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PHASE SPACE REPRESENTATIONS FOR SPIN 2

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1. Phase space representations (PSR's) in quantum theory (we mean formalisms, e.g., such as Wigner one/1,2/, where any density matrix is represented by some function of phase space variables) are not defined uniquely, unlike in classical theory. PSR's can be introduced using various forms of relevant completeness relations. For such a treatment see refs./15/(for quantum mechanics of particles), ref./16/ (for quantum field theory), refs./17,18/ (for quantum mechanics of spins $\frac{1}{2}$, 1 and $\frac{3}{2}$). Here PSR's for the spin 2 are introduced in this manner.

In PSR's expressions for expectation values, correlators, etc., are written like those in the classical probability theory, the phase space variables entering like "hidden variables". However, there are essential distinctions. In some PSR's densities are not positive definite (like the Wigner density/1/).[x)] In other PSR's densities are positive, however, expectation values, correlators, etc., include extra ("quantum") numerical factors. For example, it is mainly due to these factors expressions for the quantum correlator of components of two spins differ from that postulated by Bell/4-7/ with the reference to the classical probability theory. Now the Bell procedure /4-7/ produces quantum analogs of the Bell inequality, which are uncontradictory. Earlier Cirel'son/9/ has proposed another way to obtain quantum generalizations of the Bell inequality.

2. Main algebraic relations for spin 2 matrices are

$$\hat{s}_m\hat{s}_m = 2\cdot\mathbb{1}, \tag{1}$$

$$[\hat{s}_k\hat{s}_\ell] = i\,\varepsilon_{k\ell m}\hat{s}_m, \tag{2}$$

$$\begin{aligned}\{\hat{s}_{m_1}\hat{s}_{m_2}\hat{s}_{m_3}\hat{s}_{m_4}\hat{s}_{m_5}\} &= 10\,\delta_{m_1m_2}\{\hat{s}_{m_3}\hat{s}_{m_4}\hat{s}_{m_5}\} + 10\,\delta_{m_1m_3}\{\hat{s}_{m_2}\hat{s}_{m_4}\hat{s}_{m_5}\} +\\ &+ 10\,\delta_{m_1m_4}\{\hat{s}_{m_2}\hat{s}_{m_3}\hat{s}_{m_5}\} + 10\,\delta_{m_1m_5}\{\hat{s}_{m_2}\hat{s}_{m_3}\hat{s}_{m_4}\} +\\ &+ 10\,\delta_{m_2m_3}\{\hat{s}_{m_1}\hat{s}_{m_4}\hat{s}_{m_5}\} + 10\,\delta_{m_2m_4}\{\hat{s}_{m_1}\hat{s}_{m_3}\hat{s}_{m_5}\} + 10\,\delta_{m_2m_5}\{\hat{s}_{m_1}\hat{s}_{m_3}\hat{s}_{m_4}\} +\\ &+ 10\,\delta_{m_3m_4}\{\hat{s}_{m_1}\hat{s}_{m_2}\hat{s}_{m_5}\} + 10\,\delta_{m_3m_5}\{\hat{s}_{m_1}\hat{s}_{m_2}\hat{s}_{m_4}\} + 10\,\delta_{m_4m_5}\{\hat{s}_{m_1}\hat{s}_{m_2}\hat{s}_{m_3}\} -\end{aligned}$$

[x)] Negative probabilities were discussed by Feynman in his last papers/13,14/.

$$
\begin{aligned}
&-32(\delta_{m_2m_3}\delta_{m_4m_5}+\delta_{m_2m_4}\delta_{m_3m_5}+\delta_{m_2m_5}\delta_{m_3m_4})\hat{s}_{m_1}-\\
&-32(\delta_{m_1m_3}\delta_{m_4m_5}+\delta_{m_1m_4}\delta_{m_3m_5}+\delta_{m_1m_5}\delta_{m_3m_4})\hat{s}_{m_2}-\\
&-32(\delta_{m_1m_2}\delta_{m_4m_5}+\delta_{m_1m_4}\delta_{m_2m_5}+\delta_{m_1m_5}\delta_{m_2m_4})\hat{s}_{m_3}-\\
&-32(\delta_{m_1m_2}\delta_{m_3m_5}+\delta_{m_1m_3}\delta_{m_2m_5}+\delta_{m_1m_5}\delta_{m_2m_3})\hat{s}_{m_4}-\\
&-32(\delta_{m_1m_2}\delta_{m_3m_4}+\delta_{m_1m_3}\delta_{m_2m_4}+\delta_{m_1m_4}\delta_{m_2m_3})\hat{s}_{m_5}\,. \qquad (3)
\end{aligned}
$$

In eq.(3) and in what follows the braces mean the total symmetrization without division by $n!$:

$$\{\hat{s}_i\hat{s}_j\}=\hat{s}_i\hat{s}_j+\hat{s}_j\hat{s}_i\,, \qquad (4)$$

$$\{\hat{s}_i\hat{s}_j\hat{s}_k\}=(\hat{s}_i\hat{s}_j+\hat{s}_j\hat{s}_i)\hat{s}_k+\hat{s}_k(\hat{s}_i\hat{s}_j+\hat{s}_j\hat{s}_i)+\hat{s}_i\hat{s}_k\hat{s}_j+\hat{s}_j\hat{s}_k\hat{s}_i\,,$$

and so on. Equation (3) follows from the identity

$$\Pi_2(\vec{x}\hat{\vec{s}})=[(\vec{x}\hat{\vec{s}})^2-2^2\vec{x}^2\mathbb{1}][(\vec{x}\hat{\vec{s}})^2-\vec{x}^2\mathbb{1}](\vec{x}\hat{\vec{s}})\equiv 0\,, \qquad (5)$$

where $\Pi_2(y)$ is the minimal annihilation polynomial for $\vec{x}\hat{\vec{s}}$, $\vec{x}$ is an arbitrary unnormalized 3-vector (cf. ref. /18/, see there further information about the spin matrices).

The 5 x 5 matrices $\mathbb{1}$, $\hat{s}_i$, $\{\hat{s}_i\hat{s}_j\}$, $\{\hat{s}_i\hat{s}_j\hat{s}_k\}$ and $\{\hat{s}_i\hat{s}_j\hat{s}_k\hat{s}_l\}$ form a total basis with the completeness relation

$$
\begin{aligned}
&\frac{5}{2\cdot3}\,|\mathbb{1}|\otimes\|\mathbb{1}\|-\frac{47}{2^2\cdot3^3}\,|\hat{s}_i|\otimes\|\hat{s}_j\|-\frac{127}{2^5\cdot3^3}\,|\{\hat{s}_i\hat{s}_j\}|\otimes\|\{\hat{s}_i\hat{s}_j\}\|+\\
&+\frac{1}{2^4\cdot3^4}\,|\{\hat{s}_i\hat{s}_j\hat{s}_k\}|\otimes\|\{\hat{s}_i\hat{s}_j\hat{s}_k\}\|+\frac{1}{2^8\cdot3^4}\,|\{\hat{s}_i\hat{s}_j\hat{s}_k\hat{s}_l\}|\otimes\|\{\hat{s}_i\hat{s}_j\hat{s}_k\hat{s}_l\}\|= \qquad (6)\\
&=|\mathbb{1}\|\otimes\|\mathbb{1}|\,.
\end{aligned}
$$

Here $|\cdots|$, $\|\cdots\|$, $|\cdots\|$, $\|\cdots|$ denote matrices with pointing out the position of matrix indices: e.g., if $|\hat{s}_i|\otimes\|\hat{s}_i\|$ means $(\hat{s}_i)_{\alpha\beta}(\hat{s}_i)_{\gamma\delta}$, then $|\mathbb{1}\|\otimes\|\mathbb{1}|$ means $(\mathbb{1})_{\alpha\delta}(\mathbb{1})_{\gamma\beta}=\delta_{\alpha\delta}\delta_{\gamma\beta}$.

