(\vec{b})\hat{\rho}^{\,singlet}\right] = \frac{1}{4}(1-3\vec{a}\vec{b}). \quad (111.c)$$

The latter function can be replaced by

$$\rho_{Y'}(\vec{a},\vec{b}\,|singlet) = \frac{1}{4}\delta_{S^2}(\vec{a},-\vec{b}), \quad (111.c')$$

when integrating with the polynomials in $\vec{a}$ and $\vec{b}$ of degree $n \leq 1$.
All the densities are normalized

$$2^2\int d\mu(\vec{s}^a)\int d\mu(\vec{s}^b)\,\rho_i(\vec{s}^a,\vec{s}^b|singlet) = 1 \quad \left(i=\tfrac{1}{2},\tfrac{1}{2}',Y,Y'\right). \quad (112)$$

The correlator of components of two spins $\frac{1}{2}$ in the singlet state is expressed in terms of PSR's as

$$c(\vec{a},\vec{b}) = \underset{a}{tr}\,\underset{b}{tr}\left[(\vec{a}\hat{\vec{s}}^a)(\vec{b}\hat{\vec{s}}^b)\hat{\rho}^{\,singlet}\right] =$$

$$= 2^2 w_i^2\int d\mu(\vec{s}^a)\int d\mu(\vec{s}^b)(\vec{a}\vec{s}^a)(\vec{b}\vec{s}^b)\,\rho_i(\vec{s}^a,\vec{s}^b|singlet) = \quad (113)$$

$$= -\frac{1}{4}(\vec{a}\vec{b}),$$

where $\quad i=\frac{1}{2},\frac{1}{2}',Y,Y',\ w^2_{Y'}=w^2_Y=s(s+1)=\frac{3}{4}$. The expressions (113) with positive densities leads us (following the Bell course of derivation[8-11]) to the following quantum analogs of the Bell inequality

$$|c(\vec{a},\vec{b})-c(\vec{a},\vec{b}')|+|c(\vec{a}',\vec{b}')+c(\vec{a}',\vec{b})| \leq 2\cdot w_i^2 = 2\cdot\begin{cases}\frac{9}{4} & i=\frac{1}{2}\\ \frac{3}{4} & i=Y'\end{cases} \quad (114)$$

The latter corresponds to eq. (85).

The nonoperator representative of the product of two operators in terms of their representatives can be constructed as above, on p. 9. For the other possibility corresponding to eq. (93) see ref.[21].

The operator representatives for spin $\frac{1}{2}$ are given by[23] (p. 14)

$$s^{\ell}_j{}^{\!\!r} = \frac{1}{2}s_j + \frac{1}{2}s_k\left(s_k\frac{\partial}{\partial s_j} - s_j\frac{\partial}{\partial s_k}\right) \mp \frac{i}{2}\varepsilon_{jk\ell}\,s_k\frac{\partial}{\partial s_\ell}. \quad (115)$$

They satisfy

$$s^\ell_j s^\ell_j = s^r_j s^r_j = \frac{3}{4} \quad (116)$$

(s(s+1) for any spin s) and the commutation relations (72). However the relations

$$s^\ell_j s^\ell_k = \frac{1}{4}\delta_{jk} + \frac{i}{2}\varepsilon_{jkm}s^\ell_m\,; \qquad s^r_j s^r_k = \frac{1}{4}\delta_{jk} - \frac{i}{2}\varepsilon_{jkm}s^r_m \quad (117)$$

are satisfied only in application to polynomials in $\vec{s}$ of degree $n \leqslant 1$.

9. For spin $\frac{1}{2}$ the expectation value (108) with the positive density (102.b') (or $\delta_{S^2}(\vec{s},-\vec{a})$ for $m=-\frac{1}{2}$) may serve as "an example of what is for us a successful introduction of hidden variables" "for restricted part of quantum mechanics" (Kochen and Specker[7], however they proposed another realization). The phase space variables $\vec{s}$ enter like hidden variables. However in the same representation the correlator of components of two spins $\frac{1}{2}$ in the singlet state is described by the density (111.b), which is not positive definite.

