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DEFORMED TRACES AND COVARIANT QUANTUM ALGEBRAS FOR QUANTUM GROUPS $GL_{qp}(2)$ AND $GL_{qp}(1/1)$

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Symmetry groups and symmetry algebras have played a very significant role in the development of modern theoretical physics. Quantum groups, which are deformations (and in some sense a generalization) of usual groups have attracted a great deal of interest since the seminal papers by Drinfeld [1], Jimbo [2], Faddeev etal. [3], Woronowicz [4] and Manin [5]. These deformed (super)groups present the examples of Hopf algebras and have found application in as diverse areas of physics and mathematics as non-linear integrable models, statistical mechanics, conformal field theory, knot theory and solutions of Yang-Baxter equations etc.(see, [6-10] and references therein).

The general quantum deformations of Lie (super)groups or Lie (super)algebras are the multiparameter deformations. For instance, general quantum deformation of $GL(N)$ group has $[N(N-1)/2]+1$ deformation parameters[5]. The simple example of two-parameter quantum deformation of $GL(2)$ and its differential calculus have been considered in ref.[11]. Following the method of graded tensor product [12], the two-parameter deformation of supergroup $GL(1 \mid 1)$ has also been discussed [13].

Recently the idea of quantum orbits, one-parameter deformed quantum trace, its subsequent application to the construction of q-deformed algebras and the formulation of q-deformed Yang-Mills theory have been developed in ref.[14]. The purpose of our paper is to define quantum trace and quantum orbits for the two-parameter groups $GL_{qp}(2)$ and $GL_{qp}(1 \mid 1)$. Further, we obtain the invariants for the orbits of these groups and demonstrate that these can be succinctly expressed in terms of the deformed traces. Following the approach of ref.[15], we construct the (super)oscillator algebras covariant under the action of the $GL_{qp}(2)$ and $GL_{qp}(1 \mid 1)$ groups and show that bilinears of these (super)oscillators form the one parameter deformed covariant algebra in the case of $GL_{qp}(2)$ group and covariant extensions of $N = 2$ supersymmetric quantum mechanical algebras for the quantum group $GL_{qp}(1 \mid 1)$. The more interesting case of $GL_{qp}(2)$ leads to the construction of central extension of Witten-type q-algebra $U_q(sl(2))$ [16]. This algebra can be considered as "adjoint representation" of the quantum group $GL_{qp}(2)$.

Following the Manin's quantum hyperplane approach [5] to the general construction of quantum groups, it can be shown that the 2×2 $GL_{qp}(2)$ matrix $T_{ij} = \begin{pmatrix} ab \\ cd \end{pmatrix}$ with noncommuting elements a, b, c and d exhibits different braiding relations in rows and columns as given below [11]:

$$\begin{aligned} ab = pba, \quad cd = pdc, \\ ac = qca, \quad bd = qdb, \end{aligned} \tag{1a}$$

and the other relations are:

$$bc = \frac{q}{p}cb, \quad ad - da = (p - q^{-1})bc = (q - p^{-1})cb, \tag{1b}$$

where $q, p \in C/\{0\}$. It is easy to note that one-parameter quantum group $GL_q(2)$ corresponds to the special case of (1a,b) when $q = p$. The inverse quantum matrix

(T_{ij}^{-1}) is defined as follows [11]:

$$T_{ij}^{-1} = \mathcal{D}^{-1}\begin{pmatrix} d, & -q^{-1}b \\ -qc, & a \end{pmatrix} \equiv \begin{pmatrix} d, & -p^{-1}b \\ -pc, & a \end{pmatrix}\mathcal{D}^{-1}, \tag{2}$$

where the quantum determinant $\mathcal{D} = ad - pbc = da - q^{-1}bc = ad - qcb = da - p^{-1}cb$ is not the central element of the algebra (1a), (1b) when $q \neq p$ but obeys the following relations[11]:

$$a(\mathcal{D},\ \mathcal{D}^{-1}) = (\mathcal{D},\ \mathcal{D}^{-1})a,\ \ b(\mathcal{D},\ \mathcal{D}^{-1}) = (\frac{q}{p}\mathcal{D},\ \frac{p}{q}\mathcal{D}^{-1})b,$$

$$c(\mathcal{D},\ \mathcal{D}^{-1}) = (\frac{p}{q}\mathcal{D},\ \frac{q}{p}\mathcal{D}^{-1})c,\ \ d(\mathcal{D},\ \mathcal{D}^{-1}) = (\mathcal{D},\ \mathcal{D}^{-1})d. \tag{3}$$

