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Yu. Kubyshin [1], D. O'Connor [2], C. R. Stephens [3]

DIMENSIONAL CROSSOVER FROM NON-RENORMALIZABILITY TO RENORMALIZABILITY

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[1]Nuclear Physics Institute, Moscow State University, Moscow 119899, U.S.S.R.

[2]Address after October 1st 1991: D.I.A.S., 10 Burlington Rd., Dublin 4, Ireland

[3]Address after October 1st 1991: Inst. for Theor. Physics, Rijksuniversiteit, Princetonplein 5, P.O.Box 80006, 3508 TA Utrecht, The Netherlands

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1 Introduction

In the absence of a satisfactory theory of quantum gravity, many ideas about the structure of spacetime at the microscopic level have been suggested, one of which is that there may be more than four dimensions some of which are curled up and only visible at extremely short distances. The assumed structure of spacetime is then $E = M^4 \times B$, where M^4 is the macroscopic world and B is the compact internal space of extra dimensions, this structure is usually called multidimensional. The idea of extra dimensions dates back to the seminal works of Kaluza and Klein [1], but in recent years has become an essential ingredient in string theory, supergravity and many other models (see [2] for a review). Though the classical properties of these theories have been extensively analyzed,(see for example [3] and [4] for a review and earlier references) there are still many models which are as yet not well understood (see [5]). Intuitively one expects that if the size L of the compact additional dimensions is small enough then the contributions from these dimensions become negligible. In this context a basic question arises as to whether the contributions from the extra dimensions are really small when L is small and how small they actually are. This is the question we wish to address in this paper.

Two complications arise which make the question not that straightforward. To explain these we re-interpret the multidimensional theory in terms of four-dimensional objects, where it can be represented as a model with an *infinite* number ("tower") of particles. The simplest way of seeing this infinite tower is to make a Fourier expansion of the multidimensional field $\phi(x,y)$,

$$\phi(x,y) = \sum_N \phi_N(x) Y_N(y), \tag{1}$$

where x and y are co-ordinates of the four-dimensional and extra-dimensional parts respectively with $Y_N(y)$ being eigenfunctions of the Laplace operator on the internal space, substitute it into the action and integrate over y obtaining

$$S = -\sum_N \int_{M^4} d^4x \frac{1}{2} \phi_N(x)(\partial^2 + m^2 + M_N^2)\phi_N(x) + S_{int} \tag{2}$$

where $m^2 + M_N^2$ appear as four dimensional masses, M_N^2 being the eigenvalues of the Laplacian on B. In many models there are modes with $M_N^2 = 0$, the remaining modes have M_N^2 proportional to L^{-2}. If L^2m^2 is small then the former modes correspond to light particles whereas the latter correspond to an infinite tower of heavy particles. Now if the energy scales of our probes are less than the threshold energy for the creation of these heavy particles they will never be created in our experiments. These modes however, do make a contribution to the quantum amplitudes through loop corrections. That the tower of modes produce a finite or small contribution is not obvious due to their infinite number. If this number were not infinite but finite then we would have the decoupling theorems [6] to reassure us, however in the infinite case no such theorem has been proven, thus the first complication is whether decoupling is valid or not.

The second complication is to be seen in the nature of the infinite tower of modes refelecting the multidimensional character of the original theory, which frequently is non-renormalizable. In general the nature of the ultraviolet divergencies changes because of the infinite summation over massive modes, which can give rise to non-renormalizable terms in the Lagrangian in certain cases even when the zero mode theory is renormalizable.

To gain some feel for what one might expect to happen in this setting we appeal to experience in statistical mechanics, since euclidean quantum field theory is intimately related to statistical mechanical systems near their second order phase transition points, where similar questions arise [7]. Here the relevant phenomenon is the evolution from one critical behaviour to another, often referred to as crossover. The analog of the multidimensional problem in this setting is the crossover from the critical behaviour in one dimension to that of another via finite size effects. When the infinite tower of massive modes yields a non-renormalizable interaction the bulk system is said to be above the upper critical dimension. When crossing over from one dimension to another both of which are below the upper critical dimension the interpolation is from one renormalizable theory to another, this problem was investigated in some detail in [8] and [9]. The experience of calculations in statistical mechanics suggests that theories above the upper critical dimension (where the theory is non-renormalizable) behave essentially as mean field theories, i.e. loop corrections are negligible. An example of such a model is the Ising model which is equivalent to a scalar field theory with an infinite number of non-renormalizable interactions. Monte-Carlo simulations and other lattice calculations demonstrate the qualitative correctness of these assertions in the statistical mechanics setting [10], however in the field theory setting the crossover has not been analysed before to our knowledge.