For spin 1 in the corresponding representation we cannot avoid not positive definite densities (see above, p. 7), and this is valid for higher spins too[22,23].

In other PSR's with positive densities expectation values, correlators etc. include extra ("quantum") numerical factors (and extra terms in the representation, corresponding to the completeness relation in the form (93)).

Appendix A. The completeness relation (7) can be also written as follows

$$\frac{1}{3}|\mathbb{1}|\otimes\|\mathbb{1}\| + \frac{1}{2}|\hat{s}_j|\otimes\|\hat{s}_j\| + \frac{1}{4}|\hat{s}_j\hat{s}_k+\hat{s}_k\hat{s}_j-\tfrac{4}{3}\delta_{jk}\mathbb{1}|\otimes\|\hat{s}_j\hat{s}_k+\hat{s}_k\hat{s}_j-\tfrac{4}{3}\delta_{jk}\mathbb{1}\| =$$

$$= |\mathbb{1}\|\otimes\|\mathbb{1}|, \quad (A.1)$$

$$\frac{1}{2}|\hat{s}_j|\otimes\|\hat{s}_j\| + \frac{1}{4}|\hat{s}_j\hat{s}_k+\hat{s}_k\hat{s}_j-2\delta_{jk}\mathbb{1}|\otimes\|\hat{s}_j\hat{s}_k+\hat{s}_k\hat{s}_j-2\delta_{jk}\mathbb{1}\| = |\mathbb{1}\|\otimes\|\mathbb{1}|, \quad (A.2)$$

$$\frac{1}{2}|\hat{s}_j|\otimes\|\hat{s}_j\| + \frac{1}{4}|\hat{s}_j\hat{s}_k+\hat{s}_k\hat{s}_j-\tfrac{2}{3}\delta_{jk}\mathbb{1}|\otimes\|\hat{s}_j\hat{s}_k+\hat{s}_k\hat{s}_j-\tfrac{2}{3}\delta_{jk}\mathbb{1}\| = |\mathbb{1}\|\otimes\|\mathbb{1}|. \quad (A.3)$$

The term $|\mathbb{1}|\otimes\|\mathbb{1}\|$ is hidden in eqs. (A.2) and (A.3). Only the irreducible tensor operators $\mathbb{1}, \hat{s}_j$ and $\hat{s}_j\hat{s}_k+\hat{s}_k\hat{s}_j-\frac{4}{3}\delta_{jk}\mathbb{1}$ (with zero convolution of j and k) enter in eq.(A.1). Except for $\mathbb{1}$, they are traceless. These operators are mutually orthogonal

$$tr[\mathbb{1}\cdot\hat{s}_j]=0, \qquad tr[\mathbb{1}\cdot(\hat{s}_j\hat{s}_k+\hat{s}_k\hat{s}_j-\tfrac{4}{3}\delta_{jk}\mathbb{1})]=0,$$

$$tr[\hat{s}_j(\hat{s}_k\hat{s}_\ell+\hat{s}_\ell\hat{s}_k-\tfrac{4}{3}\delta_{\ell k}\mathbb{1})]=0,$$

$$tr[(\hat{s}_j\hat{s}_k+\hat{s}_k\hat{s}_j-\tfrac{4}{3}\delta_{jk}\mathbb{1})(\hat{s}_\ell\hat{s}_m+\hat{s}_m\hat{s}_\ell-\tfrac{4}{3}\delta_{\ell m}\mathbb{1})] =$$

$$= 2\delta_{j\ell}\delta_{km} + 2\delta_{jm}\delta_{k\ell} - \frac{4}{3}\delta_{jk}\delta_{\ell m}. \quad (A.4)$$

Note that the completeness relation (7) in terms of the Gell-Mann matrices takes the form

$$\frac{1}{3}|\mathbb{1}|\otimes\|\mathbb{1}\| + \frac{1}{2}\sum_{\alpha=1}^{8}|\lambda_\alpha|\otimes\|\lambda_\alpha\| = |\mathbb{1}\|\otimes\|\mathbb{1}|. \tag{A.5}$$