3. The density matrices of interest are defined by

$$(\vec{\alpha}\hat{\vec{s}})\hat{\rho}(m,\vec{\alpha})=\hat{\rho}(m,\vec{\alpha})(\vec{\alpha}\hat{\vec{s}})=m\hat{\rho}(m,\vec{\alpha})\,, \qquad (7)$$

where $m=0,1,2$; $\vec{\alpha}^2=1$. They can be constructed out as the Lagrange - Silvester polynomials, which can be expressed via the minimal annihilation polynomial :

$$\hat{\rho}(2,\vec{\alpha})=|2,\vec{\alpha}\rangle\langle 2,\vec{\alpha}|=N_2\,\frac{\Pi_2(y)}{y-2}\Big|_{y=\vec{\alpha}\hat{\vec{s}}}=$$

$$=\frac{1}{24}\,[(\vec{\alpha}\hat{\vec{s}})+2\cdot\mathbb{1}][(\vec{\alpha}\hat{\vec{s}})^2-\mathbb{1}](\vec{\alpha}\hat{\vec{s}})\,, \qquad (8)$$

$$\hat{\rho}(1,\vec{\alpha})=|1,\vec{\alpha}\rangle\langle 1,\vec{\alpha}|=N_1\,\frac{\Pi_2(y)}{y-1}\Big|_{y=\vec{\alpha}\hat{\vec{s}}}=$$

$$=-\frac{1}{6}\,[(\vec{\alpha}\hat{\vec{s}})^2-4\cdot\mathbb{1}][(\vec{\alpha}\hat{\vec{s}})+\mathbb{1}](\vec{\alpha}\hat{\vec{s}})\,, \qquad (9)$$

$$\hat{\rho}(0,\vec{\alpha})=|0,\vec{\alpha}\rangle\langle 0,\vec{\alpha}|=N_0\,\frac{\Pi_2(y)}{y}\Big|_{y=\vec{\alpha}\hat{\vec{s}}}=$$

$$=\frac{1}{4}\,[(\vec{\alpha}\hat{\vec{s}})^2-4\cdot\mathbb{1}][(\vec{\alpha}\hat{\vec{s}})^2-\mathbb{1}]\,. \qquad (10)$$

The normalization constants N_m follow from the normalization conditions

$$\mathrm{tr}\,\hat{\rho}(m,\vec{\alpha})=1\,. \qquad (11)$$

The density matrices for negative m are obtained from expressions (8) and (9) simply by replacing $\vec{\alpha}$ with $-\vec{\alpha}$. Summation over m gives

$$\sum_{m=-2,-1,0,1,2}\hat{\rho}(m,\vec{\alpha})=\mathbb{1}\,. \qquad (12)$$

Expectation values of spin components are given by

$$\mathrm{tr}\,[\hat{s}_i\,\hat{\rho}(m,\vec{\alpha})]=m\,\alpha_i \qquad (13)$$

The probability of finding the spin component n along $\vec{b}$ in the state with the spin component m along $\vec{\alpha}$ is

$$\rho(n,\vec{b};m,\vec{\alpha})=\mathrm{tr}\,[\hat{\rho}(n,\vec{b})\,\hat{\rho}(m,\vec{\alpha})]\,. \qquad (14)$$

These probabilities are given in Table 1. For their calculation see Appendix B. The sum of the probabilities equals unity:

$$\sum_{n=-2,-1,0,1,2}\rho(n,\vec{b};m,\vec{\alpha})=\sum_{m=-2,-1,0,1,2}\rho(n,\vec{b};m,\vec{\alpha})=1\,. \qquad (15)$$

4. Completeness relation (6) for the 5x5 matrices can be written in terms of the above density matrices $\hat{\rho}(m,\vec{\alpha})$ and in some other relative forms as follows

$$\sum_{m=0,1,2}v_m\int d\mu(\vec{s})\,|\hat{\rho}(m,\vec{s})|\otimes\|\hat{\rho}(m,\vec{s})\|=|\mathbb{1}|\otimes\|\mathbb{1}\|+|\mathbb{1}\|\otimes\|\mathbb{1}|\,, \qquad (16)$$

$$v_0=10\,,\quad v_1=\frac{50}{3}\,,\quad v_2=\frac{10}{3}\,, \qquad (16.a)$$

$$5\int d\mu(\vec{s})\,|\hat{\rho}(m,\vec{s})|\otimes\|\hat{X}(m,\vec{s})\|=|\mathbb{1}\|\otimes\|\mathbb{1}|\,, \qquad (17)$$

$$5\int d\mu(\vec{s})\,|\hat{Y}(\vec{s})|\otimes\|\hat{Y}(\vec{s})\|=|\mathbb{1}\|\otimes\|\mathbb{1}|\,. \qquad (18)$$

Table 1. Spin 2. $16\,\rho(n,\vec{b};m,\vec{\alpha})$

$m \backslash n$	2	1	0	-1	-2
2	$(1+\vec{a}\vec{b})^4$	$4(1+\vec{a}\vec{b})^3(1-\vec{a}\vec{b})$	$6[1-(\vec{a}\vec{b})^2]^2$	$4(1+\vec{a}\vec{b})(1-\vec{a}\vec{b})^3$	$(1-\vec{a}\vec{b})^4$
1	$4(1+\vec{a}\vec{b})^3(1-\vec{a}\vec{b})$	$4(1+\vec{a}\vec{b})^2(1-2\vec{a}\vec{b})^2$	$24(\vec{a}\vec{b})^2[1-(\vec{a}\vec{b})^2]$	$4(1-\vec{a}\vec{b})^2(1+2\vec{a}\vec{b})^2$	$4(1-\vec{a}\vec{b})^3(1+\vec{a}\vec{b})$
0	$6[1-(\vec{a}\vec{b})^2]^2$	$24(\vec{a}\vec{b})^2[1-(\vec{a}\vec{b})^2]$	$4[1-3(\vec{a}\vec{b})^2]^2$	$24(\vec{a}\vec{b})^2[1-(\vec{a}\vec{b})^2]$	$6[1-(\vec{a}\vec{b})^2]^2$
-1	$4(1-\vec{a}\vec{b})^3(1+\vec{a}\vec{b})$	$4(1-\vec{a}\vec{b})^2(1+2\vec{a}\vec{b})^2$	$24(\vec{a}\vec{b})^2[1-(\vec{a}\vec{b})^2]$	$4(1+\vec{a}\vec{b})^2(1-2\vec{a}\vec{b})^2$	$4(1+\vec{a}\vec{b})^3(1-\vec{a}\vec{b})$
-2	$(1-\vec{a}\vec{b})^4$	$4(1-\vec{a}\vec{b})^3(1+\vec{a}\vec{b})$	$6[1-(\vec{a}\vec{b})^2]^2$	$4(1-\vec{a}\vec{b})(1+\vec{a}\vec{b})^3$	$(1+\vec{a}\vec{b})^4$

Here $d\mu(\vec{s})=\frac{1}{2\pi}\delta(\vec{s}^2-1)\,d^3s$ is the measure on S^2. The sphere $S^2(\vec{s}^2=1)$ serves as a phase space, and $\vec{s}=(s_1,s_2,s_3)$ are phase space variables. Relations (17) and (18), being considered as equations in $\hat{X}(m,\vec{s})$ and $\hat{Y}(\vec{s})$, can be solved (except for $\hat{X}(0,\vec{s})$, the reasoning see below) to obtain

$$\hat{X}(1,\vec{\alpha})=\alpha\cdot\mathbb{1}+\beta(\vec{\alpha}\hat{\vec{s}})+\gamma(\vec{\alpha}\hat{\vec{s}})^2+\delta(\vec{\alpha}\hat{\vec{s}})^3+\varepsilon(\vec{\alpha}\hat{\vec{s}})^4, \tag{19}$$

$$\alpha=-\frac{1}{2},\ \beta=\frac{31}{12},\ \gamma=\frac{7\cdot 11}{16},\ \delta=-\frac{7}{12},\ \varepsilon=-\frac{3\cdot 7}{16}; \tag{19.a}$$

$$\hat{X}(2,\vec{\alpha})=\alpha\cdot\mathbb{1}+\beta(\vec{\alpha}\hat{\vec{s}})+\gamma(\vec{\alpha}\hat{\vec{s}})^2+\delta(\vec{\alpha}\hat{\vec{s}})^3+\varepsilon(\vec{\alpha}\hat{\vec{s}})^4, \tag{20}$$

$$\alpha=10,\ \beta=-\frac{11}{3},\ \gamma=-\frac{7\cdot 13}{4},\ \delta=\frac{7}{6},\ \varepsilon=\frac{3\cdot 7}{4}; \tag{20.a}$$

$$\hat{Y}(\vec{\alpha})=\alpha\cdot\mathbb{1}+\beta(\vec{\alpha}\hat{\vec{s}})+\gamma(\vec{\alpha}\hat{\vec{s}})^2+\delta(\vec{\alpha}\hat{\vec{s}})^3+\varepsilon(\vec{\alpha}\hat{\vec{s}})^4, \tag{21}$$

$$\alpha=\frac{1}{5}[-2(5\gamma+17\varepsilon)\pm 1],\quad \beta=\frac{1}{5}\left(-17\delta\pm\frac{1}{2}\sqrt{6}\right),$$

$$\gamma=-\frac{31}{7}\varepsilon\pm\frac{1}{2}\sqrt{\frac{2}{7}},\quad \delta=\pm\frac{1}{6}\sqrt{\frac{7}{2}},\quad \varepsilon=\pm\frac{1}{4}\sqrt{\frac{7}{2}}. \tag{21.a}$$