One of the aims of the present paper is to show the decoupling of massive modes from zero modes at low energies in the framework of a simple scalar model in six dimensions with two extra dimensions being compactified to a torus. This is achieved by demonstrating the interpolation of quantum attributes (renormalizations, renormalization group (RG) equations, etc.) of the theory as the scale of the extra dimensions goes to zero. In particular we study the transition from six to four dimensions.

An assumption is made that the additional dimensionless couplings requiring renormalization in the multidimensional theory are of the order of the power of the basic dimensionless coupling appearing in the first divergent diagram which contributes to their renormalization. This assumption is essential for our systematic approach to controlling the proliferation of divergences. In the current study our analysis is restricted to investigations of one loop diagrams.

The outline of the paper is as follows. In Sect. 2 we describe the calculation of the relevant one loop contributions and discuss renormalization prescriptions. The renormalization group equations are obtained and analysed in Sect. 3. where we

also discuss the decoupling mainly in the spirit of the paper [11]. Some concluding remarks are given in Sect. 4.

2 One-loop corrections and the renormalization prescriptions

As a simple model which captures many interesting features of quantum properties of multidimensional theories we consider a one component scalar field on the six-dimensional manifold $E = M^4 \times S^1 \times S^1$ with the radii of the both circles $L/2\pi$ (this is purely for presentation of the formulae). Here M^4 is Minkowski spacetime and the internal space is the two-dimensional torus $T^2 = S^1 \times S^1$. The action is given by

$$S = \int_E d^4x d^2y [\frac{1}{2}(\frac{\partial \phi_B(x,y)}{\partial x})^2 + (\frac{\partial \phi_B(x,y)}{\partial y})^2 - \frac{1}{2} m_B^2 \phi_B^2) + L_{int}], \quad (3)$$

$$L_{int} = -\frac{\hat{\lambda}_{1B}}{4!}\phi_B^4(x,y) - \frac{\hat{\lambda}_{2B}}{4!}\phi_B(x,y)\Box\phi_B^3(x,y) - \frac{\hat{\lambda}_{3B}}{6!}\phi_B^6(x,y),$$

where we use the subscript B to label bare quantities and $\Box$ is the D'Alambertian on E. We have introduced the second and third terms in L_{int} on the bare level since counterterms with such structures are necessary for subtracting one-loop ultraviolet divergencies in the theory under consideration. Substituting the Fourier expansion (1) into the action (3) and integrating out the extra coordinates we get the dimensionally reduced action on M^4 the quadratic part of which is given by (2) with

$$M_N^2 = (\frac{2\pi N}{L})^2 \quad (4)$$

where $N^2 = n_1^2 + n_2^2$ and the intergers n_1 and n_2 label the Fourier modes on the S^1's forming the internal space. Similarly

$$L_{int} = -\frac{\lambda_{1B}\Lambda^{2\epsilon}}{4!}\phi_{B0}^4(x) - \frac{\lambda_{2B}\Lambda^{-2+2\epsilon}}{4!}\phi_{B0}(x)\Box_{(4)}\phi_{B0}^3(x) - \frac{\lambda_{3B}\Lambda^{-2+4\epsilon}}{6!}\phi_{B0}^6(x) + L'_{int}, \quad (5)$$

where L'_{int} includes heavy modes with $N^2 \neq 0$, $\Box_{(4)}$ is the D'Alambertian on M^4 and the coupling constants $\lambda_{iB}, (i = 1,2,3)$ are related to the multidimensional ones by

$$\lambda_{1B}\Lambda^{2\epsilon} = \frac{\hat{\lambda}_{1B}}{L^2}, \qquad \lambda_{2B}\Lambda^{-2+2\epsilon} = \frac{\hat{\lambda}_{2B}}{L^2}, \qquad \lambda_{3B}\Lambda^{-4+4\epsilon} = \frac{\hat{\lambda}_{3B}}{L^4}, \quad (6)$$

We have extracted the scale Λ, which is some scale associated with the bare theory, to work with dimensionless couplings, since only dimensionless couplings can be large or small in and of themselves, throughout we will work only with dimensionless couplings.

In the present paper we consider the four-point function with external legs corresponding to light modes with $N^2=0$, studying in detail the four point vertex. Only such functions are relevant if the energies of the external particles are much smaller than L^{-1}. In many compactification schemes [4][12] L turns out to be of the order of the Planck length $L_{planck} \approx 10^{-33}cm$.