Let us give the spin 1 matrices in two representations:
a) in the usual one ($\hat{s}_j$ act on the Cartesian coordinates), $(\hat{s}_j)_{k\ell} = -i\epsilon_{jk\ell}$,

$$\hat{s}_1 = \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & -i \\ \cdot & i & \cdot \end{bmatrix} \quad \hat{s}_2 = \begin{bmatrix} \cdot & \cdot & i \\ \cdot & \cdot & \cdot \\ -i & \cdot & \cdot \end{bmatrix} \quad \hat{s}_3 = \begin{bmatrix} \cdot & -i & \cdot \\ i & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}$$

$$\hat{s}_1^2 = \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \end{bmatrix} \quad \hat{s}_1\hat{s}_2 = \begin{bmatrix} \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} \quad \hat{s}_1\hat{s}_3 = \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot \end{bmatrix}$$

$$\hat{s}_2\hat{s}_1 = \begin{bmatrix} \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} \quad \hat{s}_2^2 = \begin{bmatrix} 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 \end{bmatrix} \quad \hat{s}_2\hat{s}_3 = \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot \end{bmatrix}$$

$$\hat{s}_3\hat{s}_1 = \begin{bmatrix} \cdot & \cdot & -1 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} \quad \hat{s}_3\hat{s}_2 = \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 \\ \cdot & \cdot & \cdot \end{bmatrix} \quad \hat{s}_3^2 = \begin{bmatrix} 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}$$

$$\{\hat{s}_1\hat{s}_2\} = \begin{bmatrix} \cdot & -1 & \cdot \\ -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} \quad \{\hat{s}_2\hat{s}_3\} = \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 \\ \cdot & -1 & \cdot \end{bmatrix} \quad \{\hat{s}_3\hat{s}_1\} = \begin{bmatrix} \cdot & \cdot & -1 \\ \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot \end{bmatrix}$$

and b) in the canonical representation

$$\hat{s}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} \cdot & 1 & \cdot \\ 1 & \cdot & 1 \\ \cdot & 1 & \cdot \end{bmatrix} \quad \hat{s}_2 = \frac{i}{\sqrt{2}}\begin{bmatrix} \cdot & -1 & \cdot \\ 1 & \cdot & -1 \\ \cdot & 1 & \cdot \end{bmatrix} \quad \hat{s}_3 = \begin{bmatrix} 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 \end{bmatrix}$$

$$\hat{s}_1^2 = \frac{1}{2}\begin{bmatrix} 1 & \cdot & 1 \\ \cdot & 2 & \cdot \\ 1 & \cdot & 1 \end{bmatrix} \quad \hat{s}_1\hat{s}_2 = \frac{i}{2}\begin{bmatrix} 1 & \cdot & -1 \\ \cdot & \cdot & \cdot \\ 1 & \cdot & -1 \end{bmatrix} \quad \hat{s}_1\hat{s}_3 = \frac{1}{\sqrt{2}}\begin{bmatrix} \cdot & \cdot & \cdot \\ 1 & \cdot & -1 \\ \cdot & \cdot & \cdot \end{bmatrix}$$

$$\hat{s}_2\hat{s}_1 = \frac{i}{2}\begin{bmatrix} -1 & \cdot & -1 \\ \cdot & \cdot & \cdot \\ 1 & \cdot & 1 \end{bmatrix} \quad \hat{s}_2^2 = \frac{1}{2}\begin{bmatrix} 1 & \cdot & -1 \\ \cdot & 2 & \cdot \\ -1 & \cdot & 1 \end{bmatrix} \quad \hat{s}_2\hat{s}_3 = \frac{i}{\sqrt{2}}\begin{bmatrix} \cdot & \cdot & \cdot \\ 1 & \cdot & 1 \\ \cdot & \cdot & \cdot \end{bmatrix}$$