In the latter case any combination of signs is acceptable. Therefore there exist several solutions for $\hat{Y}$. The matrices $\hat{\rho}(m,\vec{s})$ (all m), $\hat{X}(m,\vec{s})$ (m = 1,2) and $\hat{Y}(\vec{s})$ satisfy the conditions

$$\mathrm{tr}\,\hat{\rho}(m,\vec{s})=1, \tag{22}$$

$$\mathrm{tr}\,\hat{X}(m,\vec{s})=5\alpha+10\gamma+34\varepsilon=1, \tag{23}$$

$$\mathrm{tr}\,\hat{Y}(\vec{s})=5\alpha+10\gamma+34\varepsilon=\pm 1, \tag{24}$$

$$5\int d\mu(\vec{s})\,\hat{\rho}(m,\vec{s})=\mathbb{1}, \tag{25}$$

$$5\int d\mu(\vec{s})\,\hat{X}(m,\vec{s})=(5\alpha+10\gamma+34\varepsilon)\cdot\mathbb{1}=\mathbb{1}, \tag{26}$$

$$5\int d\mu(\vec{s})\,\hat{Y}(\vec{s})=(5\alpha+10\gamma+34\varepsilon)\cdot\mathbb{1}=\pm\mathbb{1}. \tag{27}$$

In the r.h.s, of eqs. (24) and (27) the signs correspond to those in α (21.a). Note that eq. (25) is the completeness relation for the spin states $|m,\vec{s}\rangle$ (see eqs. (8)-(10)), treated as coherent states.[x] It is the usual property of many sets of coherent states, that expectation values of any operator in the coherent states represent uniquely this operator. This is the case for the states $|1,\vec{s}\rangle$ and $|2,\vec{s}\rangle$, but for $|0,\vec{s}\rangle$. It is clear from eq. (13): the zero expectation values correspond not only to the operator zero, $\mathbf{0}$, but also to the operators $\hat{S}_i$. This situation is common for any integer spin. It is due to this fact that eq. (17) cannot be solved for $\hat{X}(0,\vec{s})$.

x) See refs. /10-12/.

5. Definition of PSR's. The completeness relation (16) leads to the three-component representative for any observable

$$\left(F_0(\vec{s}),F_1(\vec{s}),F_2(\vec{s})\right)=\left(tr(\hat{\rho}(0,\vec{s})\hat{F}),tr(\hat{\rho}(1,\vec{s})\hat{F}),tr(\hat{\rho}(2,\vec{s})\hat{F})\right), \tag{28}$$

while the completeness relations (17) and (18) permit us to introduce the one-component representatives

$$F_1(\vec{s})=tr(\hat{\rho}(1,\vec{s})\hat{F}), \tag{29}$$

$$F_2(\vec{s})=tr(\hat{\rho}(2,\vec{s})\hat{F}), \tag{30}$$

$$F_{1'}(\vec{s})=tr(\hat{X}(1,\vec{s})\hat{F}), \tag{31}$$

$$F_{2'}(\vec{s})=tr(\hat{X}(2,\vec{s})\hat{F}), \tag{32}$$

$$F_Y(\vec{s})=tr(\hat{Y}(\vec{s})\hat{F}). \tag{33}$$

Each of these functions can represent the operator $\hat{F}$.

Restoration theorems. The completeness relations (16)-(18) guarantee that any operator can be restored via its representatives

$$\hat{F}=-\mathbb{1}\cdot tr\hat{F}+\sum_{m=0,1,2}v_m\int d\mu(\vec{s})\hat{\rho}(m,\vec{s})F_m(\vec{s})= \tag{34.a}$$

$$=5\int d\mu(\vec{s})\hat{X}(1,\vec{s})F_1(\vec{s})=5\int d\mu(\vec{s})\hat{X}(2,\vec{s})F_2(\vec{s})=$$
$$=5\int d\mu(\vec{s})\hat{\rho}(1,\vec{s})F_{1'}(\vec{s})=5\int d\mu(\vec{s})\hat{\rho}(2,\vec{s})F_{2'}(\vec{s})=$$
$$=5\int d\mu(\vec{s})\hat{Y}(\vec{s})F_Y(\vec{s}). \tag{34.b}$$

The trace of $\hat{F}$ can be expressed via any of its representatives (via any component of its three-component representative in the first case):

$$tr\hat{F}=5\int d\mu(\vec{s})F_m(\vec{s})= \qquad m=0,1,2,$$
$$=5\int d\mu(\vec{s})F_{m'}(\vec{s})= \qquad m'=1',2',$$
$$=5\int d\mu(\vec{s})F_Y(\vec{s}). \tag{35}$$

The trace of the product of two operators in terms of their representatives is given by

$$tr(\hat{F}\hat{G})=-tr\hat{F}\cdot tr\hat{G}+\sum_{m=0,1,2}v_m\int d\mu(\vec{s})F_m(\vec{s})G_m(\vec{s})= \tag{36.a}$$

$$=5\int d\mu(\vec{s})F_{1'}(\vec{s})G_1(\vec{s})=5\int d\mu(\vec{s})F_1(\vec{s})G_{1'}(\vec{s})=$$
$$=5\int d\mu(\vec{s})F_{2'}(\vec{s})G_2(\vec{s})=5\int d\mu(\vec{s})F_2(\vec{s})G_{2'}(\vec{s})=$$
$$=5\int d\mu(\vec{s})F_Y(\vec{s})G_Y(\vec{s}). \tag{36.b}$$

Let us give as examples the following expectation values

$$tr(\hat{F}\hat{\rho}(m,\vec{\alpha}))=-tr\hat{F}+\sum_{n=0,1,2}v_n\int d\mu(\vec{s})F_n(\vec{s})\rho(n,\vec{s};m,\vec{\alpha})= \tag{37.a}$$

$$=5\int d\mu(\vec{s})F_{1'}(\vec{s})\rho_1(\vec{s};m,\vec{\alpha})=5\int d\mu(\vec{s})F_1(\vec{s})\rho_{1'}(\vec{s};m,\vec{\alpha})=$$
$$=5\int d\mu(\vec{s})F_{2'}(\vec{s})\rho_2(\vec{s};m,\vec{\alpha})=5\int d\mu(\vec{s})F_2(\vec{s})\rho_{2'}(\vec{s};m,\vec{\alpha})=$$
$$=5\int d\mu(\vec{s})F_Y(\vec{s})\rho_Y(\vec{s};m,\vec{\alpha}). \tag{37.b}$$

The representatives of the density matrices $\hat{\rho}(0,\vec{\alpha})$, $\hat{\rho}(1,\vec{\alpha})$ and $\hat{\rho}(2,\vec{\alpha})$ are given by (PSR's A),...,F))

A) $$\{\rho(0,\vec{s};0,\vec{\alpha}),\rho(1,\vec{s};0,\vec{\alpha}),\rho(2,\vec{s};0,\vec{\alpha})\}=$$
$$=\left\{\tfrac{1}{4}[1-3(\vec{s}\vec{\alpha})^2]^2,\ \tfrac{3}{2}(\vec{s}\vec{\alpha})^2[1-(\vec{s}\vec{\alpha})^2],\ \tfrac{3}{8}[1-(\vec{s}\vec{\alpha})^2]^2\right\}, \tag{38}$$

$$\{\rho(0,\vec{s};1,\vec{\alpha}),\rho(1,\vec{s};1,\vec{\alpha}),\rho(2,\vec{s};1,\vec{\alpha})\}=$$
$$=\left\{\tfrac{3}{2}(\vec{s}\vec{\alpha})^2[1-(\vec{s}\vec{\alpha})^2],\ \tfrac{1}{4}(1+\vec{s}\vec{\alpha})^2(1-2\vec{s}\vec{\alpha})^2,\ \tfrac{1}{4}(1+\vec{s}\vec{\alpha})^3(1-\vec{s}\vec{\alpha})\right\}, \tag{39}$$

$$\{\rho(0,\vec{s};2,\vec{\alpha}),\rho(1,\vec{s};2,\vec{\alpha}),\rho(2,\vec{s};2,\vec{\alpha})\}=$$
$$=\left\{\tfrac{3}{8}[1-(\vec{s}\vec{\alpha})^2]^2,\ \tfrac{1}{4}(1+\vec{s}\vec{\alpha})^3(1-\vec{s}\vec{\alpha}),\ \tfrac{1}{16}(1+\vec{s}\vec{\alpha})^4\right\}, \tag{40}$$

