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ON A P-ADIC METRICAL DIMENSION OF SPACE

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*No point is more central than this,  
that empty space is not empty.*

*J.A.Wheeler*

The Universe mainly has two types of structures: the homogeneous and the hierarchical ones<sup>/1/</sup>. The former is convenient to describe by real number field (and its multidimensional extensions), the latter - by p-adic (non-Archimedean) number fields<sup>/2/</sup>. Nowadays due to nonlinear dynamical systems (with strange attractor phase space pictures<sup>/3/</sup>) and unified field or string theories<sup>/4/</sup>, dynamical change of the dimension of space (when time and/or space scales of the considered phenomena change) becomes actual<sup>/5/</sup>.

In the dimensional regularization technique<sup>/6/</sup> we (formally) consider non-integer and sometimes even negative dimensions of space<sup>/7/</sup>. For models of random surfaces sometimes negative values of dimension were considered<sup>/8/</sup>.

The topological (inductive) definitions of dimension<sup>/9/</sup> say that a set has (integer) dimension equal to  $n$  if its boundary has dimension  $n-1$ .

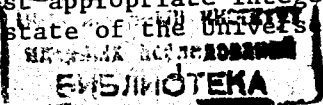
In the metrical definition of dimension<sup>/10/</sup> we count the number of covering elements (e.g. Spheres)  $N(a)$  with linear size  $a$ , and try to find a value of the parameter  $d$  from the equation

$$N(a) a^d = \text{const}, \quad a \rightarrow 0. \quad (1)$$

For this metrical dimension we have

$$d = \lim_{a \rightarrow 0} \frac{\ln N}{\ln(1/a)}. \quad (2)$$

For any set of finite number of points  $N(a) \rightarrow \text{const}$ , so  $d = 0$ . For empty set  $N(a) = 0$  and we cannot use equation (1) to determine the value of dimension which is equal to  $-1$ , by topological definition. In this case it will be convenient to use some "regularization", with  $N(a) \rightarrow 0$ . Then from (1) we see that  $d < 0$  and the simplest appropriate integer is  $-1$ . If we define the Void<sup>/11/</sup> as the state of the universe just before God cre-



ated the empty vacuum, then we can assign to the Void dimension  $-2^*$ .

But how can the integer  $N(a)$  behave almost continually near the zero, when  $a \rightarrow 0$ ? Here we can use  $p$ -adic valuation<sup>/2/</sup> of integers

$$|N|_p = \begin{cases} 0, & \text{if } N=0 \\ 1/p^{\text{ord}_p N}, & N \neq 0, \end{cases} \quad (3)$$

where for a given value of the prime number  $p$  we have the unique representation

$$N = p^{\text{ord}_p N} n, \quad (4)$$

$n$  does not contain factor  $p$ .

From definition (3), (4) we see that

$$0 \leq |N|_p \leq 1,$$

so

$$d_p = \lim_{a \rightarrow 0} \frac{\ln |N|_p}{\ln 1/a} \leq 0. \quad (5)$$

If we take, for example,

$$N = p^{mk} n, \quad a = p^{-lk}, \quad k \rightarrow \infty,$$

where  $m, k, l$  are positive integer numbers, then

$$d_p = -\frac{m}{l}. \quad (6)$$

For any non-zero rational  $x$  we have an adelic construction<sup>/2/</sup>

$$\prod_{p \geq 0} |x|_p = 1, \quad x \neq 0, \quad (7)$$

where by definition

$$|x|_0 = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

is a usual absolute valuation.

\* I thank P.Frampton and I.Volovich for the discussion of this point.

Now from (7) for  $x=N$  and definition (2) and (5) we obtain

$$d = - \sum_{p \geq 2} d_p. \quad (8)$$

So it is possible to calculate "real" fractal dimension of a set (2) by summing  $p$ -adic fractal dimensions (5) for every prime  $p$ .

2. For quantum (fluctuating) geometry, there are different ways of introduction of dimension<sup>/5/</sup>

$$d_1 = \langle \frac{\ln N}{\ln 1/a} \rangle, \quad d_2 = \frac{\ln \langle N \rangle}{\ln 1/a}. \quad (9)$$

It is easy to show that  $d_2 \geq d_1$ , using the following inequality\*

$$\sum_i P_i N_i \geq \prod_i N_i^{P_i}, \quad (9)$$

where

$$\sum_i P_i = 1, \quad P_i \geq 0.$$

This inequality is, for rational values of  $P_i = \frac{m_i}{M}$ ,  $\sum_i m_i = M$ , a consequence of the well-known inequality

$$\frac{a_1 + a_2 + \dots + a_M}{M} \geq \sqrt[M]{a_1 a_2 \dots a_M}. \quad (10)$$

when  $a_1 = a_2 = \dots = a_{m_1} = N_1$ ,  $a_{m_1+1} = \dots = a_{m_1+m_2} = N_2, \dots$

Indeed,

$$\langle N \rangle = \sum_i P_i N_i > \prod_i N_i^{P_i} = e^{\sum_i P_i \ln N_i} = e^{\langle \ln N \rangle}$$

so

$$\ln \langle N \rangle \geq \langle \ln N \rangle$$

Note, when  $\langle N \rangle < 1$ , e.g.  $P_1 = 1 - \epsilon$ ,  $P_2 = \epsilon$ ,  $N_1 = 0$ ,  $N_2 \neq 0$ ,

$\epsilon < \frac{1}{N_2}$ ,  $\langle N \rangle = \epsilon N_2 < 1$ , then  $d_2 < 0$ . In this case  $d_1$  is not defined, so there is no contradiction with inequality  $d_2 \geq d_1 \geq 0$ .

The negative dimension was invented before by B.Mandelbrot (see, e.g.,<sup>/12/</sup>). I added the last note to the manuscript, when I had seen that paper<sup>/12/</sup>.

\* This proof was stimulated by the discussion with A.Berenstein.

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