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TOROIDAL SOLENOIDS IN THE ELECTROMAGNETIC FIELD AND AHARONOV-CASHER EFFECT

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§ 1. Introduction

In ref./1/ we have analyzed the electromagnetic properties of the toroidal solenoid (TS). The present paper deals with physical applications of the formalism presented there. We are planning our exposition as follows. In § 2 the information on the TS which will be used in the subsequent sections is presented. In § 3 we consider a number of current distributions (or magnetizations) which generate quite different (gauge non-equivalent) vector potentials and lead nonetheless to the same quantum mechanical scattering of charged particles. In § 4 the motion of TS in the external magnetic field is analyzed and the physical meaning of the VP obtained in the previous section is clarified. In § 5 we study the interaction of two TS. In § 6 we prove the existence of the Aharonov – Casher /2/ effect for the TS and propose experiments testing it.

§ 2. Preliminary considerations on the toroidal solenoid

Consider the torus T

$$(\rho - d)^2 + z^2 = R^2. \tag{2.1}$$

Let the constant poloidal current (fig.1) flow over its surface. To write out the current density explicitly, we introduce the coordinates $\tilde{R}, \psi$

$$\rho = d + \tilde{R}\cos\psi, \quad z = \tilde{R}\sin\psi.$$

The value $\tilde{R} = R$ corresponds to the torus T. The infinitesimal volume and surface (of the torus T) elements have the form

$$d\tilde{V}=\tilde{R}\cdot(d+\tilde{R}\cos\Psi)\,d\tilde{R}\cdot d\Psi\cdot d\varphi,\quad dS=R\cdot(d+R\cos\Psi)\,d\Psi\,d\varphi. \qquad (2.2)$$

The density of the aforementioned poloidal current being expressed in $\tilde{R},\Psi$ coordinates is

$$\vec{j}=-\frac{gc}{4\pi}\,\frac{\delta(\tilde{R}-R)}{d+R\cos\Psi}\,\vec{n}_\Psi. \qquad (2.3)$$

Here $g=2NI/c$, N is the total number of turns in the toroidal coil, $\vec{n}_\Psi$ is the unit vector defining the current direction on the torus surface

$$\vec{n}_\Psi=\vec{n}_z\cdot\cos\Psi-(\vec{n}_x\cdot\cos\varphi+\vec{n}_y\cdot\sin\varphi)\cdot\sin\Psi. \qquad (2.4)$$

The constant g may be also expressed through the magnetic flux Φ and geometrical dimensions of the torus (2.1): $g=\Phi\cdot[2\pi\cdot(d-\sqrt{d^2-R^2})]^{-1}$. In what follows we shall extensively use toroidal coordinates μ,θ,φ. They are introduced as follows

$$x=\frac{a\,\mathrm{sh}\mu\cos\varphi}{\mathrm{ch}\mu-\cos\theta},\quad y=\frac{a\,\mathrm{sh}\mu\sin\varphi}{\mathrm{ch}\mu-\cos\theta},\quad z=\frac{a\sin\theta}{\mathrm{ch}\mu-\cos\theta}$$

$$(0<\mu<\infty,\ -\pi<\theta<\pi,\ 0<\varphi<2\pi). \qquad (2.5)$$

Let $\mu=\mu_0$ correspond to the torus T (fig.2). Then for $\mu>\mu_0$ ($\mu<\mu_0$) the point $P(x,y,z)$ (where x,y,z are defined by Eqs. (2.5)) lies inside (outside) T. For μ fixed (say, $\mu=\mu_0$) the points $P(x,y,z)$ fill the surface of the torus T with parameters $d=a\cdot\mathrm{cth}\,\mu_0$, $R=a/\mathrm{sh}\,\mu_0$. The value of the angle θ jumps from $-\pi$ to π when one intersects the circle of the radius $d-R$ lying in $z=0$ plane. The volume

element (2.2), current density (2.3) and unit vector (2.4) being expressed in toroidal coordinates are:

$$dV=\frac{a^3\,\mathrm{sh}\mu\,d\theta\,d\mu\,d\varphi}{(\mathrm{ch}\mu-\cos\theta)^3},\quad dS=\frac{a^2\,\mathrm{sh}\mu_0\,d\theta\,d\varphi}{(\mathrm{ch}\mu_0-\cos\theta)^2},$$

$$\vec{j}=-\frac{gc}{4\pi a^2}\,\frac{\delta(\mu-\mu_0)}{\mathrm{sh}\mu_0}(\mathrm{ch}\mu_0-\cos\theta)^2\,\vec{n}_\theta, \qquad (2.6)$$

$$\vec{n}_\theta=[(\vec{n}_x\cos\varphi+\vec{n}_y\cdot\sin\varphi)\,\mathrm{sh}\mu_0\sin\theta+\vec{n}_z(1-\mathrm{ch}\mu_0\cos\theta)]\cdot\frac{1}{\mathrm{ch}\mu_0-\cos\theta}.$$

In the stationary case the magnetic field (MF) $\vec{H}$ equals zero outside the toroidal solenoid (TS). Inside it only φ component of $\vec{H}$ differs from zero: $H_\varphi=\frac{g}{\rho}$. Here ρ is the distance from the solenoid's symmetry axis ($\rho=d+\tilde{R}\cos\Psi=a\cdot\mathrm{sh}\mu/(\mathrm{ch}\mu-\cos\theta)$). The vector potential (VP) of the TS was obtained in ref. /3/. Its properties were discussed in /4/. In the integral form the nonvanishing cylindrical components of VP are

$$A_z=\frac{g\sqrt{R}}{2\pi}\int_0^{2\pi}d\varphi\,\frac{d-\rho\cos\varphi}{q^{3/2}}\,Q_{\frac12}(\mathrm{ch}\nu),$$

$$A_\rho=\frac{g\sqrt{R}}{2\pi}\,z\int_0^{2\pi}d\varphi\,\frac{\cos\varphi}{q^{3/2}}\,Q_{\frac12}(\mathrm{ch}\nu) \qquad (2.7)$$