B) $$\rho_1(\vec{s};0,\vec{\alpha})=tr(\hat{\rho}(1,\vec{s})\hat{\rho}(0,\vec{\alpha}))=\tfrac{3}{2}(\vec{s}\vec{\alpha})^2[1-(\vec{s}\vec{\alpha})^2], \tag{41}$$

$$\rho_1(\vec{s};1,\vec{\alpha})=tr(\hat{\rho}(1,\vec{s})\hat{\rho}(1,\vec{\alpha}))=\tfrac{1}{4}(1+\vec{s}\vec{\alpha})^2(1-2\vec{s}\vec{\alpha})^2, \tag{42}$$

$$\rho_1(\vec{s};2,\vec{\alpha})=tr(\hat{\rho}(1,\vec{s})\hat{\rho}(2,\vec{\alpha}))=\tfrac{1}{4}(1+\vec{s}\vec{\alpha})^3(1-\vec{s}\vec{\alpha}), \tag{43}$$

C) $$\rho_2(\vec{s};0,\vec{\alpha})=tr(\hat{\rho}(2,\vec{s})\hat{\rho}(0,\vec{\alpha}))=\tfrac{3}{8}[1-(\vec{s}\vec{\alpha})^2]^2, \tag{44}$$

$$\rho_2(\vec{s};1,\vec{\alpha})=tr(\hat{\rho}(2,\vec{s})\hat{\rho}(1,\vec{\alpha}))=\tfrac{1}{4}(1+\vec{s}\vec{\alpha})^3(1-\vec{s}\vec{\alpha}), \tag{45}$$

$$\rho_2(\vec{s};2,\vec{\alpha})=tr(\hat{\rho}(2,\vec{s})\hat{\rho}(2,\vec{\alpha}))=\tfrac{1}{16}(1+\vec{s}\vec{\alpha})^4, \tag{46}$$

D) $$\rho_{1'}(\vec{s};0,\vec{\alpha})=tr(\hat{X}(1,\vec{s})\hat{\rho}(0,\vec{\alpha}))$$
$$=\tfrac{1}{16}[-29+2\cdot3\cdot5\cdot7(\vec{s}\vec{\alpha})^2-3^3\cdot7(\vec{s}\vec{\alpha})^4], \tag{47}$$

$$\rho_{1'}(\vec{s};1,\vec{\alpha})=tr(\hat{X}(1,\vec{s})\hat{\rho}(1,\vec{\alpha}))=$$
$$=\tfrac{1}{8}[3-12(\vec{s}\vec{\alpha})-2\cdot3\cdot7(\vec{s}\vec{\alpha})^2+4\cdot7(\vec{s}\vec{\alpha})^3+3^2\cdot7(\vec{s}\vec{\alpha})^4]=$$
$$=\tfrac{1}{5}[1+3P_1(\vec{s}\vec{\alpha})+5P_2(\vec{s}\vec{\alpha})+7P_3(\vec{s}\vec{\alpha})+9P_4(\vec{s}\vec{\alpha})]. \tag{48}$$

This function can be replaced by

$$\rho_{1''}(\vec{s};1,\vec{\alpha}) = \frac{1}{5}\,\delta_{S^2}(\vec{s},\vec{\alpha}) = \frac{1}{5}\lim_{\eta\to 1_-}\frac{1-\eta^2}{[1-2\eta(\vec{s}\vec{\alpha})+\eta^2]^{3/2}}\ {}^{x)}$$

$$= \frac{1}{5}\lim_{\eta\to 1_-}\Big[1+\sum_{\ell=1}^{\infty}(2\ell+1)\eta^{\ell}P_{\ell}(\vec{s}\vec{\alpha})\Big] = \frac{1}{5}\Big[1+\sum_{\ell=1}^{\infty}(2\ell+1)P_{\ell}(\vec{s}\vec{\alpha})\Big], \quad (49)$$

$$\rho_{1'}(\vec{s};2,\vec{\alpha}) = tr\big(\hat{X}(1,\vec{s})\,\hat{\rho}(2,\vec{\alpha})\big) =$$

$$= \frac{1}{32}\big[33+2^3\cdot 3^2(\vec{s}\vec{\alpha})-42(\vec{s}\vec{\alpha})^2-2^3\cdot 7(\vec{s}\vec{\alpha})^3-3^2\cdot 7(\vec{s}\vec{\alpha})^4\big], \quad (50)$$

E) $$\rho_{2'}(\vec{s};0,\vec{\alpha}) = tr\big(\hat{X}(2,\vec{s})\,\hat{\rho}(0,\vec{\alpha})\big) =$$

$$= \frac{1}{4}\big[19-2^3\cdot 3\cdot 7(\vec{s}\vec{\alpha})^2+3^3\cdot 7(\vec{s}\vec{\alpha})^4\big], \quad (51)$$

$$\rho_{2'}(\vec{s};1,\vec{\alpha}) = tr\big(\hat{X}(2,\vec{s})\,\hat{\rho}(1,\vec{\alpha})\big) =$$

$$= \frac{1}{4}\big[-9+18(\vec{s}\vec{\alpha})+3\cdot 5\cdot 7(\vec{s}\vec{\alpha})^2-2^2\cdot 7(\vec{s}\vec{\alpha})^3-2\cdot 3^2\cdot 7(\vec{s}\vec{\alpha})^4\big] \quad (52)$$

$$\rho_{2'}(\vec{s};2,\vec{\alpha}) = tr\big(\hat{X}(2,\vec{s})\,\hat{\rho}(2,\vec{\alpha})\big) =$$

$$= \frac{1}{8}\big[3-12(\vec{s}\vec{\alpha})-2\cdot 3\cdot 7(\vec{s}\vec{\alpha})^2+2^2\cdot 7(\vec{s}\vec{\alpha})^3+3^2\cdot 7(\vec{s}\vec{\alpha})^4\big] =$$

$$= \frac{1}{5}\big[1+3P_1(\vec{s}\vec{\alpha})+5P_2(\vec{s}\vec{\alpha})+7P_3(\vec{s}\vec{\alpha})+9P_4(\vec{s}\vec{\alpha})\big]. \quad (53)$$

This function can be replaced by

$$\rho_{2''}(\vec{s};2,\vec{\alpha}) = \frac{1}{5}\,\delta_{S^2}(\vec{s},\vec{\alpha}) \quad (54)$$

F) $$\rho_{Y}(\vec{s};0,\vec{\alpha}) = tr\big(\hat{Y}(\vec{s})\,\hat{\rho}(0,\vec{\alpha})\big) =$$

$$= \alpha+3\gamma+12\varepsilon-3(\gamma+7\varepsilon)(\vec{s}\vec{\alpha})^2+9\varepsilon(\vec{s}\vec{\alpha})^4. \quad (55)$$

$$\rho_{Y}(\vec{s};1,\vec{\alpha}) = tr\big(\hat{Y}(\vec{s})\,\hat{\rho}(1,\vec{\alpha})\big) =$$

$$= \alpha+\frac{5}{2}\gamma+\frac{17}{2}\varepsilon+(\beta+7\delta)(\vec{s}\vec{\alpha})-\frac{3}{2}(\gamma+\varepsilon)(\vec{s}\vec{\alpha})^2-6\delta(\vec{s}\vec{\alpha})^3-6\varepsilon(\vec{s}\vec{\alpha})^4 \quad (56)$$

$$\rho_{Y}(\vec{s};2,\vec{\alpha}) = tr\big(\hat{Y}(\vec{s})\,\hat{\rho}(2,\vec{\alpha})\big) =$$

$$= \alpha+\gamma+\frac{5}{2}\varepsilon+(2\beta+5\delta)(\vec{s}\vec{\alpha})+3(\gamma+4\varepsilon)(\vec{s}\vec{\alpha})^2+3\delta(\vec{s}\vec{\alpha})^3+\frac{3}{2}\varepsilon(\vec{s}\vec{\alpha})^4. \quad (57)$$

Here $P_l(x)$ are the Legendre polynomials, and $\delta_{S^2}(\vec{s},\vec{\alpha})$ is the δ-function on the sphere S^2. The replacements indicated are possible since the representatives ρ are always integrated with polynomials in $\vec{s}$ maximum to fourth power. In the representation F), $\alpha,\beta,\gamma,\delta,\varepsilon$ are given by eq. (21.a).