We consider the four-point vertex function Γ^4 evaluated at the symmetric point $p_i p_j = \frac{p^2}{4}(4\delta_{ij}-1)$, where p_i are incoming four-momenta. Calculating the standard one loop diagram proportional to λ_{1B}^2 using dimensional regularization we get

$$\Gamma^4(p,L,m_B,\lambda_{1B},\lambda_{2B}) = \lambda_{1B}\Lambda^{2\epsilon} + p^2\lambda_{2B}\Lambda^{-2+2\epsilon} - \lambda_{1B}^2\Lambda^{4\epsilon}p^{-2\epsilon}I(Lp,\frac{m}{p}), \tag{7}$$

where ϵ is the regularization parameter and

$$I(Lp,\frac{m}{p}) = p^{2\epsilon}\frac{3}{2}\sum_{n_1,n_2=-\infty}^{\infty}\int\frac{d^{4-2\epsilon}k}{(2\pi)^{4-2\epsilon}}\frac{1}{(k^2+m^2+M_N^2)((p-k)^2+m^2+M_N^2)}$$

Performing the momentum integration, introducing the Feynmann parameter t we obtain,

$$I(Lp,\frac{m}{p}) = \frac{3}{2}\frac{\Gamma(\epsilon)}{(4\pi)^{2-\epsilon}}\sum_{n_1,n_2=-\infty}^{\infty}\int_0^1\frac{dt}{(t(1-t)+(\frac{2\pi N}{Lp})^2+(\frac{m}{p})^2)^\epsilon}. \tag{8}$$

Here we assume that $\lambda_{2B} \simeq \lambda_{1B}^2$, so that the one loop diagrams proportional to $\lambda_{1B}\lambda_{2B}$ and λ_{2B}^2 can be neglected. The consistency of this assumption is discussed in Sect. 3. Note that if we do not make this assumption the proliferation of divergences becomes uncontrollable.

Now we are ready to discuss our renormalization prescription. Analysing the one-loop contribution in (7) we see that the function $I(pL,m/p)$ given by (8) and its first derivative with respect to p^2 are divergent when ϵ goes to zero (the sum can be understood in the sense of ζ-function regularization [13]). The divergence of the derivative reflects the six-dimensional character of the original theory. Thus there are two undetermined parameters which correspond to the renormalizations of the operators ϕ_B^4 and $\phi_B\Box_{(4)}\phi_B^3$ in the bare Lagrangian. We will define corresponding dimensionless renormalized coupling constants by the following normalization conditions:

$$\begin{gathered}\frac{\partial\Gamma^4}{\partial p^2}|_{p^2=\kappa^2} = \lambda_2\kappa^{-2+2\epsilon}\\ (1-p^2\frac{\partial}{\partial p^2})\Gamma^4|_{p^2=\kappa^2} = \lambda_1\kappa^{2\epsilon}\end{gathered} \tag{9}$$

Implementing these conditions we obtain relations between the bare and the renormalized couplings. In the current case these yield

$$\begin{aligned}\lambda_1\kappa^{2\epsilon} &= \lambda_{1B}\Lambda^{2\epsilon} + \lambda_{1B}^2\Lambda^{4\epsilon}\kappa^{-2\epsilon}\tfrac{1}{2}A_{\epsilon+1}I(\kappa L,\tfrac{m}{\kappa}),\\ \lambda_2\kappa^{-2+2\epsilon} &= \lambda_{2B}\Lambda^{-2+2\epsilon} - \lambda_{1B}^2\Lambda^{4\epsilon}\kappa^{-2-2\epsilon}\tfrac{1}{2}A_\epsilon I(\kappa L,\tfrac{m}{\kappa}),\end{aligned} \tag{10}$$

where we have introduced $A_\nu = \kappa\partial_\kappa - 2\nu$. Note the following useful identities: $[A_{\nu_1},A_{\nu_2}]=0$ and $\kappa^{2\nu}A_0\kappa^{-2\nu} = A_\nu$.

Inverting the expansions (10) to obtain the bare coupling constants in terms of the renormalized ones and substituting into eq. (7) we get

$$\begin{aligned}\Gamma^4 = \;& \lambda_1\kappa^{2\epsilon} + \tfrac{p^2}{\kappa^2}\lambda_2\kappa^{2\epsilon}\\ & -\lambda_1^2\kappa^{2\epsilon}\left[(\tfrac{\kappa^2}{p^2})^\epsilon I(Lp,\tfrac{m}{p}) + \tfrac{1}{2}A_{1+\epsilon}I(\kappa L,\tfrac{m}{\kappa}) - \tfrac{1}{2}(\tfrac{p^2}{\kappa^2})A_\epsilon I(\kappa L,\tfrac{m}{\kappa})\right].\end{aligned} \tag{11}$$

It can be readily shown that the expression in the square brackets is finite when ϵ goes to zero. Note we have used the fact that to one loop no wavefunction renormalization is required.