$(\mathrm{ch}\nu=(r^2+d^2+R^2-2d\rho\cos\varphi)/2Rq$, $q=[(\rho\cos\varphi-d)^2+z^2]^{1/2}$, $r^2=\rho^2+z^2$, $Q_\nu(x)$ is the Legendre function of the 2nd kind). For the infinitely thin TS ($R\ll d$) these integrals can be taken in a closed form

$$A_z = \frac{R^2}{2\cdot(d\rho)^{3/2}}\frac{1}{\operatorname{sh}\mu_1}\cdot\left[\rho\cdot Q^1_{\frac{1}{2}}(\operatorname{ch}\mu_1) - d\cdot Q^1_{-\frac{1}{2}}(\operatorname{ch}\mu_1)\right],$$

$$A_\rho = -\frac{R^2 z}{2(d\rho)^{3/2}}\frac{1}{\operatorname{sh}\mu_1}\cdot Q^1_{\frac{1}{2}}(\operatorname{ch}\mu_1),\quad \operatorname{ch}\mu_1 = \frac{r^2+d^2}{2d\rho}$$

At large distances VP falls as r^{-3}

$$A_z \sim \frac{\pi g d R^2}{8}\,\frac{1+3\cos 2\theta_s}{r^3},\quad A_\rho \sim \frac{3\pi g d R^2}{8}\,\frac{\sin 2\theta_s}{r^3} \tag{2.8}$$

(r, θ_s are the usual spherical coordinates).

Instead of currents (2.3) or (2.6) one may introduce magnetization: $c\cdot \operatorname{rot}\vec{M} = \vec{j}$. It is confined entirely inside the TS and is given by

$$\vec{M} = M\cdot\vec{n}_\varphi,\quad M = \frac{g}{4\pi}\,\frac{\theta(R-\tilde{R})}{d+\tilde{R}\cos\psi} = \frac{g}{4\pi a}\,\frac{\operatorname{ch}\mu-\cos\theta}{\operatorname{sh}\mu}\,\theta(\mu-\mu_0) \tag{2.9}$$

Here θ is the step function ($\theta(x)=0$ for $x<0$ and 1 for $x>0$). As the current and magnetization formalisms are entirely equivalent /5,6/, one may forget about the solenoid's current and treat TS as a magnetized ring with magnetization given by Eq. (2.9). Its technical realization (ferromagnetic ring with magnetization independent of applied fields) was used /7/ for the experimental verification of the Aharonov – Bohm effect.

§3. More general current distributions

We have mentioned that for the current densities (2.3) or (2.6) the MF $\vec{H}$ equals zero outside TS. Whether this choice

of current density is unique? It turns out that more general current distribution having the same property is

$$\vec{j} = -\frac{1}{\operatorname{sh}\mu}\cdot f(\mu)\,\theta(\mu-\mu_1)\cdot(\operatorname{ch}\mu-\cos\theta)^2\,\vec{n}_\theta. \tag{3.1}$$

Here $f(\mu)$ is an arbitrary continuous function. The occurrence of the step function in (3.1) means that the currents are contained inside the torus of the radius $a/\operatorname{sh}\mu_1$. The current distribution (3.1) may be treated as the continuous superposition of the δ-type distributions (2.6). The VP corresponding to the current (3.1) is given in Appendix 1. At large distances it falls like r^{-3}

$$A_\rho \sim \frac{3}{2}\,\frac{\pi^2 a^5}{r^3 c}\sin 2\theta_s\cdot d(\mu_1),\quad A_z \sim \frac{\pi^2 a^5}{2r^3 c}\cdot(1+\cos 2\theta_s)\cdot d(\mu_1)$$

$$d(\mu_1) = \int_{\mu_1}^{\infty} d\mu\cdot\frac{\operatorname{ch}\mu}{\operatorname{sh}^3\mu}\cdot f(\mu). \tag{3.2}$$

Inside TS only φ components of magnetic field $\vec{H}$ and magnetization $\vec{M}$ are different from zero

$$H_\varphi = \frac{4\pi a}{c}\,\frac{\operatorname{ch}\mu-\cos\theta}{\operatorname{sh}\mu}\int_{\mu_1}^{\mu} f(\mu)\,d\mu,\quad M_\varphi = H_\varphi/4\pi. \tag{3.3}$$

The magnetic flux through TS cross-section is

$$\Phi = \iint H_\varphi\,d\rho\,dz = \frac{8\pi^2 a^3}{c}\int_{\mu_1}^{\infty}(\operatorname{ch}\mu-1)\,f(\mu)\,d\mu.$$

For the sake of completeness we present here the current distribution, magnetic field and its flux in $\tilde{R},\psi$ coordinates:

$$\vec{j} = -\frac{f_1(\tilde{R})\cdot\Theta(R-\tilde{R})}{d+\tilde{R}\cos\Psi}\vec{n}_\Psi, \quad H_\varphi = \frac{4\pi}{c}\frac{\Theta(R-\tilde{R})}{d+\tilde{R}\cos\Psi}\int_{\tilde{R}}^{R} f_1(x)\,dx, \tag{3.4}$$

$$M_\varphi = \frac{H_\varphi}{4\pi}, \quad \Phi = \frac{8\pi^2}{c}\int_0^R (d-\sqrt{d^2-x^2})\cdot f_1(x)\,dx.$$

Here f_1 is an arbitrary function defining the current distribution inside TS. Up to now we have dealt with VP which fall as r^{-3} at large distances (see Eqs. (2.8) and (3.2)). The freedom in the choice of the function f (or f_1) gives a hint that asymptotic behaviour of VP may be changed. To prove this we need the terms of higher orders in the asymptotic expansion of VP. For the current density (2.3) this expansion was obtained in /1/. It has the form

$$A_z = -\frac{1}{2}gR\sum_{\ell=2}^{\infty}\frac{1}{r^{\ell+1}}\cdot P_\ell(\cos\Theta_s)\cdot f_\ell^0,$$
$$A_\rho = \frac{1}{2}gR\sum_{\ell=2}^{\infty}\frac{1}{r^{\ell+1}}\frac{1}{\ell(\ell+1)}P_\ell^1(\cos\Theta_s)\cdot f_\ell^1. \tag{3.5}$$