Now let us introduce a 2×2 quantum matrix E_{ij} with noncommuting elements. Following transformations of E_{ij}:

$$E_{ij} \longrightarrow T_{ik}E_{kl}T_{lj}^{-1} \tag{4}$$

define, for all possible $T_{ij} \in GL_{qp}(2)$, the quantum orbit in the space of 2×2 q-matrices E_{ij} if the elements of this matrix commute with that of T_{ij} (i.e. $[T_{ij}, E_{kl}] = 0$). Since the elements of T are noncommuting objects, the usual trace of matrix E is not invariant under transformations (4). However, it turns out that following expression:

$$tr_{qp}(E) = tr_{qp}(TET^{-1}) = r^{-1}E_{11} + rE_{22}\ , \tag{5}$$

where $r = \sqrt{qp}$, remains invariant under (4). We will call this as the quantum (qp)-trace. It is straightforward to see that the case $q = p$ in (5) yields the one-parameter trace of ref.[14] and, $q = p = 1$ corresponds to the usual undeformed trace. It may be noticed that all other invariants for the orbit (4) can be written as $c^n = Tr_{qp}(E^n)$.

Now let us construct the two parametric covariant quantum oscillators for $GL_{qp}(2)$: To this end in mind, we introduce two sets of q-oscillators A_i and $\tilde{A}_i$ $(i = 1, 2)$. In the language of differential geometry on the quantum hyper plane [17], these operators correspond to co-ordinates and derivatives. It is clear that following relations:

$$\begin{aligned} A_1A_2 - qA_2A_1 &= 0\ , \\ \tilde{A}_1\tilde{A}_2 - p^{-1}\tilde{A}_2\tilde{A}_1 &= 0 \end{aligned} \tag{6}$$

remain invariant under $GL_{qp}(2)$ transformations:

$$A_i \rightarrow T_{ij}A_j\ ,\ \tilde{A}_i \rightarrow \tilde{A}_jT_{ji}^{-1}\ , \tag{7}$$

where $A_i = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$ and $\tilde{A}_i = (\tilde{A}_1,\ \tilde{A}_2)$. Consistent with relations (6), following oscillator algebra also remain invariant under transformations(7):

$$\begin{aligned} A_2\tilde{A}_1 - \tfrac{(\alpha-\beta)}{p}\tilde{A}_1A_2 &= 0, \\ A_1\tilde{A}_2 - \tfrac{(\alpha-\beta)}{q}\tilde{A}_2A_1 &= 0, \\ A_2\tilde{A}_2 - \alpha\tilde{A}_2A_2 &= 1 + (\alpha - \tfrac{(\alpha-\beta)}{r^2})\tilde{A}_1A_1, \end{aligned} \tag{8a}$$

if we postulate the validity of following general bilinear oscillator relation:

$$A_1\tilde{A}_1 - \alpha\tilde{A}_1A_1 = 1 + \beta\tilde{A}_2A_2\ . \tag{8b}$$

Here α and β are c-number parameters which can be fixed by requiring associativity of the oscillator algebra (6) and (8). In fact, the oscillator algebra (6) and (8) give us the possibility to reorder a product of oscillators A_i and $\tilde{A}_i$ such that all waved operators can be brought to the left side of the products. For example let us consider the product $A_1\tilde{A}_1\tilde{A}_2$. There are two possible ways to reorder this expression as given below:

$$(A_1(\tilde{A}_1\tilde{A}_2)) = \frac{(\alpha-\beta)}{r^2}(\tilde{A}_2 + \alpha\tilde{A}_2\tilde{A}_1A_1 + \beta\tilde{A}_2\tilde{A}_2A_2), \tag{9a}$$

$$((A_1\tilde{A}_1)\tilde{A}_2) = (1+\beta)\tilde{A}_2 + (\frac{(\alpha-\beta)^2}{r^2} + \alpha\beta)\tilde{A}_2\tilde{A}_1A_1 + \alpha\beta\tilde{A}_2\tilde{A}_2A_2. \tag{9b}$$