$$\hat{s}_3\hat{s}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot \end{bmatrix} \quad \hat{s}_3\hat{s}_2 = \frac{i}{\sqrt{2}}\begin{bmatrix} \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot \end{bmatrix} \quad \hat{s}_3^2 = \begin{bmatrix} 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 \end{bmatrix}$$

$$\{\hat{s}_1\hat{s}_2\} = i\begin{bmatrix} \cdot & \cdot & -1 \\ \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot \end{bmatrix} \quad \{\hat{s}_2\hat{s}_3\} = \frac{i}{\sqrt{2}}\begin{bmatrix} \cdot & -1 & \cdot \\ 1 & \cdot & 1 \\ \cdot & -1 & \cdot \end{bmatrix} \quad \{\hat{s}_3\hat{s}_1\} = \frac{1}{\sqrt{2}}\begin{bmatrix} \cdot & 1 & \cdot \\ 1 & \cdot & -1 \\ \cdot & -1 & \cdot \end{bmatrix}$$

In the first representation the squares of the spin matrices are diagonal, and therefore, it is obvious that they commute with each other

$$[\hat{s}_i^2, \hat{s}_k^2] = 0 \qquad \text{(no summations).} \tag{A.6}$$

Hence they can be measured simultaneously (like for spin $\frac{1}{2}$) with three different (unlike for spin $\frac{1}{2}$) outcomes: (0,1,1), (1,0,1) or (1,1,0)[7].

Note that both the proof, using the first representation, and the following "coordinate-free" proof:

$$[\hat{s}_j^2\, \hat{s}_k^2] = (\hat{s}_k\hat{s}_j\hat{s}_j + \hat{s}_j\hat{s}_j\hat{s}_k)\hat{s}_k - \hat{s}_k(\hat{s}_j\hat{s}_j\hat{s}_k + \hat{s}_k\hat{s}_j\hat{s}_j) =$$
$$= (\delta_{jk}\hat{s}_j + \hat{s}_k)\hat{s}_k - \hat{s}_k(\delta_{jk}\hat{s}_j + \hat{s}_k) = 0, \tag{A.7}$$

using the Duffin-Kemmer algebra (3), are clearly simpler than those, given in ref[7].

Appendix B. Derivation of eqs. (18)-(20). Integrating, we get

$$2\cdot 3\int d\mu(\vec{s})\, \hat{\rho}(0,\vec{s})\otimes\hat{\rho}(0,\vec{s}) = -\frac{2}{5}\mathbb{1}\otimes\mathbb{1} + \frac{1}{5}\{\hat{s}_i\hat{s}_j\}\otimes\{\hat{s}_i\hat{s}_j\} \tag{B.1}$$

$$2\cdot 3\int d\mu(\vec{s})\, \hat{\rho}(1,\vec{s})\otimes\hat{\rho}(1,\vec{s}) = \frac{1}{2}\left[\frac{4}{5}\mathbb{1}\otimes\mathbb{1} + \hat{s}_i\otimes\hat{s}_i + \frac{1}{10}\{\hat{s}_i\hat{s}_j\}\otimes\{\hat{s}_i\hat{s}_j\}\right]. \tag{B.2}$$

Hence it is clear that the completeness relation (7) can be represented in terms of these integrals by eq. (18).

In eqs. (19) and (20) we need the itegral

$$3\int d\mu(\vec{\alpha})\left[\alpha'\cdot\mathbb{1} + \beta'(\vec{\alpha}\hat{\vec{s}}) + \gamma'(\vec{\alpha}\hat{\vec{s}})^2\right]\otimes\left[\alpha\cdot\mathbb{1} + \beta(\vec{\alpha}\hat{\vec{s}}) + \gamma(\vec{\alpha}\hat{\vec{s}})^2\right] =$$
$$= 3\left[\alpha'\alpha + \frac{1}{3}(\alpha'\gamma + \gamma'\alpha)s(s+1) + \frac{1}{3\cdot 5}\gamma'\gamma(s(s+1))^2\right]\cdot\mathbb{1} +$$
$$+ \beta'\beta\,\hat{s}_i\otimes\hat{s}_i + \frac{1}{10}\gamma'\gamma\{\hat{s}_i\hat{s}_j\}\otimes\{\hat{s}_i\hat{s}_j\}. \tag{B.3}$$