Here $f_\ell^0 = \int d\Psi\cos\Psi\,\rho^\ell\cdot P_\ell(R\sin\Psi/\rho)$, $f_\ell^1 = \int d\Psi\cdot\sin\Psi\,\rho^\ell P_\ell^1(R\sin\Psi/\rho)$, $\rho = (d^2+R^2+2dR\cos\Psi)^{1/2}$. The summing in (3.5) is performed over the even values of ℓ. The deviation of these Eqs. (and all others containing Legendre polynomials) from those presented in /1/ is due to the fact that normalized to unity $\left(\int_{-1}^{1}[P_\ell^m(x)]^2dx = 1\right)$ Legendre functions were used there. Here we adopt the usual normalization $\left(\int_{-1}^{1}[P_\ell^m(x)]^2dx = \frac{2}{2\ell+1}\frac{(\ell+m)!}{(\ell-m)!}\right)$.

The first two terms in the expansion (3.5) are

$$A_z \approx \frac{\pi g dR^2}{2r^3}\left[P_2(\cos\Theta_s) - \frac{3}{2}\frac{d^2-R^2/4}{r^2}P_4(\cos\Theta_s)\right],$$
$$A_\rho \approx \frac{\pi g dR^2}{4r^3}\left[P_2^1(\cos\Theta_s) - \frac{3}{4}\frac{d^2-R^2/4}{r^2}P_4^1(\cos\Theta_s)\right]. \tag{3.6}$$

Consider two concentric solenoids TS_1 and TS_2 (see fig.3b) with parameters d_1, R_1, Φ_1 and d_2, R_2, Φ_2. The total VP equals

$$A_z^{(1,2)} = A_z^{(1)} + A_z^{(2)}, \quad A_\rho^{(1,2)} = A_\rho^{(1)} + A_\rho^{(2)}. \tag{3.7}$$

Now we adjust the solenoid's parameters in such a way as to cancel the leading terms ($\sim r^{-3}$) in the asymptotics of VP. This happens if

$$g_1 d_1 R_1^2 + g_2 d_2 R_2^2 = 0. \tag{3.8}$$

Substituting here the explicit values of g, d, R for each of the solenoids ($d_{1,2} = a\cdot\mathrm{cth}\,\mu_{1,2}$, $R_{1,2} = a/\mathrm{sh}\,\mu_{1,2}$, $g_{12} = \Phi_{1,2}\cdot[a\cdot(\mathrm{cth}\,\mu_{1,2}-1)]^{-1}$) and putting $y_{1,2} = \mathrm{cth}\,\mu_{1,2}$ we obtain the following equation

$$\Phi_1 y_1(y_1+1) + \Phi_2 y_2(y_2+1) = 0.$$

This Eq. may be resolved wrt y_2

$$y_2 = -\frac{1}{4} + \left[\frac{1}{4} - \frac{\Phi_1}{\Phi_2}y_1(y_1+1)\right]^{1/2}. \tag{3.9}$$

The total VP (3.7) now falls as r^{-5}

$$A_z^{(1,2)} \approx -\frac{9}{32}\Phi_1 a^2 y_1(y_1+1)(y_1^2-y_2^2)\frac{1}{r^5}P_4(\cos\theta_s),$$

$$A_\rho^{(1,2)} \approx -\frac{9}{128}\Phi_1 a^2 y_1(y_1+1)\cdot(y_1^2-y_2^2)\frac{1}{r^5}P_4^1(\cos\theta_s). \tag{3.10}$$

Now we surround TS_1 and TS_2 by the impenetrable (for the incoming charged particles) torus T. Let the total magnetic flux of TS_1 and TS_2 be equal Φ (that is $\Phi = \Phi_1 + \Phi_2$). The quantum scattering cross-sections (CS) of charged particles depends only on the geometrical parameters of the impenetrable torus T and the magnetic flux inside T. This means that situations presented in figs. 3a (one TS with magnetic flux Φ inside T) and 3b (two TS with magnetic flux Φ inside T) are physically indistinguishable inspite of the quite different asymptotic behaviour of VP ($\sim r^{-3}$ for fig. 3a and $\sim r^{-5}$ for fig.3b). The transition between VP corresponding to figs. 3a and 3b cannot be performed by means of gauge transformation (there are different $\vec{H}$ inside T for figs. 3a and 3b). Our game may be continued further. Take (in addition to TS_1 and TS_2) solenoids TS_3 and TS_4 with parameters satisfying relation $g_3 d_3 R_3^2 + g_4 d_4 R_4^2 = 0$. The complete VP of TS_3 and TS_4 is given by:

$$A_z^{(3,4)} = -\frac{9}{32}\Phi_3 a^2 y_3(y_3+1)(y_3^2-y_4^2)\frac{1}{r^5}P_4(\cos\theta_s),$$

$$A_\rho^{(3,4)} = -\frac{9}{128}\Phi_3 a^2 y_3(y_3+1)(y_3^2-y_4^2)\frac{1}{r^5}P_4^1(\cos\theta_s). \tag{3.11}$$

Here $y_3 = \operatorname{cth}\mu_3$ and $y_4 = \operatorname{cth}\mu_4$. The complete VP generated by solenoids $TS_1 - TS_4$ is

$$A_z = A_z^{(1,2)} + A_z^{(3,4)}, \quad A_\rho = A_\rho^{(1,2)} + A_\rho^{(3,4)}.$$

We require now the equality of the total magnetic flux to Φ ($\Phi_1+\Phi_2+\Phi_3+\Phi_4 = \Phi$) and the cancellation of coefficients at r^{-5}. The latter happens if

$$\Phi_1 y_1(y_1+1)(y_1^2-y_2^2) + \Phi_3 y_3(y_3+1)(y_3^2-y_4^2) = 0. \tag{3.12}$$