As can be seen, we obtain two different results on the right hand sides of (9a) and (9b) which are found to coincide only in two cases:

$$i.)\ \ \alpha = r^2,\ \beta = 0; \tag{10a}$$

$$ii.)\ \ \alpha = 1/r^2,\ \beta = (1-r^2)/r^2. \tag{10b}$$

The case (10a) leads to the following oscillator algebra:

$$\begin{aligned} A_2\tilde{A}_1 - q\tilde{A}_1A_2 &= 0, \\ A_1\tilde{A}_2 - p\tilde{A}_2A_1 &= 0, \\ A_2\tilde{A}_2 - qp\tilde{A}_2A_2 &= 1 + (qp-1)\tilde{A}_1A_1, \\ A_1\tilde{A}_1 - qp\tilde{A}_1A_1 &= 1, \\ A_1A_2 - qA_2A_1 &= 0, \\ \tilde{A}_1\tilde{A}_2 - p^{-1}\tilde{A}_2\tilde{A}_1 &= 0, \end{aligned} \tag{11}$$

while the algebra corresponding to (10b) can be obtained from equations (11) by replacements $i = 1 \leftrightarrow i = 2$ and $q, p \leftrightarrow q^{-1}, p^{-1}$. It is worth noting that the algebras of covariant pair of q-oscillators (ref.[15]) can be obtained from (11) by substitution $q = p$, $\tilde{A}_i = A_i^{\dagger}$ and replacements corresponding to the solution (10b). Here we stress that the procedure of obtaining conditions (10) is equivalent to the procedure of deducing and solving Yang-Baxter equations.

It can now be seen from equation (7) that quantum matrix:

$$E_{ij} = A_i\tilde{A}_j \tag{12}$$

satisfies the transformation law (4) of the quantum orbit. Furthermore, the invariance of the trace (5) under transformations (4) leads to the following $GL_{qp}(2)$ invariant Hamiltonian (H_{qp}) in bilinears of the covariant oscillators:

$$H_{qp} = r^{-1}A_1\tilde{A}_1 + rA_2\tilde{A}_2\ . \tag{13}$$

From the q-oscillator algebra (6) and (8), one can see that Hamiltonian (13) is related to the trivially $GL_{qp}(2)$ invariant Hamiltonian $(\tilde{H})$ by the following equation:

$$\tilde{H} = \sum_{i=1}^{2} \tilde{A}_i A_i = \frac{1}{r\alpha + r^{-1}\beta}[H_{qp} - (r + r^{-1})], \tag{14}$$

where α and β are defined in equations (10).

Now we would like to define the "adjoint representation" of the quantum group $GL_{qp}(2)$ and establish that the corresponding space of representation can be realized as a central extension of Witten-type $U_q(sl(2))$ algebra. For this purpose, the q-matrix E_{ij} can be rewritten in the following form:

$$E = \frac{1}{r+r^{-1}} \left\{ \begin{pmatrix} Tr_{qp}E & 0 \\ 0 & Tr_{qp}E \end{pmatrix} + \begin{pmatrix} rE_0, & (r+r^{-1})E_{12} \\ (r+r^{-1})E_{21}, & -r^{-1}E_0 \end{pmatrix} \right\} =$$

$$= \frac{1}{r+r^{-1}} (Tr_{qp}(E) + \tilde{E}), \tag{15}$$

where $E_0 = E_{11} - E_{22}$ and $\tilde{E} = \begin{pmatrix} rE_0, & (r+r^{-1})E_{12} \\ (r+r^{-1})E_{21}, & -r^{-1}E_0 \end{pmatrix}$. From the expression (15), it is clear that transformations (4) lead to following "four dimensional representation" of $GL_{qp}(2)$:

$$\begin{pmatrix} E_0' \\ E_{12}' \\ E_{21}' \\ H_{qp}' \end{pmatrix} = \frac{1}{\mathcal{D}} \begin{pmatrix} \mathcal{D} + (p+q^{-1})bc, & -(q+p^{-1})ac, & (p+q^{-1})db, & 0 \\ -pq^{-1}ba, & a^2, & -pq^{-2}b^2, & 0 \\ qp^{-1}cd, & -q^2p^{-1}c^2, & d^2, & 0 \\ 0, & 0, & 0, & \mathcal{D} \end{pmatrix} \begin{pmatrix} E_0 \\ E_{12} \\ E_{21} \\ H_{qp} \end{pmatrix}. \tag{16}$$