We are specifically interested in the limit of small κL so we develop an expansion for $I(\kappa L,\frac{m}{\kappa})$ to assist our further discussion. We obtain from

$$\begin{aligned}I(\kappa L,\tfrac{m}{\kappa}) = \;& I_0(\tfrac{m}{\kappa})\\ & + \tfrac{3}{2(4\pi)^{2-\epsilon}}\textstyle\sum_{k=0}^{\infty}\sum_{l=0}^{k}(-)^k\tfrac{\Gamma(k-l+1)^2\Gamma(\epsilon+k)\zeta(\epsilon+k)}{\Gamma(k+1)\Gamma(2k-2l+2)}(\tfrac{m}{\kappa})^{2l}[(\tfrac{\kappa L}{2\pi})^2]^{\epsilon+k}\end{aligned} \tag{12}$$

where $I_0(\frac{m}{\kappa})$ is the $N^2=0$ term in (8) (the four dimensional result), and

$$\zeta(\nu) = \sum_{N^2\neq 0}(N^2)^{-\nu}$$

is the generalized *zeta*-function, (see [13] for a discussion of its properties). When $m^2=0$ (12) simplifies to

$$I(\kappa L) = I_0 + \frac{3}{2(4\pi)^{2-\epsilon}}\sum_{k=0}^{\infty}(-)^k\frac{\Gamma(k+1)}{\Gamma(2k+2)}\Gamma(\epsilon+k)\zeta(\epsilon+k)[(\frac{\kappa L}{2\pi})^2]^{\epsilon+k} \tag{13}$$

I_0 being a numerical constant (though divergent when $\epsilon\to 0$) given by

$$I_0 = \frac{3}{2}\frac{\Gamma(\epsilon)}{(4\pi)^{2-\epsilon}}\frac{\Gamma(1-\epsilon)^2}{\Gamma(2-2\epsilon)}. \tag{14}$$

Analysing the relations (10) in the limit of small κL we see that the coupling constant λ_2 still has a finite renormalization. Since in this limit the internal space disappears we would expect to recover standard four-dimensional formulae where λ_2 is not renormalized. This can be achieved by another choice of the normalization conditions. For example one can choose them as follows in the $m^2=0$ limit (which we restrict ourselves to in the remaining discussion)

$$g_2\kappa^{-2+2\epsilon} = \frac{1}{1+\epsilon}\left[\frac{\partial}{\partial p^2} + \frac{p^2}{1+\epsilon}(\frac{\partial}{\partial p^2})^2\right]\Gamma^4|_{p^2=\kappa^2}, \tag{15}$$

$$g_1\kappa^{2\epsilon} = \frac{1}{1+\epsilon}\left[1 - \frac{\partial}{\partial p^2} - \frac{p^2}{1+\epsilon}(\frac{\partial}{\partial p^2})^2\right]\Gamma^4|_{p^2=\kappa^2}. \tag{16}$$

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Implementing these conditions we obtain relations between the bare and the renormalized couplings. In the current case these yield

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where we have introduced $A_\nu = \kappa\partial_\kappa - 2\nu$. Note the following useful identities: $[A_{\nu_1}, A_{\nu_2}] = 0$ and $\kappa^{2\nu}A_0\kappa^{-2\nu} = A_\nu$.

Inverting the expansions (10) to obtain the bare coupling constants in terms of the renormalized ones and substituting into eq. (7) we get

$$\begin{aligned} \Gamma^4 = {} & \lambda_1\kappa^{2\epsilon} + \tfrac{p^2}{\kappa^2}\lambda_2\kappa^{2\epsilon} \\ & -\lambda_1^2\kappa^{2\epsilon}\left[(\tfrac{\kappa^2}{p^2})^{\epsilon}I(Lp, \tfrac{m}{p}) + \tfrac{1}{2}A_{1+\epsilon}I(\kappa L, \tfrac{m}{\kappa}) - \tfrac{1}{2}(\tfrac{p^2}{\kappa^2})A_{\epsilon}I(\kappa L, \tfrac{m}{\kappa})\right]. \end{aligned} \tag{11}$$

It can be readily shown that the expression in the square brackets is finite when ϵ goes to zero. Note we have used the fact that to one loop no wavefunction renormalization is required.