Thus obtained configuration of four TS (fig.3c) generates VP falling like r^{-7} at large distances. Being imbedded into the same impenetrable torus T it undergoes the same quantum mechanical scattering as TS configurations shown in fig.3a,b. The coincidence of quantum mechanical CS reflects merely the non-singlevaluedness of either quantum inverse problem (the same CS for different current distributions inside T) or classical electrodynamic problem (the same $\vec{E}, \vec{H}$ are generated by the different currents). For each of the current configurations shown in figs. 3a-3c the magnetic flux equals Φ. This means that integral $\oint A_\ell d\ell$ taken along any closed contour passing through the hole of T also equals Φ. Particularly this is valid for the integral $\int A_z dz$ taken along the z axis. As VP A_z falls faster and faster as one pushes from top to bottom in fig.3, so VP is concentrated more and more at the neighbourhood of T.

§ 4. The motion of toroidal moment in the external magnetic field

The interaction energy of TS with the external MF equals/1/:

$$\mathcal{U} = -\int \vec{H}_{ext} \cdot \vec{M} \, dV. \tag{4.1}$$

Here $\vec{M}$ is the magnetization of TS. As only φ component of $\vec{M}$ differs from zero (see Eq. (2.9)) so TS interacts with external MF if the latter has nonzero projection on the equatorial plane of TS and nonvanishing overlapping with M_φ. As an example consider the interaction of TS with the linear conductor which carries current I (fig.4). It turns out that the interaction energy equals $4\pi g I/c$ if the conductor passes through TS' hole and zero otherwise. If the source of MF is sufficiently separated from TS then MF may be developed (near TS) into the series

$$\vec{H}_{ext}(\vec{r}_s) = \vec{H}_{ext}(\vec{r}_0) + (\vec{r}\,\vec{\nabla}_0)\vec{H}_{ext}(\vec{r}_0). \tag{4.2}$$

Here $\vec{r}_0$ defines some point at the neighbourhood of TS (e.g., its centre-of-mass (CM)), $\vec{r}$ is the vector going from $\vec{r}_0$ to the particular point inside TS. Inserting this expansion into (4.1) we get

$$\mathcal{U} = -\vec{\mu}_d \cdot \vec{H}(\vec{r}_0) - \frac{1}{2}\vec{\mu}_t \cdot \mathrm{rot}\,\vec{H}(\vec{r}_0). \tag{4.3}$$

Here $\vec{\mu}_d = \int \vec{M}\, dV$ is the magnetic dipole moment. It equals zero for TS. The quantity

$$\vec{\mu}_t = \int (\vec{r} \times \vec{M})\, dV \tag{4.4}$$

up to a constant coincides with the so-called toroidal dipole moment (TDM) /1,8/. For the magnetization given by Eq. (2.9) only z component of $\vec{\mu}_t$ differs from zero:

$$\mu_t = \frac{1}{2}\pi g d R^2. \tag{4.5}$$

For the arbitrary orientation (θ, φ) of the solenoid symmetry axis $\vec{\mu}_t = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)\frac{1}{2}\pi g d R^2$. To study the motion of TS in the magnetic field one should write out Lagrangian L. One has $L = T - \mathcal{U}$. The kinetic energy is the sum of CM energy $T_{CM} = \frac{1}{2}M(\dot{x}_0^2 + \dot{y}_0^2 + \dot{z}_0^2)$ and the rotation energy $T_{rot} = \frac{1}{2}(A\omega_x^2 + B\omega_y^2 + C\omega_z^2)$. Here M is the mass of TS; x_0, y_0, z_0 are CM coordinates of TS; $\omega_x, \omega_y, \omega_z$ are the angular velocity projections on the body fixed axes $\omega_x = \dot{\theta}\sin\psi - \dot{\varphi}\sin\theta\cos\psi$, $\omega_y = \dot{\theta}\cos\psi + \dot{\varphi}\sin\theta\sin\psi$, $\omega_z = \dot{\varphi}\cos\psi + \dot{\psi}$; angles φ, θ, ψ determine the orientation of the body fixed coordinate system wrt laboratory one; dots over these angles mean time derivatives. A, B, C are the moments of inertia for TS ($A = B = \pi^2 d^3 R^2 M(1 + 5R^2/4d^2)$, $C = 2\pi^2 d^3 R^2 M(1 + 3R^2/4d^2)$. Using the Maxwell Equation $\mathrm{rot}\,\vec{H}_{ext} = \frac{1}{c}\dot{\vec{E}}_{ext} + \frac{4\pi}{c}\vec{j}_{ext}$ and bearing in mind that expansion (4.2) holds at large distances from the MF source where $\vec{j}_{ext} = 0$ one gets

$$\mathcal{U} = -\frac{1}{2}\dot{\vec{E}} \cdot \vec{\mu}_t \quad \left(\dot{\vec{E}} = \frac{\partial \vec{E}}{\partial t}\right).$$

If $\dot{\vec{E}}_{ext}$ (or $\mathrm{rot}\,H_{ext}$) is directed along z axis then this Eq. is simplified

$$\mathcal{U} = -\frac{1}{2c}\dot{E}_z(\vec{r}_0)\mu_t \cos\theta. \tag{4.6}$$

Here θ is the angle between $\vec{\mu}_t$ and z axis. It follows from (4.6) that potential energy of TDM in the external MF depends both on TDM position and its orientation. The Lagrange Eqs. ($\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$, $q_i = x_0, y_0, z_0, \varphi, \theta, \psi$) lead to the coupled system of Eqs. which uniquely determine the motion of TDM.