This representation is reducible because we have one dimensional (H_{qp}) and three dimensional (E_0, E_{12}, E_{21}) invariant subspaces. We can redefine operators $E_+ = E_{12} = A_1\bar{A}_2$ and $E_- = E_{21} = A_2\bar{A}_1$ and show that the oscillator relations (11) lead to:

$$[H_{qp}, E_\pm] = [H_{pq}, E_0] = 0\,, \tag{17}$$

$$\begin{aligned} [E_-, E_+] &= \left(\tfrac{r^2-1}{r^2+1}\right) E_0^2 + \left(\tfrac{1}{r^2} + \tfrac{r^2-1}{r^3+r} H_{qp}\right) E_0\,, \\ [E_\pm, E_0]_{(r^{\mp1}, r^{\pm1})} &\equiv r^{\mp1} E_\pm E_0 - r^{\pm1} E_0 E_\pm = \\ &= \pm \left(\tfrac{r^2-1}{r^2} H_{qp} + \tfrac{r^2+1}{r^3}\right) E_\pm. \end{aligned} \tag{18}$$

This algebra turns out to be invariant under adjoint-rotations (16).

For a given algebra, it is of utmost importance to obtain the Casimir operator(s) because the eigen values of such operator(s) designate the representation of the algebra. The qp-deformed quadratic Casimir operator for the algebra (18) turns out to be related to the invariant:

$$c_2 = Tr_{qp}(E^2)\,. \tag{19}$$

In fact, it can be expressed in terms of the generators of (18) as given below:

$$c = c_2 - \frac{H_{qp}^2}{r + r^{-1}} = (r^{-1}E_+E_- + rE_-E_+) + \frac{E_0^2}{r + r^{-1}}\,. \tag{20}$$

We can rewrite the commutation relations (18) in the concise form as follows:

$$\tilde{E}_{ij}\tilde{E}_{jk} = (r + r^{-1})c\delta_{ik} - \kappa\tilde{E}_{ik}, \tag{21}$$

where $\kappa = \frac{r^2-1}{r^2}H_{qp} + \frac{r^2+1}{r^3}$, c is the Casimir operator (20) and matrix $\tilde{E}$ is defined in (15). Now the invariance of algebra (18) under adjoint rotations (16) becomes transparant in veiw of the transformation law $\tilde{E} \to T\tilde{E}T^{-1}$.

It is worth noting that the algebra (18) depend only on a single parameter $r = \sqrt{pq}$ and, therefore, is consistent with the Drinfeld's uniqueness theorem(see, for instance [11]). The algebras (17) and (18) in terms of generators $E_\pm$, E_0 and H_{pq} give us the q-deformation of $gl(2)$. This covariant algebra is the central extension of the Witten-type deformation of the $sl(2)$ algebra if we consider H_{qp} as the central element. We see, therefore, that the "adjoint-representation" of the group $GL_{qp}(2)$ leads to the generalization of the algebra obtained in ref.[16] (see also [14]).

Now we shall discuss the deformed trace and orbits for the supersymmetric case of $GL_{qp}(1 \mid 1)$. It is known that the 2×2 quantum matrix $T^s_{ij} = \begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix}$ describes $GL_{qp}(1 \mid 1)$ group, if noncommuting two odd elements β and γ and two even elements a and d satisfy different braiding relations in rows and columns as given below [13]:

$$\begin{aligned} &a\beta = p\beta a,\ d\beta = p\beta d, \quad a\gamma = q\gamma a,\ d\gamma = q\gamma d, \beta\gamma = -(q/p)\gamma\beta, \\ &[a, d] = (q - p^{-1})\gamma\beta = \ (q^{-1} - p)\beta\gamma,\ \beta^2 = \gamma^2 = 0, \\ &\qquad p,\ q \in \mathcal{C}/\{0\}. \end{aligned} \tag{22}$$

These relations reduce to the one-parameter case of $GL_q(1 \mid 1)$ in the limit $q = p$. The antipode $(S_{ij} = (T^s)^{-1}_{ij})$ and the quantum superdeterminant $\mathcal{D}^s$ (q-berezinian) are obtained by applying Borel-Gauss decomposition on the matrix T^s. These are given as follows [13,18]:

$$(T^s)^{-1}_{ij} = \begin{pmatrix} a^{-1} + a^{-1}\beta d^{-1}\gamma a^{-1}, & -a^{-1}\beta d^{-1} \\ -d^{-1}\gamma a^{-1}, & d^{-1} - d^{-1}\beta a^{-1}\gamma d^{-1} \end{pmatrix},$$

$$\mathcal{D}^s = ad^{-1} - \beta d^{-1}\gamma d^{-1} = d^{-1}a - d^{-1}\beta d^{-1}\gamma. \tag{23}$$