We are specifically interested in the limit of small κL so we develop an expansion for $I(\kappa L, \frac{m}{\kappa})$ to assist our further discussion. We obtain from

$$\begin{aligned} I(\kappa L, \tfrac{m}{\kappa}) = {} & I_0(\tfrac{m}{\kappa}) \\ & + \tfrac{3}{2(4\pi)^{2-\epsilon}}\sum_{k=0}^{\infty}\sum_{l=0}^{k}(-)^k\tfrac{\Gamma(k-l+1)^2\Gamma(\epsilon+k)\zeta(\epsilon+k)}{\Gamma(k+1)\Gamma(2k-2l+2)}(\tfrac{m}{\kappa})^{2l}[(\tfrac{\kappa L}{2\pi})^2]^{\epsilon+k} \end{aligned} \tag{12}$$

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Analysing the relations (10) in the limit of small κL we see that the coupling constant λ_2 still has a finite renormalization. Since in this limit the internal space disappears we would expect to recover standard four-dimensional formulae where λ_2 is not renormalized. This can be achieved by another choice of the normalization conditions. For example one can choose them as follows in the $m^2 = 0$ limit (which we restrict ourselves to in the remaining discussion)

$$g_2\kappa^{-2+2\epsilon} = \frac{1}{1+\epsilon}\left[\frac{\partial}{\partial p^2} + \frac{p^2}{1+\epsilon}(\frac{\partial}{\partial p^2})^2\right]\Gamma^4|_{p^2=\kappa^2}, \tag{15}$$

$$g_1\kappa^{2\epsilon} = \frac{1}{1+\epsilon}\left[1 - \frac{\partial}{\partial p^2} - \frac{p^2}{1+\epsilon}(\frac{\partial}{\partial p^2})^2\right]\Gamma^4|_{p^2=\kappa^2}. \tag{16}$$

The $\frac{1}{1+\epsilon}$ arises as $\frac{2}{d_2-2d_1}$ where d_1 and d_2 are the powers of κ with which the couplings λ_1 and λ_2 enter into Γ^4, in this case -2ϵ and $2-2\epsilon$ respectively. The set of coupling constants (g_1, g_2) differs from the set (λ_1, λ_2) considered before by a finite renormalization, the explicit relation between these sets to one-loop is

$$\frac{\lambda_1}{1+\epsilon} = g_1 - \tfrac{1}{4}g_1^2 A_\epsilon A_{1+\epsilon} I(\kappa L),$$
$$\frac{\lambda_2}{1+\epsilon} = g_2 + \tfrac{1}{4}g_1^2 A_\epsilon A_{1+\epsilon} I(\kappa L). \tag{17}$$

These formulae imply that

$$(1+\epsilon)g_1\kappa^{2\epsilon} = \lambda_{1B}\Lambda^{2\epsilon} - \lambda_{1B}^2\Lambda^{4\epsilon}\kappa^{-2\epsilon}[1 - \tfrac{1}{4(1+\epsilon)}A_0A_\epsilon]I(\kappa L),$$
$$(1+\epsilon)g_2\kappa^{-2+2\epsilon} = \lambda_{2B}\Lambda^{-2+2\epsilon} - \lambda_{1B}^2\Lambda^{4\epsilon}\tfrac{\kappa^{-2-2\epsilon}}{4(1+\epsilon)}A_0A_\epsilon I(\kappa L). \tag{18}$$

It can be readily checked that the renormalization of g_2 in (10) vanishes when $\kappa L \to 0$.

Observe from the normalization conditions (10) and (18) that $\lambda_1 + \lambda_2 = (1+\epsilon)(g_1+g_2) = \Gamma^4(\kappa, L, \lambda_{1B}, \lambda_{2B})\kappa^{-2\epsilon}$ where λ_{1B} and λ_{2B} are understood as functions of either λ_1 and λ_2 or g_1 and g_2.

Γ^4 can be written in the form

$$\Gamma^4(\frac{p^2}{\kappa^2}, \kappa L) = \Gamma_0^4(\frac{p^2}{\kappa^2}) + \delta\Gamma^4(\frac{p^2}{\kappa^2}, \kappa L)$$

which for $\epsilon = 0$ gives

$$\Gamma_0^4(\tfrac{p^2}{\kappa^2}) = g_1 + \tfrac{p^2}{\kappa^2}g_2 + \tfrac{3}{2}\tfrac{g_1^2}{32\pi^2}\ln\tfrac{p^2}{\kappa^2}$$
$$\delta\Gamma^4(\tfrac{p^2}{\kappa^2}, \kappa L) = -g_1^2\tfrac{3}{32\pi^2}\textstyle\sum_{k=2}(-)^k\tfrac{\Gamma(k)\Gamma(k+1)}{\Gamma(2k+2)}\zeta(2k)(\tfrac{\kappa L}{2\pi})^{2k}\{(\tfrac{p^2}{\kappa^2})^k - 1 + k^2(1-\tfrac{p^2}{\kappa^2})\} \tag{19}$$

It can be shown that the series above is convergent and goes to zero when $\kappa L \to 0$. If however in the above we had used minimal subtraction instead of the normalization conditions implemented above we would find that the finite contribution to Γ^4 become infinite in the limit of $\kappa L \to 0$.