We have seen that TDM is an important characteristic of the infinitely thin TS. In fact the position of CM and the direction of TDM completely determine the dynamics of small TS. We turn now to the situation shown in fig.3b. The total magnetization is given by

$$M_{12} = \frac{1}{4\pi a} \frac{ch\mu - \cos\theta}{sh\mu} \left[g_1 \cdot \theta(\mu - \mu_1) + g_2 \theta(\mu - \mu_2) \right]. \tag{4.7}$$

The corresponding TDM equals

$$\mu_t^{(1,2)} = \frac{1}{2} \pi (g_1 d_1 R_1^2 + g_2 d_2 R_2^2). \tag{4.8}$$

This quantity vanishes for the configuration presented in fig.3b (see Eq. (3.8)). This means that next term in the expansion of $\vec{H}_{ext}$ should be taken into account. It equals $\frac{1}{2}(\vec{r}\cdot\vec{\nabla}_0)^2 \cdot \vec{H}_{ext}(\vec{r}_0)$. Substituting it into Eq. (4.1) one arrives to

$$U = -\frac{1}{3} \frac{\partial}{\partial x_{0i}} (rot\, \vec{H}_{ext})_j \int x_i (\vec{r} \times \vec{M}_{12})_j \, dV. \tag{4.9}$$

It is easy to check that integrals in the RHS of this Eq. disappear for the magnetization (4.7). The next term in the expansion of $\vec{H}_{ext}$ is

$$\frac{1}{6} (\vec{r}\,\vec{\nabla}_0)^3 \cdot \vec{H}_{ext}(\vec{r}_0).$$

Inserting this into Eq. (4.1) one obtains after some manipulations

$$U = -\frac{1}{8} \frac{\partial^2 (rot\, \vec{H}_{ext}(\vec{r}_0))_k}{\partial x_{0i} \partial x_{0j}} \int x_i x_j (\vec{r} \times \vec{M}_{12})_k \, dV. \tag{4.10}$$

The nonvanishing integrals occurring here are

$$\int x^2 (\vec{r} \times \vec{M}_{12})_z \, dV = \int y^2 (\vec{r} \times \vec{M}_{12})_z \, dV = \frac{7}{32} \Phi_1 a^4 y_1 (y_1+1)(y_1^2 - y_2^2),$$

$$\int z^2 (\vec{r} \times \vec{M}_{12})_z \, dV = \frac{1}{16} \Phi_1 a^4 y_1 (y_1+1)(y_1^2 - y_2^2), \tag{4.11}$$

$$\int xz (\vec{r} \times \vec{M}_{12})_x \, dV = \int yz (\vec{r} \times \vec{M}_{12})_y \, dV =$$

$$= -\frac{1}{32} \Phi_1 a^4 y_1 (y_1+1)(y_1^2 - y_2^2), \quad y_i = cth\, \mu_i$$

Thus, for the configuration shown in fig.3b the interaction energy is given by Eq. (4.10). The integrals entering the RHS of (4.10) are usually referred to as toroidal moments of higher orders (or multipolarities) /8,9/. Turning now to the current configuration indicated on fig.3c we have instead of (4.10)

$$U = -\frac{1}{8} \frac{\partial^2 (rot\, \vec{H}_{ext})_k}{\partial x_{0i} \partial x_{0j}} \int x_i x_j (\vec{r} \times (\vec{M}_{12} + \vec{M}_{34}))_k \, dV. \tag{4.12}$$

Here M_{34} is given by Eq. similar to (4.7). From the Eq. (3.12) it follows at once that interaction term (4.12) disappears for the configuration presented in fig.3c. The interaction terms reappear when two further steps in the expansion of $\vec{H}_{ext}$ are made. Thus, we obtain the physical interpretation of the current configurations shown in fig.3: they generate the toroidal moments of different multipolarities.

If the conditions (5.4) are not satisfied one may use exact expressions for EMF strengtgs which are valid outside the sphere of the radius R_1+d_1 . For H_φ one has /1/

$$H_\varphi = \frac{i\pi^{3/2}K_1^2}{c\sqrt{2}} \sum_{l=1}^{\infty} \frac{2l+1}{l(l+1)} h_l(Kr) \cdot P_l^1(\cos\theta_s) \cdot a_l(E). \quad (5.7)$$

Here h_l is the spherical Hankel function $\left(h_l(x) = H^{(1)}_{l+\frac{1}{2}}(x)/\sqrt{x}\right)$, $a_l(E)$ is determined by the current density

$$a_l(E) = -\frac{g_1 R_1 c}{2\sqrt{2\pi}} \frac{1}{\sqrt{l}} \sqrt{\frac{2l+1}{2l+3}} \left(\sqrt{l+1} \cdot F^0_{l+1} - \sqrt{l+2} \cdot F^1_{l+1}\right), \quad (5.8)$$

$$F_l^0 = \int d\psi \cos\psi \cdot g_l \cdot P_l, \quad F_l^1 = \int d\psi \cdot \sin\psi \cdot g_l \cdot P_l^1,$$

$g_l(x) = J_{l+1/2}(x)/\sqrt{x}$ is the spherical Bessel function. The arguments of the Bessel and Legendre functions in (5.8) are $K\rho_1$ and $R_1\sin\psi/\rho_1$, resp. Here $\rho_1 = (d_1^2+R_1^2+2d_1R_1\cos\psi)^{1/2}$. The summing in (5.7) is performed over the odd values of l . Inserting (5.1) and (5.7) into (5.3) one arrives to

$$U = -\frac{\pi^{3/2} i K^2 Q_2 d_2}{2\sqrt{2}} \sum \frac{2l+1}{l(l+1)} h_l(Kr_2) \, P_l^1\left(\frac{z}{r_2}\right) a_l(E). \quad (5.9)$$

To obtain the final answer one should multiply this Eq. by the factor $\exp(-i\omega t)$ and then take a real part. Thus, there exists nonvanishing interaction between two TS with time-dependent currents. Consider now the case when the symmetry axes of TS_1 and TS_2 are mutually orthogonal (fig.6). In this case the overlapping integral (5.3) disappears. We conclude: the interaction energy of two TS with alternate currents is proportional to the cosine of angle between their symmetry axes (or between their toroidal moments).