All the above densities are normalized as follows

$$5\int d\mu(\vec{s})\,\rho(n,\vec{s};m,\vec{\alpha}) = 1 \qquad n=0,1,2 \quad (58)$$

$$5\int d\mu(\vec{s})\,\rho_i(\vec{s};m,\vec{\alpha}) = 1\,, \quad (59)$$

where i = 1,2,1',1" (for m = 1), 2',2" (for m = 2), Y.

x) The power $\frac{3}{2}$ of the denominator was missed in eq.(71) of ref./18/.

The corresponding (in the sense of eq. (36)) representatives of any component of spin 2 are given by

A) $$\{tr[(\vec{b}\hat{\vec{s}})\hat{\rho}(0,\vec{s})],\ tr[(\vec{b}\hat{\vec{s}})\hat{\rho}(1,\vec{s})],\ tr[(\vec{b}\hat{\vec{s}})\hat{\rho}(2,\vec{s})]\} = \{0,(\vec{b}\vec{s}),2(\vec{b}\vec{s})\}, \quad (60)$$

B) $$(\vec{b}\hat{\vec{s}})_{1'} = tr[(\vec{b}\hat{\vec{s}})\hat{X}(1,\vec{s})] = 2(5\beta+17\delta)(\vec{b}\vec{s}) = 6(\vec{b}\vec{s}), \quad (61)$$

C) $$(\vec{b}\hat{\vec{s}})_{2'} = tr[(\vec{b}\hat{\vec{s}})\hat{X}(2,\vec{s})] = 2(5\beta+17\delta)(\vec{b}\vec{s}) = 3(\vec{b}\vec{s}), \quad (62)$$

D) $$(\vec{b}\hat{\vec{s}})_{1} = tr[(\vec{b}\hat{\vec{s}})\hat{\rho}(1,\vec{s})] = (\vec{b}\vec{s}), \quad (63)$$

E) $$(\vec{b}\hat{\vec{s}})_{2} = tr[(\vec{b}\hat{\vec{s}})\hat{\rho}(2,\vec{s})] = 2(\vec{b}\vec{s}), \quad (64)$$

F) $$(\vec{b}\hat{\vec{s}})_{Y} = tr[(\vec{b}\hat{\vec{s}})\hat{Y}(\vec{s})] = 2(5\beta+17\delta)(\vec{b}\vec{s}) = \pm\sqrt{s(s+1)}(\vec{b}\vec{s}) = \pm\sqrt{6}(\vec{b}\vec{s}) \quad (65)$$

In the latter case the signs correspond to those in β (21.a).

In PSR's the expectation values of the spin component $(\vec{b}\hat{\vec{s}})$ can be represented as follows

$$tr[(\vec{b}\hat{\vec{s}})\hat{\rho}(m,\vec{\alpha})] = \int d\mu(\vec{s})(\vec{b}\vec{s})\big[v_1\,\rho(1,\vec{s};m,\vec{\alpha})+2v_2\,\rho(2,\vec{s};m,\vec{\alpha})\big] = \quad (66.a)$$

$$= 5w_i\int d\mu(\vec{s})(\vec{b}\vec{s})\,\rho_i(\vec{s};m,\vec{\alpha}), \quad (66.b)$$

where v_1 and v_2 are given by eqs. (16.a), i = 1,2,1' (or 1" for m = 1), 2' (or 2" for m = 2), Y,

$$w_1=6\,,\ w_2=3\,,\ w_{1'}=w_{1''}=1,\ w_{2'}=w_{2''}=2$$

$$w_Y = 2(5\beta+17\delta) = \pm\sqrt{s(s+1)} = \pm\sqrt{6} \quad (67)$$

$$w_i w_{i'} = s(s+1) = 6\,,\quad w_Y^2 = s(s+1) = 6. \quad (68)$$

This is the case for any spin s.

6. The singlet state of two spins 2 in PSR's. The singlet state is defined by

$$(\hat{s}^a_m+\hat{s}^b_m)|singlet\rangle = 0 \qquad (m=1,2,3), \quad (69)$$

$$(\hat{s}^a_m+\hat{s}^b_m)\hat{\rho}^{\,singlet} = \hat{\rho}^{\,singlet}(\hat{s}^a_m+\hat{s}^b_m) = 0, \quad (70)$$

$$|singlet\rangle = \frac{1}{\sqrt{5}}\Big(u^a(2)\otimes u^b(-2)-u^a(1)\otimes u^b(-1)+u^a(0)\otimes u^b(0)- -u^a(-1)\otimes u^b(1)+u^a(-2)\otimes u^b(2)\Big), \quad (71)$$

$$\hat{\rho}^{\,singlet} = |singlet\rangle\langle singlet|. \quad (72)$$

In fact, the singlet state is independent of any quantization axis, and the density matrix $\hat{\rho}^{\,singlet}$ can be written manifestly independent of it (cf. refs./17/ and /18/). However, the state vector in the form (71) implies the use of some quantization axis. Nevertheless, this defect can be employed to simplify calculations (see Appendix B and ref./18/).

The probabilities to find definite components of two spins 2 in the singlet state are expressed via the one-spin probabilities

$$\rho(m,\vec a;n,\vec b\,|\,\text{singlet}) = \operatorname*{tr}_a \operatorname*{tr}_b\left[\hat\rho^{a}(m,\vec a)\hat\rho^{b}(n,\vec b)\hat\rho^{\,\text{singlet}}\right] =$$
$$= \hat\rho^{\,a}_{\alpha'\alpha}(m,\vec a)\hat\rho^{\,b}_{\beta\beta'}(n,\vec b)\,\hat\rho^{\,\text{singlet}}_{\alpha\beta,\alpha'\beta'} =$$
$$= \frac{1}{5}\rho(m,\vec a;-n,\vec b) = \frac{1}{5}\rho(m,\vec a;n,-\vec b) =$$
$$= \frac{1}{5}\rho(-m,\vec a;n,\vec b) = \frac{1}{5}\rho(m,-\vec a;n,\vec b) \tag{73}$$

(see, e.g., refs.[8,18])

$$\sum_{m,n=-2,-1,0,1,2}\ \sum \rho(m,\vec a;n,\vec b\,|\,\text{singlet}) = 1\,, \tag{74}$$

$$5^2\int d\mu(\vec a)\int d\mu(\vec b)\,\rho(m,\vec a;n,\vec b\,|\,\text{singlet}) = 1. \tag{75}$$

The probabilities $\rho(m,\vec a;n,\vec b\,|\,\text{singlet})$ are given explicitly in Table 2.

The singlet state can be represented in PSR's

A) by the 9-component representative

$$\rho(m,\vec a;n,\vec b\,|\,\text{singlet}) \qquad m,n=0,1,2, \tag{76}$$

or by the following 1-component representatives

B) $$\rho_1(\vec a,\vec b\,|\,\text{singlet}) = \rho(1,\vec a;1,\vec b\,|\,\text{singlet}) = \frac{1}{20}(1-\vec a\vec b)^2(1+2\vec a\vec b)^2, \tag{77}$$

C) $$\rho_2(\vec a,\vec b\,|\,\text{singlet}) = \rho(2,\vec a;2,\vec b\,|\,\text{singlet}) = \frac{1}{80}(1-\vec a\vec b)^4, \tag{78}$$

D) $$\rho_{1'}(\vec a,\vec b\,|\,\text{singlet}) = \operatorname*{tr}_a \operatorname*{tr}_b\left[\hat X^{a}(1,\vec a)\hat X^{b}(1,\vec b)\hat\rho^{\,\text{singlet}}\right] =$$
$$= \frac{1}{5}\Big[5\alpha^2+20\alpha\gamma+68\alpha\varepsilon+13\gamma^2+74\gamma\varepsilon+97\varepsilon^2-2(5\beta^2+34\beta\delta+47\delta^2)(\vec a\vec b)$$
$$+3(7\gamma^2+62\gamma\varepsilon+127\varepsilon^2)(\vec a\vec b)^2-6^2\delta^2(\vec a\vec b)^3+6^2\varepsilon^2(\vec a\vec b)^4\Big] =$$
$$= \frac{1}{5}\left[-\frac{5\cdot 19}{2^6}+\frac{3\cdot 5}{2^2}(\vec a\vec b)-\frac{3\cdot 7^3}{2^5}(\vec a\vec b)^2-\frac{7^2}{2^2}(\vec a\vec b)^3+\frac{3^4\cdot 7^2}{2^6}(\vec a\vec b)^4\right] \tag{79}$$