Here $\mathcal{D}^s$ is the center for the algebra (22). The quantum orbits for the supergroup $GL_{qp}(1 \mid 1)$ can be defined through the transformations (4) for a 2×2 super q-matrix E_{ij} whose elements (anti)commute with that of T^s_{ij}. Even though elements of T^s_{ij} follow graded commutation relations (22), the following supertrace:

$$Str_{qp}(E) = E_{11} - E_{22} = Str_{qp}(T^s E (T^s)^{-1}) \tag{24}$$

remains invariant under transformations (4). It is interesting to note that equation (24) coincides with supertraces for undeformed and one-parameter deformed supergroup $GL(1 \mid 1)$. It can be shown that all invariants of the quantum "super-orbit" $(E \to T^s E (T^s)^{-1})$ can be expressed as:

$$c_n = Str_{qp}(E^n) = Str(E^n). \qquad (24a)$$

We now introduce the set of bosonic $(A, \tilde{A})$ and fermionic $(B, \tilde{B})$ variables to study the system of qp-superoscillators covariant under coaction of the quantum supergroup $GL_{qp}(1 \mid 1)$. It is straightforward to demonstrate that following relations:

$$\begin{aligned} AB - qBA &= 0, \\ \tilde{A}\tilde{B} - p^{-1}\tilde{B}\tilde{A} &= 0, \\ B^2 = \tilde{B}^2 &= 0, \end{aligned} \qquad (25)$$

remain invariant under $GL_{qp}(1 \mid 1)$ transformations:

$$\begin{pmatrix} A \\ B \end{pmatrix} \longrightarrow \begin{pmatrix} a, & \beta \\ \gamma, & d \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = T^s \begin{pmatrix} A \\ B \end{pmatrix}, \qquad (26a)$$

$$(\tilde{A},\, \tilde{B}) \longrightarrow (\tilde{A},\, \tilde{B}) \begin{pmatrix} a^{-1} + a^{-1}\beta d^{-1}\gamma a^{-1}, & -a^{-1}\beta d^{-1} \\ -d^{-1}\gamma a^{-1}, & d^{-1} - d^{-1}\beta a^{-1}\gamma d^{-1} \end{pmatrix} = (\tilde{A},\, \tilde{B})(T^s)^{-1}. \qquad (26b)$$

Consistent with the oscillator algebra (25), the other general qp-superoscillator relations are as follows:

$$\begin{aligned} A\tilde{A} - (\lambda + \tfrac{\lambda+\nu}{pq})\tilde{A}A &= 1 - (\nu - \tfrac{\nu+\lambda}{pq})\tilde{B}B, \\ A\tilde{B} &= \tfrac{(\lambda+\nu)}{q}\tilde{B}A, \\ B\tilde{A} &= \tfrac{(\lambda+\nu)}{p}\tilde{A}B, \end{aligned} \qquad (27)$$

if we postulate the validity of the following deformed anticommutator:

$$\{B, \tilde{B}\}_{(1,\nu)} \equiv B\tilde{B} + \nu\tilde{B}B = 1 + \lambda\tilde{A}A\,, \qquad (28)$$

where ν and λ are c-number parameters which can be determined by requiring associativity of the algebra (25), (27) and (28). Indeed, we can reorder the waved oscillators to the left of the product $A\tilde{A}\tilde{B}$ in two different ways which lead to two different results on the right hand side. To obtain the unique result on the right hand side, we have to take:

$$i.)\ \nu = 1,\ \lambda = r^2 - 1, \qquad (29a)$$

$$ii.)\ \nu = 1,\ \lambda = 0. \qquad (29b)$$

The two parametric quantum superoscillator algebras corresponding to the solution (29a) and consistent with equations (25) are as follows:

$$\begin{aligned} A\tilde{B} - p\tilde{B}A &= 0, \\ B\tilde{A} - q\tilde{A}B &= 0, \\ A\tilde{A} - pq\tilde{A}A &= 1, \\ B\tilde{B} + \tilde{B}B &= 1 + (pq-1)\tilde{A}A. \end{aligned} \qquad (30a)$$

