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ON A P-ADIC METRICAL DIMENSION OF SPACE

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No point is more central than this,
that empty space is not empty.

J.A.Wheeler

The Universe mainly has two types of structures: the homogeneous and the hierarchical ones[1]. The former is convenient to describe by real number field (and its multidimensional extensions), the latter - by p-adic (non-Archimedian) number fields[2]. Nowadays due to nonlinear dynamical systems (with strange attractor phase space pictures[3]) and unified field or string theories[4], dynamical change of the dimension of space (when time and/or space scales of the considered phenomena change) becomes actual[5].

In the dimensional regularization technique[6] we (formally) consider non-integer and sometimes even negative dimensions of space[7]. For models of random surfaces sometimes negative values of dimension were considered[8].

The topological (inductive) definitions of dimension[9] say that a set has (integer) dimension equal to n if its boundary has dimension n-1.

In the metrical definition of dimension[10] we count the number of covering elements (e.g. Shperes) N(a) with linear size a, and try to find a value of the parameter d from the equation

$$N(a)\,a^{d} = \text{const}, \quad a \to 0. \tag{1}$$

For this metrical dimension we have

$$d = \lim_{a \to 0} \frac{\ln N}{\ln(1/a)}. \tag{2}$$

For any set of finite number of points $N(a) \to \text{const}$, so $d = 0$. For empty set $N(a) = 0$ and we cannot use equation (1) to determine the value of dimension which is equal to -1, by topological definition. In this case it will be convenient to use some "regularization", with $N(a) \to 0$. Then from (1) we see that $d < 0$ and the simplest appropriate integer is -1. If we define the Void[11] as the state of the Universe just before God created the empty vacuum, then we can assign to the Void dimenstion -2*.

But how can the integer $N(a)$ behave almost continually near the zero, when $a \to 0$? Here we can use p-adic valuation [2] of integers

$$|N|_p = \begin{cases} 0, & \text{if } N = 0 \\ 1/p^{\mathrm{ord}_p N}, & N \neq 0, \end{cases} \tag{3}$$

where for a given value of the prime number p we have the unique representation

$$N = p^{\mathrm{ord}_p N} n, \tag{4}$$

n does not contain factor p.

From definition (3), (4) we see that

$0 \le |N|_p \le 1$,

so

$$d_p = \lim_{a \to 0} \frac{\ln |N|_p}{\ln 1/a} \le 0. \tag{5}$$

If we take, for example,

$N = p^{mk} n, \quad a = p^{-\ell k}, \quad k \nearrow \infty,$

where m, k, ℓ are positive integer numbers, then

$$d_p = -\frac{m}{\ell}. \tag{6}$$

For any non-zero rational x we have an adelic construction [2]

$$\prod_{p \ge 0} |x|_p = 1, \quad x \neq 0, \tag{7}$$

where by definition

$$|x|_0 = \begin{cases} x, & x \ge 0 \\ -x, & x < 0 \end{cases}$$

is a usual absolute valuation.

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* I thank P.Frampton and I.Volovich for the discussion of this point.

Now from (7) for $x = N$ and definition (2) and (5) we obtain

$$d = -\sum_{p \ge 2} d_p. \tag{8}$$

So it is possible to calculate "real" fractal dimension of a set (2) by summing p-adic fractal dimensions (5) for every prime p.

2. For quantum (fluctuating) geometry, there are different ways of introduction of dimension [5]

$$d_1 = \left\langle \frac{\ln N}{\ln 1/a} \right\rangle, \quad d_2 = \frac{\ln \langle N \rangle}{\ln 1/a}. \tag{9}$$

It is easy to show that $d_2 \ge d_1$, using the following inequality*

$$\sum_i P_i N_i \ge \prod_i N_i^{P_i}, \tag{9}$$

where

$\sum_i P_i = 1, \; P_i \ge 0.$

This inequality is, for rational values of $P_i = \frac{m_i}{M}$, $\sum_i m_i = M$, a consequence of the well-known inequality

$$\frac{a_1 + a_2 + \ldots + a_M}{M} \ge \sqrt[M]{a_1 a_2 \ldots a_M}, \tag{10}$$

when $a_1 = a_2 = \ldots = a_{m_1} = N_1$, $a_{m_1+1} = \ldots = a_{m_1+m_2} = N_2, \ldots$

Indeed,

$$\langle N \rangle = \sum_i P_i N_i > \prod_i N_i^{P_i} = e^{\sum P_i \ln N_i} = e^{\langle \ln N \rangle}$$

so

$\ln \langle M \rangle \ge \langle \ln N \rangle$ ■

Note, when $\langle N \rangle < 1$, e.g. $P_1 = 1 - \epsilon$, $P_2 = \epsilon$, $N_1 = 0$, $N_2 \neq 0$, $\epsilon < \frac{1}{N_2}$, $\langle N \rangle = \epsilon N_2 < 1$, then $d_2 < 0$. In this case d_1 is not defined, so there is no contradiction with inequality $d_2 \ge d_1 \ge 0$.

The negative dimension was invented before by B.Mandelbrot (see, e.g., [12]). I added the last note to the manuscript, when I had seen that paper [12].

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REFERENCES

1. Peebles P.J.E. - Large-Scale Structures of the Universe, Princeton Univ. Press, Princeton, 1980.
2. Koblitz N. - p-adic Numbers, p-adic Analysis and Zeta Functions, Springer Verlag, 1984.
3. Lichtenberg A.J., Liberman M.A. - Regular and Stochastic Motion, Springer-Verlag, 1983.
4. Green M.B., Schwarz J.H., Witten E. - Superstring Theory, Cambridge Univ. Press, Cambridge, 1987.
5. Makhaldiani N.V. - Communication of the JINR, P2-87-306, Dubna, 1987.
6. Collins J.C. - Renormalization, Cambridge Univ. Press, Cambridge, 1984.
7. Halliday I.G., Ricotta R.M. - Phys. Lett. B, 1987, 193, p.241.
8. Kazakov V.A., Kostov I.K., Migdal A.A. - Phys. Lett. B, 1985, 157, p.295.
9. Hurewicz W., Wallman Wh. - Dimension Theory, 1941.
10. Mandelbrot B. - The Fractal Geometry of Nature, Freeman, San Francisco, 1982.
11. Frampton P.H., Volovich I.V. - Mod. Phys. Lett. A, 1990, 5, p.1825.
12. Mandelbrot B. - Physica A, 1990, 163, p.306.

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