E) $$\rho_{2'}(\vec a,\vec b\,|\,\text{singlet}) = \operatorname*{tr}_a \operatorname*{tr}_b\left[\hat X^{a}(2,\vec a)\hat X^{b}(2,\vec b)\hat\rho^{\,\text{singlet}}\right] =$$
$$= \frac{1}{20}\left[2\cdot 167+2\cdot 3\cdot 19(\vec a\vec b)-3\cdot 7^2\cdot 23(\vec a\vec b)^2-4\cdot 7^2(\vec a\vec b)^3+3^4\cdot 7^2(\vec a\vec b)^4\right] \tag{80}$$

F) $$\rho_Y(\vec a,\vec b\,|\,\text{singlet}) = \operatorname*{tr}_a \operatorname*{tr}_b\left[\hat Y^{a}(\vec a)\hat Y^{b}(\vec b)\hat\rho^{\,\text{singlet}}\right] =$$
$$= \frac{1}{2^3\cdot 5}\left[3+2^2\cdot 3(\vec a\vec b)-2\cdot 3\cdot 7(\vec a\vec b)^2-2^2\cdot 7(\vec a\vec b)^3+3^2\cdot 7(\vec a\vec b)^4\right]$$
$$= \frac{1}{5^2}\left[1-3P_1(\vec a\vec b)+5P_2(\vec a\vec b)-7P_3(\vec a\vec b)+9P_4(\vec a\vec b)\right] \tag{81}$$

$$\rho_{Y'}(\vec a,\vec b\,|\,\text{singlet}) = \frac{1}{5^2}\,\delta_{S^2}(\vec a,-\vec b) \tag{82}$$

Table 2. Spin 2. $80\,\rho(m,\vec a;n,\vec b\,|\,\text{singlet})$

$m \backslash n$	2	1	0	−1	−2
2	$(1-\vec a\vec b)^4$	$4(1-\vec a\vec b)^3(1+\vec a\vec b)$	$6[1-(\vec a\vec b)^2]^2$	$4(1-\vec a\vec b)(1+\vec a\vec b)^3$	$(1+\vec a\vec b)^4$
1	$4(1-\vec a\vec b)^3(1+\vec a\vec b)$	$4(1-\vec a\vec b)^2(1+2\vec a\vec b)^2$	$24(\vec a\vec b)^2[1-(\vec a\vec b)^2]$	$4(1+\vec a\vec b)^2(1-2\vec a\vec b)^2$	$4(1+\vec a\vec b)^3(1-\vec a\vec b)$
0	$6[1-(\vec a\vec b)^2]^2$	$24(\vec a\vec b)^2[1-(\vec a\vec b)^2]$	$4[1-3(\vec a\vec b)^2]^2$	$24(\vec a\vec b)^2[1-(\vec a\vec b)^2]$	$6[1-(\vec a\vec b)^2]^2$
−1	$4(1+\vec a\vec b)^3(1-\vec a\vec b)$	$4(1+\vec a\vec b)^2(1-2\vec a\vec b)^2$	$24(\vec a\vec b)^2[1-(\vec a\vec b)^2]$	$4(1-\vec a\vec b)^2(1+2\vec a\vec b)^2$	$4(1-\vec a\vec b)^3(1+\vec a\vec b)$
−2	$(1+\vec a\vec b)^4$	$4(1+\vec a\vec b)^3(1-\vec a\vec b)$	$6[1-(\vec a\vec b)^2]^2$	$4(1+\vec a\vec b)(1-\vec a\vec b)^3$	$(1-\vec a\vec b)^4$

The function ρ_Y is independent of the choice of Y (i.e., of the signs in eqs. (21.a) for $\alpha, \beta, \gamma, \delta, \varepsilon$). The functins $\rho_{1'}, \rho_{2'}$ and ρ_Y are not positive definite. However, ρ_Y can be replaced by the positive definite function $\rho_{Y'}$. All the densities are normalized as follows

$$5^2 \int d\mu(\vec{s}^a) \int d\mu(\vec{s}^b) \rho_i(\vec{s}^a, \vec{s}^b | \text{singlet}) = 1. \tag{83}$$

In terms of these densities the correlator of components of two spins 2 in the singlet state can be written as

$$c(\vec{a},\vec{b}) = \operatorname*{tr}_a \operatorname*{tr}_b [(\vec{a}\hat{\vec{s}}^a)(\vec{b}\hat{\vec{s}}^b)\hat{\rho}^{\,\text{singlet}}] =$$

$$= \int d\mu(\vec{s}^a) \int d\mu(\vec{s}^b) (\vec{a}\vec{s}^a)(\vec{b}\vec{s}^a) \sum_{m,n=0,1,2} \sum v_m v_n mn\, \rho(m, \vec{s}^a; n, \vec{s}^b | \text{singlet}) =$$

$$= 5^2 \left(\frac{2}{3}\right)^2 \int d\mu(\vec{s}^a) \int d\mu(\vec{s}^b) (\vec{a}\vec{s}^a)(\vec{b}\vec{s}^b) [25 \rho(1, \vec{s}^a; 1, \vec{s}^b | \text{singlet}) +$$

$$+ 20 \rho(1, \vec{s}^a; 2, \vec{s}^b | \text{singlet}) + 4 \rho(2, \vec{s}^a; 2, \vec{s}^b | \text{singlet})] = \tag{84.a}$$

$$= 5^2 w_i^2 \int d\mu(\vec{s}^a) \int d\mu(\vec{s}^b) (\vec{a}\vec{s}^a)(\vec{b}\vec{s}^b) \rho_i(\vec{s}^a, \vec{s}^b | \text{singlet}) \tag{84.b}$$

$$= -2(\vec{a}\vec{b}),$$

where $i = 1, 2, 1', 2', Y, Y'$; w_i are given by eqs. (67), $w_{Y'} = w_Y$, $w_{Y'}^2 = w_Y^2 = s(s+1) = 6$. In all the representations correlator (84) resembles its classical counterpart assumed by Bell. However, there are essential distinctions. In representations D), E) and F) the densities are not positive definite. In cases A), B), C) and F) with $\rho_{Y'}$ instead of ρ_Y the densities are positive, however, extra numerical factors exclude reducing to classics.

Expressions (84) with the positive densities admit the Bell type derivation of inequalities for the correlator. This way lead us to the following quantum analogs of the Bell inequality

$$|c(\vec{a},\vec{b}) - c(\vec{a},\vec{b}')| + |c(\vec{a}',\vec{b}') + c(\vec{a}',\vec{b})| \leq 2 w_i^2 = 2 \cdot \begin{cases} \left(\frac{2\cdot 7}{3}\right)^2 & \\ 6^2 & i=1 \\ 3^2 & i=2 \\ 6 & i=Y', \end{cases} \tag{85}$$

where the factor $\left(\frac{2\cdot 7}{3}\right)^2$ corresponds to the expression (84.a) for $c(\vec{a},\vec{b})$ (PSR A)). Among these estimations the last is the best. It is also universal for all the spins s (see ref. /18/, p. 14).

7. Let us proceed with some other aspects of PSR's. A representative of the product of two observables $\hat{F}$ and $\hat{G}$ can be expressed via their representatives as follows

$$F(\vec{s}) * G(\vec{s}) \equiv \operatorname{tr}(\hat{Z}(\vec{s})\hat{F}\hat{G}) =$$

$$= 5^2 \int d\mu(\vec{s}') \int d\mu(\vec{s}'') K(\vec{s}; \vec{s}', \vec{s}'') F(\vec{s}') G(\vec{s}'') \tag{86}$$

after putting expressions (34.b) for the operators $\hat{F}$ and $\hat{G}$. The kernel K is

$$K(\vec{s}, \vec{s}', \vec{s}'') = \operatorname{tr}(\hat{Z}(\vec{s})\hat{Z}'(\vec{s}')\hat{Z}''(\vec{s}'')), \tag{87}$$

where Z, Z' and Z" are matrices entering eqs. (34.b), e.g., $\hat{Z}(\vec{s}) = \hat{\rho}(m, \vec{s})$, $\hat{Z}'(\vec{s}) = \hat{Z}''(\vec{s}) = \hat{X}(m, \vec{s})$.