3 Renormalization group equations

In this section we derive the renormalization group equations for the coupling constants introduced in the previous section and examine their solution. We conclude the section with a discussion of decoupling of the extra dimensions as a generalization of the usual decoupling theorems.

Then β-functions for these couplings, where $\beta(X) = A_0X$, are the following

$$\beta(\lambda_1) = -2\epsilon\lambda_1 + \lambda_1^2 J(\kappa L)$$
$$\beta(\lambda_2) = (2-2\epsilon)\lambda_2 - \lambda_1^2 J(\kappa L) \tag{20}$$

where we found it convenient to define a function

$$J(\kappa L) = \tfrac{1}{2}A_{\epsilon+1}A_\epsilon I(\kappa L)$$
$$= \tfrac{3\Gamma(2+\epsilon)}{(4\pi)^{2-\epsilon}}\textstyle\sum_{n_1,n_2=-\infty}^{\infty}\int_0^1 \tfrac{dt}{(t(1-t)+(\frac{2\pi N}{Lp})^2+(\frac{m}{p})^2)^\epsilon}. \tag{21}$$

which has the series expansion for small κL

$$J(\kappa L) = a_2 + \frac{3}{(4\pi)^{2-\epsilon}}\sum_{k=2}^{\infty}(-)^k\frac{k(k-1)\Gamma(k+1)}{\Gamma(2k+2)}\Gamma(\epsilon+k)\zeta(\epsilon+k)[(\frac{\kappa L}{2\pi})^2]^{\epsilon+k} \tag{22}$$

which is a well defined function when $\epsilon \to 0$.

$$a_2 = \frac{3\Gamma(2+\epsilon)}{(4\pi)^{2-\epsilon}}\frac{\Gamma(1-\epsilon)^2}{\Gamma(2-2\epsilon)} \tag{23}$$

and is the usual four dimensional coefficient. We can now consider the beta functions for the alternative couplings. Let us consider first the couplings (g_1, g_2) defined in the previous section. We find that the β-functions for these couplings are

$$\beta(g_1) = -2\epsilon g_1 - g_1^2A_\epsilon[1+\epsilon-\tfrac{1}{4}A_0A_\epsilon]I(\kappa L)$$
$$\beta(g_2) = (2-2\epsilon)g_2 - g_1^2\tfrac{1}{4}A_{1+\epsilon}A_0A_\epsilon I(\kappa L) \tag{24}$$

Some rearranging yields

$$\beta(g_1) = -2\epsilon g_1 + g_1^2\tilde{J}(\kappa L)$$
$$\beta(g_2) = (2-2\epsilon)g_2 - g_1^2 S(\kappa L) \tag{25}$$

For convenience we have defined

$$S(\kappa L) = \frac{1}{4}A_0A_\epsilon A_{1+\epsilon}I(\kappa L) = \frac{1}{2}A_0J(\kappa L)$$

and

$$\tilde{J}(\kappa L) = J(\kappa L) + S(\kappa L)$$

which have the series representations for small κL

$$S(\kappa L) = \frac{3}{(4\pi)^{2-\epsilon}}\sum_{k=2}^{\infty}(-)^k\frac{(k+\epsilon)k(k-1)\Gamma(k+1)}{\Gamma(2k+2)}\Gamma(\epsilon+k)\zeta(\epsilon+k)[(\frac{\kappa L}{2\pi})^2]^{\epsilon+k} \tag{26}$$

and

$$\tilde{J}(\kappa L) = a_2$$
$$+\tfrac{3}{(4\pi)^{2-\epsilon}}\textstyle\sum_{k=2}^{\infty}(-)^k\tfrac{(k+1+\epsilon)k(k-1)\Gamma(k+1)}{\Gamma(2k+2)}\Gamma(\epsilon+k)\zeta(\epsilon+k)[(\tfrac{\kappa L}{2\pi})^2]^{\epsilon+k} \tag{27}$$

We readily see from the series expansions that these renormalization group equations have the property that when $\kappa L \to 0$ the coupling constant g_2 is not renormalized, since in this limit $S(\kappa L) \to 0$. This is a desirable feature since this coupling is not necessary in the four dimensional theory where it does not undergo an infinite renormalization. Similarly in this limit the renormalization of g_1 reduces to the four dimensional result, since a_2 is all that survives, and is exactly the result obtained by doing the calculation purely in four dimensions. That this is true is essentially a demonstration of the decoupling of the infinite tower of massive modes.