The EMF of TS with time-dependent currents and their interaction was experimentally studied in ref. /12/. The following remarkable fact was observed there. Consider torus T (fig.7). Now "dress" it by the toroidal solenoids ts_i (instead of the poloidal turns). These solenoids are feeded by the alternate current. For the very thin solenoids ts_i one obtains dense covering of the torus T surface. According to ref. /12/ the electric field E differs from zero only inside torus T. Thus one gets electrical solenoid of the finite dimensions. The present author has not been able to justify theoretically this fact. Up to now there are known only infinitesimal realizations of electrical solenoids /13/ as well as nonphysical realization via the current of magnetic monopoles /14/.

§6. Aharonov – Casher effect for the toroidal solenoid

At first we repeat well-known arguments /2/ for the cylindrical solenoid (CS). Consider the charged particle e in the field of the infinite resting CS. The term in Lagrangian describing their interaction is

$$\frac{e}{c} \vec{v}_e \vec{A}(\vec{r}_e - \vec{r}_s). \quad (6.1)$$

Here $\vec{r}_e$ and $\vec{v}_e$ are the radius-vector and velocity of the charged particle, $\vec{r}_s$ is the radius-vector of CS. The Galilean invariance of the Lagrangian permits one to write out explicitly the interaction when both charge and solenoid are in motion

$$\frac{e}{c}(\vec{v}_e - \vec{v}_s)\,\vec{A}(\vec{r}_e - \vec{r}_s). \tag{6.2}$$

Here $\vec{v}_s$ it the velocity of CS. As was shown by Aharonov and Casher /2/ the added term

$$-\frac{e}{c}\,\vec{v}_s\,\vec{A}(\vec{r}_e - \vec{r}_s) \tag{6.3}$$

describes the scattering of neutral particles with the magnetic dipole moment on the infinite charged filament. The experiment/15/ in which the neutrons were scattered by the electric field of charged filament was performed in 1989 . It has confirmed the existence of the AC effect.

Now we turn to the TS. The interaction of a charged particle with resting TS is described by the same Eq. (6.1) where under $\vec{A}$ one should understand the VP of the TS (see § 2 and refs. therein). The experiments in which the electrons were scattered by the VP of TS were performed by Tonomura et al /7/. Their theoretical interpretation may be found in refs. /16/. The same requirement of the Galilean invariance oblige us to choose Eq. (6.2) as an interaction between the moving charge and TS. It is our nearest goal to obtain and interpret the added term (6.3). Let TS with the poloidal current given by Eq. (2.3) moves in an external electric field with scalar potential φ and field strength $\vec{E} = -\operatorname{grad}\varphi$. According to Special Relativity the moving current $\vec{j}$ generates charge density $\rho = \gamma(\vec{v}_s \cdot \vec{j})/c^2$ $(\gamma = (1-\beta^2)^{-1/2})$. The factor γ can be disregarded if we limit ourselves to the Galilean invariant theory. The interaction of moving TS with an external electric field is given by

$$U = \int \varphi(\vec{r}_s - \vec{r}_e)\,\rho(\vec{r}_s)\,dV_s = \frac{1}{c^2}\int \varphi\,(\vec{v}_s \cdot \vec{j})\,dV_s. \tag{6.4}$$

We change current $\vec{j}$ by the equivalent magnetization $(\vec{j} = c \cdot \operatorname{rot}\vec{M})$ and integrate by parts

$$U = \frac{1}{c}\,\vec{v}_s \int \vec{E}(\vec{r}_s - \vec{r}_e) \times \vec{M}\,dV. \tag{6.5}$$

At large distances from the source (or for small dimensions of TS) the electrical field $\vec{E}$ may be developed into the series

$$\vec{E}(\vec{r}_s - \vec{r}_e) = \vec{E}(\vec{r}_0 - \vec{r}_e) + (\vec{r}\,\vec{\nabla}_0)\cdot\vec{E}(\vec{r}_0 - \vec{r}_e). \tag{6.6}$$

Here $\vec{r}_0$ and $\vec{r}$ are the same as in Eq.(4.2). Substitute this Eq. into (6.5)

$$U = -\frac{1}{2c}\,(\vec{v}_s \cdot \vec{\nabla}_0)\cdot(\vec{E} \cdot \vec{\mu}_t). \tag{6.7}$$

Here $\vec{\mu}_t$ is the TDM $(\vec{\mu}_t = \int \vec{r} \times \vec{M}\,dV)$. In the derivation of (6.7) it was also taken into account that dipole magnetic moment $\vec{\mu}_d = 0$ for the TS (see § 4). We ask now: what electric field $\vec{E}$ should be substituted into (6.7) in order to obtain term (6.3) restoring the Galilean symmetry of Lagrangian? We equate Eqs. (6.3) and (6.7)

$$e\,\vec{v}_s \cdot \vec{A}(\vec{r}_0 - \vec{r}_e) = -\frac{1}{2}(\vec{v}_s\,\vec{\nabla}_0)\cdot(\vec{E}(\vec{r}_0 - \vec{r}_e)\cdot\vec{\mu}_t). \tag{6.8}$$

The sign minus appears here because the potential energy enters into the Lagrangian with negative sign, it is also taken into account that VP of the TS is an even function of coordinates/17/

(contrary to the CS case). Now compare coefficients at the particular cartesian component of $\vec{v}_s$

$$e A_i (\vec{r}_0 - \vec{r}_e) = -\frac{1}{2} (\vec{\mu}_t)_k \frac{\partial E_k (\vec{r}_0 - \vec{r}_e)}{\partial x_{0i}} \qquad (6.9)$$

Without loss of generality we assume that symmetry axes of TS in both sides of Eq. (6.9) are parallel to the z axis. In this case only z component of $\vec{\mu}_t$ differs from zero ($\mu_t = \frac{1}{2}\pi g d R^2$, see§ 4). As expansion (6.6) is valid at large distance from electric field source one should use in LHS of (6.9) the asymptotic values of A_i defined by Eq.(2.8). This gives

$$\frac{e}{2} \frac{r^2 + 3(x^2 - y^2)}{r^5} = -\frac{\partial E_z}{\partial z},$$

$$3e \frac{xz}{r^5} = -\frac{\partial E_z}{\partial x}, \quad 3e \frac{yz}{r^5} = -\frac{\partial E_z}{\partial y}. \qquad (6.10)$$