Comparison with eq. (7) leads us to the set of equations

$$3\left[\alpha'\alpha + \frac{2}{3}(\alpha'\gamma + \gamma'\alpha) + \frac{4}{3\cdot 5}\gamma'\gamma\right] = -1,$$
$$\beta'\beta = \frac{1}{2},$$
$$\frac{1}{10}\gamma'\gamma = \frac{1}{4}, \tag{B.4}$$

which must be solved for α, β, γ

with $\alpha' = 0,\ \beta' = \frac{1}{2},\ \gamma' = \frac{1}{2}$, (B.5)

and with $\alpha' = \alpha,\ \beta' = \beta,\ \gamma' = \gamma$ (B.6)

to obtain eqs. (19) and (20), respectively.

References

1. Wigner E.P. Phys.Rev., 1932, 40, p.749.
2. Neumann, von, J. Mathematische Grundlagen der Quantenmechanik. Springer, Berlin, 1932.
3. Einstein A., Podolsky B., Rosen N., Phys.Rev. 1935, 47, p.777.
4. Moyal J.E. Proc. Cambr. Phil. Soc., 1949, 45, p.99.

5. Bohm D. Quantum Theory, Prentice Hall, Englewood Cliffs, 1951.
Bohm D., Aharonov Y., Phys.Rev. 1957, 108, p.1070.

6. Feynman R.P., Hibbs A.R. Quantum mechanics and path integrals. McGraw-Hill, New York, 1965.

7. Kochen S., Specker E.P. Journ.of Math. and Mech. 1967, 17, N1,p.59.

8. Bell J.S. Physika 1964, 1, p.195.

9. Bell J.S. Rev. Mod. Phys. 1966, 38, p.447.

10. Bell J.S. Foundations of Quantum Mechanics (ed.B.Espagnat). Academic Press, New York, 1971, p.171.

11. Clauser J.P., Shimoni A. Reports on Progress in Physics, 1978, 41, p.1881.

12. Mermin N.D. Phys. Rev. 1980, D22, p.356.

13. Cirel'son B.S. Lett. Math. Phys. 1990, 4, p.93.

14. Arecchi F.T., Courtens E., Gilmore R., Thomas H. Phys. Rev., 1972, A6, p.2211.

15. Klauder J.P. Journ. Phys. 1983, A16, p.L317.

16. Переломов А.М. Обобщенные когерентные состояния и их приложения. "Наука", М., 1987.

17. Feynman R.P. Intern. Journ. of Theor. Phys., 1982, 21, p.467.

18. Feynman R.P. Negative Probability. Caltech preprint. Pasadena, 1984.

19. Polubarinov I.V. JINR E2-11478, Dubna, 1978; E2-81-240, Dubna, 1981; E2-83-688, Dubna, 1983.

20. Полубаринов И.В. В кн.: Теоретико-групповые методы в физике. Наука, М., 1983, т.2, стр.3 и 189.
Polubarinov I.V. In: Group Theoretical Methods in Physics, Harwood Academic Publishers, Chur, London, Paris, New York, 1985, v.1, pp. 167 and 577.
Polubarinov I.V. JINR E2-82-800, Dubna, 1982.

21. Polubarinov I.V. JINR E2-88-80, Dubna, 1988.

22. Polubarinov I.V. JINR E4-91-247, Dubna, 1991. Correction: in this paper in the first expression for $\delta_{S2}(\vec{s},\vec{\alpha})$ (eq. (71)) the power $\frac{3}{2}$ of the denominator was missed (cf. eq. (43) on p. 6 here).

23. Polubarinov I.V. JINR E2-91-462, Dubna, 1991.

24. Копылов Г.И., Полубаринов И.В., Семашко Г.Л. ЖЭТФ, 1964, 46,p.1320.

Received by Publishing Department
on December 29, 1991.