8. The equations of motion for the density matrix $\hat{\rho}$ and any observable $\hat{F}$, which does not depend explicitly on time, i.e., the von Neumann and Heisenberg-Born-Jordan-Dirac equations

$$\hbar \frac{d}{dt} \hat{\rho}(t) = -i[\hat{H}, \hat{\rho}(t)], \quad \frac{d}{dt}\hat{F} = 0 \tag{88}$$

$$\hbar \frac{d}{dt} \hat{F}(t) = i[\hat{H}, \hat{F}(t)], \quad \frac{d}{dt}\hat{\rho} = 0 \tag{89}$$

in the Schrödinger and Heisenberg pictures, respectively, take in PSR's the form

$$\hbar \frac{d}{dt} \rho(\vec{s}, t) = -i\left(H(\vec{s}) * \rho(\vec{s}, t) - \rho(\vec{s}, t) * H(\vec{s})\right), \quad \frac{d}{dt} F(\vec{s}) = 0 \tag{90}$$

$$\hbar \frac{d}{dt} F(\vec{s}, t) = i\left(H(\vec{s}) * F(\vec{s}, t) - F(\vec{s}, t) * H(\vec{s})\right), \quad \frac{d}{dt} \rho(\vec{s}) = 0 \tag{91}$$

9. Left and right operator representatives. Besides the above nonoperator representatives, in some cases operator representatives can be introduced:

$$\operatorname{tr}(\hat{Z}(\vec{s})\hat{F}\hat{G}) = F^{\ell} \operatorname{tr}(\hat{Z}(\vec{s})\hat{G}) = F^{\ell} G(\vec{s}) \tag{92.a}$$

$$= G^{r} \operatorname{tr}(\hat{Z}(\vec{s})\hat{G}) = G^{r} F(\vec{s}) \tag{92.b}$$

(cf. refs. /15-17/). The left and right operator representatives F^l and F^r are partial differential operators acting on $\vec{s}$ (on functions of $\vec{s}$). Note that

$$\operatorname{tr}(\hat{Z}(\vec{s})\hat{F}) = F^{\ell}_{r} \operatorname{tr}(\hat{Z}(\vec{s})) = F^{\ell}_{r} \cdot 1 = F(\vec{s}) \qquad (\operatorname{tr}\hat{Z}(\vec{s}) = 1) \tag{93}$$

Equations (92.a) and (92.b) supply us (if F^l and F^r exist) with two more expressions for the nonoperator representative of product of two operators in addition to eq. (86).

The left representatives are multiplied in the same order as original operators, while the right representatives in the inverse order:

$$\operatorname{tr}(\hat{Z}(\vec{s})\hat{F}_1\hat{F}_2\hat{G}) = F_1^{\ell} F_2^{\ell} \operatorname{tr}(\hat{Z}(\vec{s})\hat{G}), \tag{94}$$

$$\operatorname{tr}(\hat{Z}(\vec{s})\hat{G}\hat{F}_1\hat{F}_2) = F_2^{r} F_1^{r} \operatorname{tr}(\hat{Z}(\vec{s})\hat{G}). \tag{95}$$

The left representatives commute with the right ones:

$$[F^{\ell}, G^{r}] = 0 \tag{96}$$

as a general rule for all associative theories (unlike nonassociative ones).

The operator representatives can be introduced in the representation with $\hat{Z}(\vec{s})=\hat{\rho}(s,\vec{s})$ (m=s) for all the spins $s = \frac{1}{2}, 1, \frac{3}{2}, 2, \dots$

$$tr\left(\hat{\rho}(s,\vec{s})\hat{s}_j\hat{F}\right)=s_j^l\, tr\left(\hat{\rho}(s,\vec{s})\hat{F}\right), \tag{97}$$

$$tr\left(\hat{\rho}(s,\vec{s})\hat{F}\,\hat{s}_j\right)=s_j^r\, tr\left(\hat{\rho}(s,\vec{s})\hat{F}\right). \tag{98}$$

To find s_j^l we solve the equations

$$s_j^l\,\hat{\rho}(s,\vec{s})=\hat{\rho}(s,\vec{s})\hat{s}_j\ , \tag{99}$$

$$s_j^r\,\hat{\rho}(s,\vec{s})=\hat{s}_j\,\hat{\rho}(s,\vec{s}), \tag{100}$$

supposing that

$$s_j^{l,r}=\xi s_j+\eta s_k\left(s_k\frac{\partial}{\partial s_j}-s_j\frac{\partial}{\partial s_k}\right)+\zeta\varepsilon_{jkl}s_k\frac{\partial}{\partial s_l} \tag{101}$$

where ξ, η, ζ are unknown (only inner operators on S^2 are admissible). Thus we obtain$^{x)}$

$$s_j^{l,r}=s\,s_j+\frac{1}{2}s_k\left(s_k\frac{\partial}{\partial s_j}-s_j\frac{\partial}{\partial s_k}\right)\mp\frac{i}{2}\varepsilon_{jkl}s_k\frac{\partial}{\partial s_l} \tag{102}$$

for spins $s = \frac{1}{2}, 1, \frac{3}{2}, 2, \dots$. These operators satisfy the usual commutation relations with additional minus for the right representatives

$$[s_j^l, s_k^l]=i\varepsilon_{jkl}s_l^l,\quad [s_j^r,s_k^r]=-i\varepsilon_{jkl}s_l^r,\quad [s_j^l,s_k^r]=0 \tag{103}$$

The representatives F^l and F^r of any operator $\hat{F}$ may be obtained explicitly by replacing in $\hat{F}$ the spin matrices $\hat{s}_j$ by their left and right representatives, respectively, taking into account the above rule of order of factors.

10. Equations of motion (88) and (89) take in these terms the form of the Liouville equation

$$\frac{\partial}{\partial t}\rho(\vec{s},t)=-\mathcal{L}\rho(\vec{s},t),\quad \frac{\partial}{\partial t}F(\vec{s})=0 \quad \text{(Schr. pict.)} \tag{104}$$

$$\frac{\partial}{\partial t}F(\vec{s},t)=\mathcal{L}F(\vec{s},t),\quad \frac{\partial}{\partial t}\rho(\vec{s})=0 \quad \text{(Heis. pict.)} \tag{105}$$

where

$$\rho(\vec{s},t)=tr\left(\hat{\rho}(s,\vec{s})\hat{\rho}(t)\right),\qquad F(\vec{s})=tr\left(\hat{\rho}(s,\vec{s})\hat{F}\right), \tag{106}$$

$$F(\vec{s},t)=tr\left(\hat{\rho}(s,\vec{s})\hat{F}(t)\right),\qquad \rho(\vec{s})=tr\left(\hat{\rho}(s,\vec{s})\hat{\rho}\right) \tag{107}$$

for any spin $s = \frac{1}{2}, 1, \frac{3}{2}, 2, \dots$ and

$$\mathcal{L}=i\hbar^{-1}(H^l-H^r) \tag{108}$$

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x) The equations $s_j^l\,\hat{Z}(\vec{s})=\hat{Z}(\vec{s})\hat{s}_j$, $s_j^r\,\hat{Z}(\vec{s})=\hat{s}_j\hat{Z}(\vec{s})$ with $\hat{Z}(\vec{s})=\alpha\cdot 1+\beta(\vec{s}\hat{\vec{s}})+\gamma(\vec{s}\hat{\vec{s}})^2+\dots$ have solutions only for the special values of $\alpha, \beta, \gamma, \dots$.

is a Liouvillian, a partial differential operator. The formal solutions of eqs. (104) and (105) are

$$\rho(\vec{s},t)=e^{-\mathcal{L}t}\rho(\vec{s},0), \tag{109}$$

$$F(\vec{s},t)=e^{\mathcal{L}t}F(\vec{s},0) \tag{110}$$

if the Hamiltonian, and therefore the Liouvillian are independent of time. (The solutions (109) and (110) correspond to the formal solutions of eqs. (88) and (89)

$$\hat{\rho}(t)=e^{-i\hbar^{-1}\hat{H}t}\hat{\rho}(0)\,e^{i\hbar^{-1}\hat{H}t},\quad \hat{F}(t)=e^{i\hbar^{-1}\hat{H}t}\hat{F}(0)\,e^{-i\hbar^{-1}\hat{H}t}) \tag{111}$$