Here $x = x_0 - x_e$ etc. It is easy to check that these Eqs. are satisfied for $E_z = ez/r^3$. But this is just z component of $\vec{E} = e\vec{r}/r^3$. From this it follows at once that at large distances the aforementioned term (6.3) describes scattering of toroidal magnetic moments on the Coulomb field. In order to find electric field at finite distances we turn to Eq.(6.4) in which the expansion of electric field was not yet performed. Again we require the coincidence of Eqs. (6.4) and (6.3)

$$\frac{e}{c} \vec{v}_s \vec{A} = \frac{1}{c^2} \int \varphi (\vec{v}_s \cdot \vec{j}) \, dV \qquad (6.11)$$

or comparing coefficients at $\vec{v}_s$

$$e\vec{A}(\vec{r}_0 - \vec{r}_e) = \frac{1}{c} \int \varphi (\vec{r}_0 - \vec{r}_e - \vec{r}) \, \vec{j}(\vec{r}) \, dV. \qquad (6.12)$$

Now we remind that VP $\vec{A}$ satisfies Poisson Eq.:

$$\Delta \vec{A} = -\frac{4\pi}{c} \vec{j}$$

Its solution is

$$\vec{A}(\vec{r}) = \frac{1}{c} \int \frac{1}{|\vec{r} - \vec{r}'|} \vec{j}(\vec{r}') \, dV'. \qquad (6.13)$$

By comparing Eqs. (6.12) and (6.13) we get

$$\varphi = e / |\vec{r}_0 - \vec{r}_e|. \qquad (6.14)$$

But this is just the scalar potential of the point charge. Now we write out the classical Lagrangian

$$L = \frac{1}{2} m_s v_s^2 + \frac{1}{2} m_e v_e^2 + \frac{e}{c} (\vec{v}_e - \vec{v}_s) \vec{A}(\vec{r}_e - \vec{r}_s). \qquad (6.15)$$

There is no classical scattering as Eqs. $\dot{\vec{v}}_e = 0$, $\dot{\vec{v}}_s = 0$ follow from this Lagrangian. Now we fix the position of the Coulomb centre ($\vec{v}_e = 0$, $\vec{r}_e = 0$, $\vec{v}_s = \vec{v}$, $\vec{r}_s = \vec{r}$, $m_s = m$) Then

$$L = \frac{1}{2} m v^2 - \frac{e}{c} \vec{v} \cdot \vec{A}(\vec{r}). \qquad (6.16)$$

For the infinitely small TS it reduces to

$$L = \frac{1}{2} m v^2 + \frac{1}{2c} (\vec{v} \vec{\nabla}) \cdot (\vec{E} \cdot \vec{\mu}_t), \quad \vec{E} = \frac{e\vec{r}}{r^3}. \qquad (6.17)$$

Again there is no classical scattering for these Lagrangians. The corresponding Schroedinger Eqs. are

$$\frac{1}{2m}\left(\vec{p}+\frac{e}{c}\vec{A}\right)^2\Psi=\mathcal{E}\Psi, \tag{6.18}$$

$$\frac{1}{2m}\left(\vec{p}-\frac{1}{2c}\vec{\nabla}(\vec{E}\cdot\vec{\mu}_t)\right)^2\Psi=\mathcal{E}\Psi. \tag{6.19}$$

The second of these Eqs. holds at sufficiently large (comparatively with the TS dimensions) distances from the Coulomb center. Schroedinger Eq.(6.19) describes quantum scattering of toroidal moment by the Coulomb field. How to realize this experiment? One should find neutral particles having nonvanishing toroidal moment (and zero magnetic dipole one). It was claimed in ref. /18/ that Majorana neutrinos are just such particles. The second way is the scattering of ferromagnetic microparticles by the Coulomb field. According to ref. /19/ these microparticles carry the toroidal dipole moment.

It should be noted that Eqs. (6.12), (6.16) and (6.18) are valid for the toroidal moments of arbitrary multipolarities (see § 4). In this case $\vec{A}$ is the VP corresponding to the chosen multipolarity (see § 3). Eqs. (6.7) and (6.19) are referred only to the toroidal dipole moments.

7. Conclusion

We briefly summarize the main results obtained.

1. The gauge nonequivalent vector potentials are obtained which lead to the same quantum mechanical scattering. It turns out that current distributions (or magnetizations) generating these vector potentials carry toroidal moments of different multipolarities.

2. The equations are obtained which describe the motion of toroidal moment in an external magnetic field.

3. It is shown that there exists nonzero interaction between two solenoids with alternate current in their coils.

4. It is proved the existence of the Aharonov – Casher effect for the toroidal solenoids and moments. The experiments testing it are suggested.

Appendix 1

Here are the nonvanishing cylindrical components of VP corresponding to the current distribution (3.1):

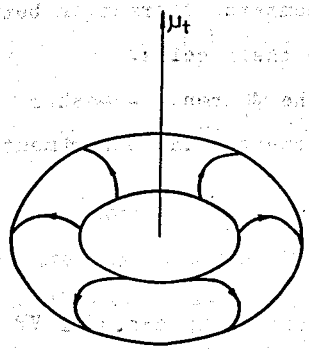
$$A_z=\frac{8\sqrt{2}a^2}{c}(\mathrm{ch}\,\mu-\cos\theta)^{1/2}\sum\frac{\cos n\theta}{1+\delta_{n0}}\cdot P_{n-\frac{1}{2}}\int_{\mu_1}^{\infty}d\mu\cdot f(\mu)\cdot Q_{n-\frac{1}{2}}\cdot q_n^{(1)},$$