Two other couplings related to the above are

$$h_1 = \lambda_1 J(\kappa L)$$
$$h_2 = \lambda_2 J(\kappa L) \tag{28}$$

These have beta functions

$$\beta(h_1) = -2\epsilon(\kappa L)h_1 + h_1^2$$
$$\beta(h_2) = (2 - 2\epsilon(\kappa L))h_2 - h_1^2 \tag{29}$$

where $\epsilon(\kappa L)$ is a function that interpolates between ϵ and $\epsilon + 1$. These couplings are the most illuminating for the dimensional crossover since both terms in the above expressions do not vary significantly. More explicitly

$$\epsilon(\kappa L) = \epsilon - \frac{1}{2J(\kappa L)}\kappa\partial_\kappa J(\kappa L) \tag{30}$$

The corresponding set of couplings to (g_1, g_2) give

$$\beta(h_1') = -2\epsilon'(\kappa L)h_1' + h_1'^2$$
$$\beta(h_2') = (2 - 2\epsilon'(\kappa L))h_2' - h_1'^2\frac{S(\kappa L)}{\tilde{J}(\kappa L)} \tag{31}$$

where

$$\epsilon'(\kappa L) = \frac{1}{2\tilde{J}}\kappa\frac{\partial}{\partial\kappa}\tilde{J}(\kappa L) \tag{32}$$

Before turning to the solution of these renormalization group equations let us examine the large κL limit of these equations. We note that in this limit the functions $J(\kappa L)$, $\tilde{J}(\kappa L)$ and $S(\kappa L)$ have the following asymptotic forms

$$J(\kappa L) = (\kappa L)^2(b + exponentially \quad small \quad terms)$$
$$\tilde{J}(\kappa L) = (\kappa L)^2(2b + exponentially \quad small \quad terms) \tag{33}$$
$$S(\kappa L) = (\kappa L)^2(b + exponentially \quad small \quad terms)$$

where

$$b = \frac{3\Gamma(1+\epsilon)\Gamma(2-\epsilon)^2}{(4\pi)^{3-\epsilon}\Gamma(4-2\epsilon)}$$

These functions are diverging in this limit because the volume of the internal manifold was absorbed into the couplings when we did the Fourier transform and this volume is now diverging. Thus the couplings (g_1, g_2) and (λ_1, λ_2) which are natural for the four-dimensional limit ($\kappa L \to 0$) are inappropriate for the six-dimensional limit ($\kappa L \to \infty$). We define six-dimensional couplings

$$\bar{g}_1 = (\kappa L)^2 g_1$$
$$\bar{g}_2 = (\kappa L)^2 g_2 \tag{34}$$

and equivalently for the set (λ_1, λ_2). Similarly we define functions

$$J'(\kappa L) = \frac{J(\kappa L)}{(\kappa L)^2}$$
$$\tilde{J}'(\kappa L) = \frac{J(\kappa L)}{(\kappa L)^2}$$

and

$$S'(\kappa L) = \frac{S(\kappa L)}{(\kappa L)^2}$$

Note that the coupling constants (h_1, h_2) and (h_1', h_2') naturally incorporate the above re-definitions in taking these limits. The renormalization group equations in this limit in terms of $(\bar{g}_1, \bar{g}_2)$ in six dimensions have the form

$$\beta(\bar{g}_1) = (2 - 2\epsilon)\bar{g}_1 + \bar{g}_1^2\tilde{J}'(\kappa L)$$
$$\beta(\bar{g}_2) = (4 - 2\epsilon)\bar{g}_2 - \bar{g}_1^2 S'(\kappa L) \tag{35}$$

where in the limit $\kappa L \to \infty$, $\tilde{J}'(\kappa L) \to b$ a constant. These are the natural six dimensional renormalization group equations for this system.

Since we have performed a renormalization of all terms necessary to make the results finite in the six-dimensional case and recover the four-dimensional renormalization group equations in the $\kappa L \to 0$ limit, we have renormalization group equations that interpolate, in what we believe to be a natural way, between the four and six dimensional theories. The non-triviality of the renormalization group equation for the additional coupling g_2 in six dimensions reflects the non-renormalizability of the six-dimensional theory. In general in higher loop calculations we expect the same features to persist, however, a proliferation of additional parameters will arise in our prescription. The essential feature of our work is that it brings into the realm of calculability the corrections due to additional dimensions, even when working with non renormalizable theories.