Similarly, the solution (29b) yields following superoscillator relations consistent with equations (25):

$$\begin{aligned} A\tilde{B} - q^{-1}\tilde{B}A &= 0, \\ B\tilde{A} - p^{-1}\tilde{A}B &= 0, \\ A\tilde{A} - \tfrac{1}{pq}\tilde{A}A &= 1 + (\tfrac{1}{pq} - 1)\tilde{B}B, \\ B\tilde{B} + \tilde{B}B &= 1\,. \end{aligned} \qquad (30b)$$

The case $q = p$, $\tilde{A} = A^\dagger$, $\tilde{B} = B^\dagger$ for algebras (30a) and (30b) gives us the known algebras of covariant pair of q-oscillators of ref.[15]. We stress here again that the procedure of obtaining conditions (29) is equivalent to the derivation and solution of the graded Yang-Baxter equations.

We can now see that following super q-matrix:

$$E_{ij} = \begin{pmatrix} A\tilde{A}, & A\tilde{B} \\ B\tilde{A}, & B\tilde{B} \end{pmatrix} \qquad (31)$$

satisfies the transformation laws (4) if covariant superoscillators obey the $GL_{qp}(1/1)$ transformation laws (26). The invariance of qp-supertrace (24) leads to the emergence of following bilinear representation of the invariant Hamiltonian in terms of super qp-oscillators:

$$H^s_{qp} = A\tilde{A} - B\tilde{B} = \frac{\nu+\lambda}{qp}(\tilde{A}A + \tilde{B}B). \qquad (32)$$

The right hand side of equation (32) is trivially supercovariant in view of the transformations (26).

It is now obvious that the super-transformations (4) lead to the following four dimensional "adjoint representation" of $GL_{qp}(1 \mid 1)$:

$$\begin{pmatrix} Y' \\ E'_{12} \\ E'_{21} \\ H^{s\,\prime}_{pq} \end{pmatrix} = \begin{pmatrix} 1, & \gamma d^{-1}, & \beta a^{-1}, & \beta d^{-1}\gamma a^{-1} \\ 0, & \mathcal{D}^s, & 0, & -\beta d^{-1} \\ 0, & 0, & (\mathcal{D}^s)^{-1}, & \gamma a^{-1} \\ 0, & 0, & 0, & 1 \end{pmatrix} \begin{pmatrix} Y \\ E_{12} \\ E_{21} \\ H^s_{pq} \end{pmatrix}. \qquad (33)$$

Here $Y = \frac{E_{11}+\mu E_{22}}{1+\mu}$, where $\mu \neq -1$ is a c-number. The representation (33) is reducible because we have one dimensional invariant subspace with co-ordinate

$H^s_{qp} = E_{11} - E_{22}$. To obtain the covariant algebra for coordinates of this representation the key ingredient is to represent these coordinates in terms of covariant oscillators using the defining equation (31). Considering the following operators:

$$Y = \frac{A\tilde{A} + \mu B\tilde{B}}{1+\mu}, \quad H^s_{pq} = A\tilde{A} - B\tilde{B}, \quad Q = E_{12} = A\tilde{B}, \quad \bar{Q} = E_{21} = B\tilde{A}, \qquad (34)$$

the covariant superalgebra for the case of (30a) is written as:

$$\begin{gathered} [H^s_{pq}, Q] = [H^s_{pq}, \bar{Q}] = [H^s_{pq}, Y] = Q^2 = \bar{Q}^2 = 0, \\ \{Q, \bar{Q}\} = (1 + (r^2 - 1)H^s_{pq})H^s_{pq} = \mathcal{H}, \\ [Q, Y] = \{1 + (r^2-1)H^s_{pq}\}Q, \quad [\bar{Q}, Y] = -\{1 + (r^2-1)H^s_{pq}\}\bar{Q}, \end{gathered} \qquad (35a)$$

while for the case (30b) we have the following covariant superalgebra

$$\begin{aligned} \{Q, \bar{Q}\} &= H^s_{pq}, \quad [Q, Y] = Q, \quad [\bar{Q}, Y] = -\bar{Q}, \\ [H^s_{pq}, Q] &= [H^s_{pq}, \bar{Q}] = [H^s_{pq}, Y] = Q^2 = \bar{Q}^2 = 0, \end{aligned} \qquad (35b)$$

The Casimir operator c^s for the covariant algebras (35a,b) is related to the invariant c_2 (24a) and can be expressed as

$$c^s = c_2 - \frac{\mu - 1}{\mu + 1}(H^s_{pq})^2 = Q\bar{Q} - \bar{Q}Q + 2YH^s_{pq}.$$