$$A_\rho=-\frac{8\sqrt{2}a^2}{c}(\mathrm{ch}\,\mu-\cos\theta)^{1/2}\sum\sin n\theta\cdot P^1_{n-\frac{1}{2}}\int_{\mu_1}^{\infty}d\mu\cdot f(\mu)\cdot Q_{n-\frac{1}{2}}\cdot q_n^{(2)},$$

outside the toroidal solenoid ($\mu<\mu_1$) and

$$A_z=\frac{8\sqrt{2}a^2}{c}(\mathrm{ch}\,\mu-\cos\theta)^{1/2}\sum\frac{\cos n\theta}{1+\delta_{n0}}\left[Q_{n-\frac{1}{2}}\cdot\int_{\mu_1}^{\mu}d\mu\cdot f(\mu)\cdot P_{n-\frac{1}{2}}\cdot q_n^{(1)}+\right.$$
$$\left.+P_{n-\frac{1}{2}}\int_{\mu}^{\infty}d\mu\cdot f(\mu)\cdot Q_{n-\frac{1}{2}}\cdot q_n^{(1)}\right],$$

$$A_\rho=-\frac{8\sqrt{2}a^2}{c}(\mathrm{ch}\,\mu-\cos\theta)^{1/2}\sum\sin n\theta\left[P^1_{n-\frac{1}{2}}\int_{\mu}^{\infty}d\mu\cdot f(\mu)\cdot Q_{n-\frac{1}{2}}\cdot q_n^{(2)}\right.$$
$$\left.+Q^1_{n-\frac{1}{2}}\int_{\mu_1}^{\infty}d\mu\cdot f(\mu)\cdot P_{n-\frac{1}{2}}\cdot q_n^{(2)}\right]$$



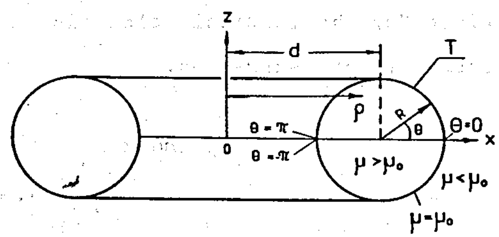


Fig.1. The poloidal current on the surface of toroidal solenoid and the associated toroidal moment.

Fig.2. The geometrical dimensions of the toroidal solenoid.

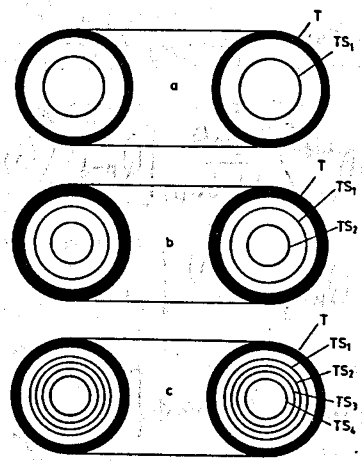


Fig.3. The different current configurations are imbedded into the impenetrable torus T. They generate gauge nonequivalent vector-potentials leading to the same quantum mechanical scattering of charged particles. The vector potentials are concentrated more and more near the torus T as one moves from a) to c).

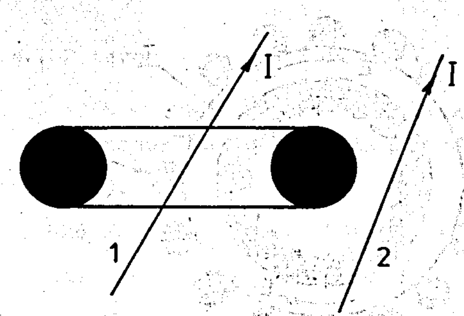


Fig.4. The linear conductor with the stationary current I interacts with toroidal solenoid when it passes through the solenoid hole (position 1). Otherwise there is no interaction between them (position 2).

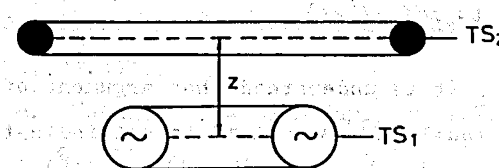


Fig.5. The toroidal solenoids interact if the alternate currents flow in their coils.

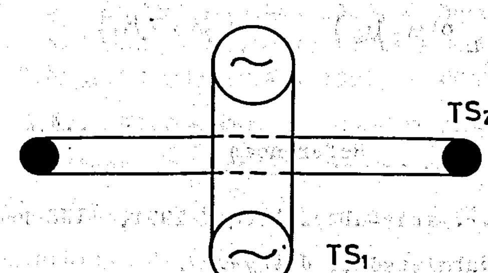


Fig.6. The toroidal solenoids do not interact if their symmetry axes are perpendicular.

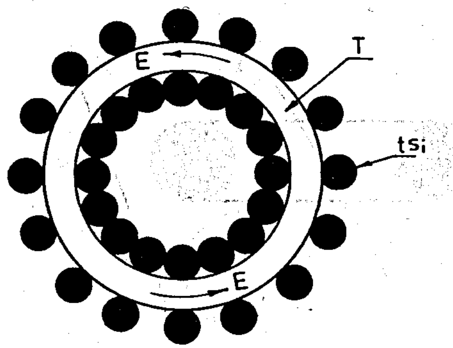


Fig.7. The torus T is "dressed" by the toroidal solenoids with alternate currents in their coils. For the very thin solenoids ts_i and dense covering of T by ts_i the electromagnetic field is confined inside T. (according to ref. /12/).

inside it $(\mu > \mu_1)$. It is understood that argument of the Legendre functions equals $ch\,\mu$ if it is not indicated. Further,

$$q_n^{(1)} = (n+\tfrac{1}{2})Q_{n+\frac{1}{2}} - (n-\tfrac{1}{2})Q_{n-\frac{3}{2}}, \quad q_n^{(2)} = Q_{n+\frac{1}{2}} - Q_{n-\frac{3}{2}}.$$

The VP obtained in /3/ corresponds to the current density (2.6). It is obtained when the following choice of function f is made:

$$f = \frac{ge}{4\pi a^2}\,\delta(\mu-\mu_0) \qquad (\mu_0 > \mu_1).$$

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