We now turn to the solutions of the renormalization group equations. We illustrate the method of solution by considering the equations (29) which we solve by noting that they can be rewritten in the form

$$\beta(h_1^{-1}exp[-\int_{\kappa_0}^{\kappa} 2\epsilon(xL)\tfrac{dx}{x}]) = -exp[-\int_{\kappa_0}^{\kappa} 2\epsilon(xL)\tfrac{dx}{x}]$$
$$\beta(h_2 exp[-\int_{\kappa_0}^{\kappa}(2-2\epsilon(xL))\tfrac{dx}{x}]) = -h_1^2 exp[-\int_{\kappa_0}^{\kappa}(2-2\epsilon(xL))\tfrac{dx}{x}] \quad (36)$$

and integrated without difficulty to obtain

$$h_1(\kappa) = \frac{h_1(\kappa_0)exp[-\int_{\kappa_0}^{\kappa} 2\epsilon(xL)\frac{dx}{x}]}{1 - h_1\int_{\kappa_0}^{\kappa}\frac{dy}{y}exp[-\int_{\kappa_0}^{y} 2\epsilon(xL)\frac{dx}{x}]}$$

$$\begin{aligned} h_2(\kappa) &= exp\left[\int_{\kappa_0}^{\kappa}(2-2\epsilon(xL))\frac{dx}{x}\right] \\ &\left[h_2(\kappa_0) - \int_{\kappa_0}^{\kappa} h_1(y)^2 exp[-\int_{\kappa_0}^{y}(2-2\epsilon(xL))\frac{dx}{x}]\frac{dy}{y}\right] \end{aligned} \quad (37)$$

In the above κ_0 is an initial renormalization point, and the solution tells us how the coupling changes as the renormalization point is changed. The solutions of the equations (25) are obtained by substituting back for the original variables. We obtain

$$\lambda_1(\rho) = \frac{\lambda_1(1)\rho^{-2\epsilon}}{1 - \lambda_1(1)\int_1^{\rho} dx x^{-2\epsilon-1}J(x\kappa_0 L)} \quad (38)$$

$$\lambda_2(\rho) = \rho^{2-2\epsilon}[\lambda_2(1) - \int_1^{\rho} dx x^{2\epsilon-3}\lambda_1^2(x)x^{-2\epsilon}J(x\kappa_0 L)] \quad (39)$$

where we have defined $\rho = \frac{\kappa}{\kappa_0}$. By direct analogy the solutions for the couplings (h_1', h_2') can be obtained yielding (g_1, g_2) to be

$$g_1(\rho) = \frac{g_1(1)\rho^{-2\epsilon}}{1 - g_1(1)\int_1^{\rho} dx x^{-2\epsilon-1}\tilde{J}(x\kappa_0 L)} \quad (40)$$

$$g_2(\rho) = \rho^{2-2\epsilon}[g_2(1) - \int_1^{\rho} dx g_1^2(x)x^{2\epsilon-3}S(x\kappa_0 L)] \quad (41)$$

Since in the four-dimensional limit the coupling g_2 does not get renormalized it seems natural to choose the normalization condition such that $g_2 = 0$ in which case the theory reduces exactly to the four-dimensional one. This of course is a form of fine tuning in six dimensions, however it is preserved by the renormalization group flow and is natural from the four dimensional point of view. Our initial assumption of imposing the relationship $g_2 \sim g_1^2$ is similar to the fine tuning of Coleman and Weinberg in the case of scalar electrodynamics [14]. Note that the non-renormalizability of the theory begins to become important when we begin to probe the theory at scales of order L.

In the limit $\kappa L \to 0$ $\tilde{J}(\kappa L) = a_2$ and (40) gives a Landau pole at $\rho = \rho^*$ [15]. If $g_2(1) = 0$ or is fine tuned to be small then $g_2(\rho)$ remains small relative to g_1 (and our assumption is valid) for $1 \leq \rho << \rho^*(\kappa L)$; otherwise our assumption is not self-consistent and other diagrams must be considered.

4 Conclusion

We have demonstrated to one-loop that a non-renormalizable theory does reduce to a renormalizable one as the extra-dimensions are shrunk to zero size. Though we have only addressed the question in the case of the four point function in this work, similar analysis can be carried out for the a general N-point function. Again one sees that our method generalizes, however additional renormalizations are necessary. By performing appropriate L dependent subtractions one can obtain renormalization group equations where each of the Wilson functions reduces in the limit $\kappa L \to 0$ to the ones obtained by a direct calculation in four dimensions. Thus we have an interpolation between the $L \to \infty$ case and the $L = 0$ case. We believe our prescription should be extendable to any order, at least in principle, though in practice this may be very tedious.

We can summarize our results as the decoupling of compact dimensions in the infrared domain, via decoupling of the infinite tower of modes. Zero modes give the leading contribution to physical amplitudes and the renormalization group equation in the limit of $\kappa L \to 0$, i.e. when heavy modes cannot be seen experimentally. It is in this sense that we have dimensional crossover from non-renormalizability to renormalizability. This makes the picture of dimensional reduction more plausible in that it appears self consistent at the quantum level.

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