It is interesting to note that in the case of super quantum group $GL_{qp}(1 \mid 1)$, we obtain $N = 2$ supersymmetric quantum mechanical algebras (with supercharges Q, $\bar{Q}$, and supersymmetric hamiltonian $\mathcal{H}$ (35a) for the case (29a) and H^s_{pq} (32) for the case (29b)) as subalgebras of the covariant algebras (35a,b). The generator Y gives the extensions of these quantum mechanical superalgebras. The q-superalgebra (35b) (and superalgebra (35a) after rescaling of the generators Q, $\bar{Q}$, Y) is isomorphic to the undeformed Lie superalgebra $gl(1 \mid 1)$. This is in agreement with one-parametric case [18].

The emergence of the algebras with generators $\{ Q, \bar{Q}, \mathcal{H}\ (H^s_{pq}) \}$ from the $GL_{qp}(1 \mid 1)$ covariant superalgebras (35a,b) tells us about "hidden q-symmetry" in supersymmetric quantum mechanical systems with Hamiltonians constructed above. It may be a strong physical motivation for the study of quantum supergroups.

To conclude, it is worthwhile to note that the (super)traces (5) (for $p = q$) and (24) can be obtained as special cases from a general supertrace defined for the one parameter deformed group $GL_q(N \mid M)$. To obtain such a general supertrace the essential ingredients are the relations between deformed (super)traces and invariant Hamiltonians that are deduced in equations (14) and (32). Moreover, it is also essential to take into account the results of ref.[15] where one parameter q-oscillator algebras and corresponding invariant Hamiltonians in terms of the bilinears have been obtained explicitly. Such a general supertrace for group $GL_q(N \mid M)$ is as follows:

$$Str_q(E) = Str_q(TET^{-1}) =$$

$$= (q^{(M-N)/2}) \sum_{i=1}^{N} (q^{-(N+1)+2i} E_{ii}) - (q^{(N-M)/2}) \sum_{s=N+1}^{N+M} (q^{(M+1)-2(s-N)} E_{ss}), \qquad (36)$$

where E_{ij} is $(N+M) \times (N+M)$ q-supermatrix and the elements of the quantum matrix T_{ij} $(i, j = 1, 2, \ldots, N+M)$ generate the quantum group $GL_q(N \mid M)$. The matrix T_{ij} acts on the co-ordinates $(A_1, A_2, \ldots, A_N, B_1, B_2, \ldots, B_M)$ defined on the quantum hyperplane. The basic relations for these co-ordinates are as follows:

$$A_i A_j = q A_j A_i\ (i < j),\ A_i B_s = q B_s A_i,\ B_r B_s = -q B_s B_r\ (r < s). \qquad (37)$$

We would like to emphasize at this juncture that the substitutions $N = 2, M = 0$ and $N = 1, M = 1$ lead to the emergence of the expressions for (super)traces obtained in (5) (for $p = q$) and (24). Furthermore, $M = 0$ in equation (36) yields the expression for q-trace presented in ref.[14] for the quantum group $GL_q(N)$. The special case of this q-trace ($N = 3$, $M = 0$) was used in [19] in the context [3,20] of the construction of the central elements for the quantum algebras.

It is possible to extend covariant (super)algebras (18) and (35) which corespond to "adjoint representation" (16), (33) and construct the higher dimensional "representations" for the quantum groups $GL_{qp}(2)$, $GL_{qp}(1 \mid 1)$. For this purpose, one has to consider the following tensors which are elements of the enveloping algebra of qp-oscillators:

$$E^{i_1,\ldots,i_n;r_1,\ldots,r_m}_{j_1,\ldots,j_{n'};s_1,\ldots,s_{m'}} = \tilde{A}_{i_1} \ldots \tilde{A}_{i_n} \bar{B}_{r_1} \ldots \bar{B}_{r_m} A_{j_1} \ldots A_{i_{n'}} B_{s_1} \ldots B_{s_{m'}}. \qquad (38)$$

Transformations presented in (7) and (26), define the action of $GL_q(1 \mid 1)$ and $GL_q(2)$ on such tensors and give rise to the higher dimensional "representations" of these quantum groups. The interesting nontrivial problem here would be to separate (38) into irreducible representations and investigate corresponding covariant infinite-dimensional algebras. We hope that the notion of the quantum trace will help in the resolution of this problem.

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