For the motion in a constant magnetic field the Hamiltonian and its nonoperator and operator representatives are

$$\hat{H}=-\hbar\vec{\omega}\hat{\vec{s}},\quad H(\vec{s})=-\hbar s\vec{\omega}\vec{s},\quad H^{l,r}=-\hbar\vec{\omega}\vec{s}^{\,l,r}\quad \left(\vec{\omega}=\frac{e}{2mc}\vec{B}\right) \tag{112}$$

and therefore the Liouvillian is

$$\mathcal{L}=-i\vec{\omega}(\vec{s}^{\,l}-\vec{s}^{\,r})=-\omega_j\varepsilon_{jkl}s_k\frac{\partial}{\partial s_l} \tag{113}$$

for any spin. Equations (104) and (105) take the form

$$\frac{\partial}{\partial t}\rho(\vec{s},t)=\omega_j\varepsilon_{jkl}s_k\frac{\partial}{\partial s_l}\rho(\vec{s},t), \tag{114}$$

$$\frac{\partial}{\partial t}F(\vec{s},t)=-\omega_j\varepsilon_{jkl}s_k\frac{\partial}{\partial s_l}F(\vec{s},t), \tag{115}$$

$$\rho(\vec{s},t)=\exp\left(t\,\omega_j\varepsilon_{jkl}s_k\frac{\partial}{\partial s_l}\right)\rho(\vec{s},0) \tag{116}$$

$$F(\vec{s},t)=\exp\left(-t\,\omega_j\varepsilon_{jkl}s_k\frac{\partial}{\partial s_l}\right)F(\vec{s},0). \tag{117}$$

One can also write eqs. (114) and (115) in terms of the nonoperator representative of Hamiltonian as follows

$$\frac{\partial}{\partial t}\rho(\vec{s},t)=(s\hbar)^{-1}\varepsilon_{jkl}s_j\frac{\partial H(\vec{s})}{\partial s_k}\frac{\partial\rho(\vec{s},t)}{\partial s_l}, \tag{118}$$

$$\frac{\partial}{\partial t}F(\vec{s},t)=-(s\hbar)^{-1}\varepsilon_{jkl}s_j\frac{\partial H(\vec{s})}{\partial s_k}\frac{\partial F(\vec{s},t)}{\partial s_l} \tag{119}$$

This form of the equations resembles the Liouville equation in the classical mechanics (see, e.g., refs./15-17/):

$$\frac{\partial}{\partial t}\rho(x,p,t)=-\mathcal{L}\rho(x,p,t)\quad \left(\text{or } \frac{\partial}{\partial t}F(x,p,t)=\mathcal{L}F(x,p,t)\right) \tag{120}$$

$$\mathcal{L}=\sum_i\left(\frac{\partial H}{\partial p_i}\frac{\partial}{\partial x_i}-\frac{\partial H}{\partial x_i}\frac{\partial}{\partial p_i}\right) \tag{121}$$

The r.h.s!s of eqs. (118) and (119) are analogs of the Poisson brackets. But here the variables s_j ($j = 1,2,3$) on the sphere S^2 ($\vec{s}^{\,2}=1$) serve as phase space variables instead of the variables x and p in the usual Poisson brackets. The Liouvillian (113) is the partial differential operator of the first order. Due to this fact equations (114) and (115) are of the classical nature. They can be solved in terms of characteristics $\vec{s}(t)$, which satisfy analogs of the Hamilton and Newton equations (for details see ref./17/, cf. refs./15,16/).

Appendix A. The spin 2 matrices in the canonical representation

$$\hat{S}_1 = \frac{1}{2}\begin{pmatrix} \cdot & 2 & \cdot & \cdot & \cdot \\ 2 & \cdot & \sqrt{6} & \cdot & \cdot \\ \cdot & \sqrt{6} & \cdot & \sqrt{6} & \cdot \\ \cdot & \cdot & \sqrt{6} & \cdot & 2 \\ \cdot & \cdot & \cdot & 2 & \cdot \end{pmatrix} \quad \hat{S}_2 = \frac{i}{2}\begin{pmatrix} \cdot & -2 & \cdot & \cdot & \cdot \\ 2 & \cdot & -\sqrt{6} & \cdot & \cdot \\ \cdot & \sqrt{6} & \cdot & -\sqrt{6} & \cdot \\ \cdot & \cdot & \sqrt{6} & \cdot & -2 \\ \cdot & \cdot & \cdot & 2 & \cdot \end{pmatrix} \quad \hat{S}_3 = \begin{pmatrix} 2 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & -2 \end{pmatrix}$$

The dots stand for zeros.

Powers of the spin matrices

$$\hat{S}_1^2 = \frac{1}{2}\begin{pmatrix} 2 & \cdot & \sqrt{6} & \cdot & \cdot \\ \cdot & 5 & \cdot & 3 & \cdot \\ \sqrt{6} & \cdot & 6 & \cdot & \sqrt{6} \\ \cdot & 3 & \cdot & 5 & \cdot \\ \cdot & \cdot & \sqrt{6} & \cdot & 2 \end{pmatrix} \quad \hat{S}_2^2 = \frac{1}{2}\begin{pmatrix} 2 & \cdot & -\sqrt{6} & \cdot & \cdot \\ \cdot & 5 & \cdot & -3 & \cdot \\ -\sqrt{6} & \cdot & 6 & \cdot & -\sqrt{6} \\ \cdot & -3 & \cdot & 5 & \cdot \\ \cdot & \cdot & -\sqrt{6} & \cdot & 2 \end{pmatrix} \quad \hat{S}_3^2 = \begin{pmatrix} 4 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 4 \end{pmatrix}$$

$$\hat{S}_1^3 = \frac{1}{2}\begin{pmatrix} \cdot & 5 & \cdot & 3 & \cdot \\ 5 & \cdot & 4\sqrt{6} & \cdot & 3 \\ \cdot & 4\sqrt{6} & \cdot & 4\sqrt{6} & \cdot \\ 3 & \cdot & 4\sqrt{6} & \cdot & 5 \\ \cdot & 3 & \cdot & 5 & \cdot \end{pmatrix} \quad \hat{S}_2^3 = \frac{i}{2}\begin{pmatrix} \cdot & -5 & \cdot & 3 & \cdot \\ 5 & \cdot & -4\sqrt{6} & \cdot & 3 \\ \cdot & 4\sqrt{6} & \cdot & -4\sqrt{6} & \cdot \\ -3 & \cdot & 4\sqrt{6} & \cdot & -5 \\ \cdot & -3 & \cdot & 5 & \cdot \end{pmatrix} \quad \hat{S}_3^3 = \begin{pmatrix} 8 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & -8 \end{pmatrix}$$

$$\hat{S}_1^4 = \frac{1}{2}\begin{pmatrix} 5 & \cdot & 4\sqrt{6} & \cdot & 3 \\ \cdot & 17 & \cdot & 15 & \cdot \\ 4\sqrt{6} & \cdot & 24 & \cdot & 4\sqrt{6} \\ \cdot & 15 & \cdot & 17 & \cdot \\ 3 & \cdot & 4\sqrt{6} & \cdot & 5 \end{pmatrix} \quad \hat{S}_2^4 = \frac{1}{2}\begin{pmatrix} 5 & \cdot & -4\sqrt{6} & \cdot & 3 \\ \cdot & 17 & \cdot & -15 & \cdot \\ -4\sqrt{6} & \cdot & 24 & \cdot & -4\sqrt{6} \\ \cdot & -15 & \cdot & 17 & \cdot \\ 3 & \cdot & -4\sqrt{6} & \cdot & 5 \end{pmatrix} \quad \hat{S}_3^4 = \begin{pmatrix} 16 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 16 \end{pmatrix}$$

Symmetrized products of the spin matrices

$$\{\hat{S}_1\hat{S}_2\} = i\begin{pmatrix} \cdot & \cdot & -\sqrt{6} & \cdot & \cdot \\ \cdot & \cdot & \cdot & -3 & \cdot \\ \sqrt{6} & \cdot & \cdot & \cdot & -\sqrt{6} \\ \cdot & 3 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \sqrt{6} & \cdot & \cdot \end{pmatrix} \quad \{\hat{S}_2\hat{S}_3\} = \frac{i}{2}\begin{pmatrix} \cdot & -6 & \cdot & \cdot & \cdot \\ 6 & \cdot & -\sqrt{6} & \cdot & \cdot \\ \cdot & \sqrt{6} & \cdot & \sqrt{6} & \cdot \\ \cdot & \cdot & -\sqrt{6} & \cdot & 6 \\ \cdot & \cdot & \cdot & -6 & \cdot \end{pmatrix} \quad \{\hat{S}_3\hat{S}_1\} = \frac{1}{2}\begin{pmatrix} \cdot & 6 & \cdot & \cdot & \cdot \\ 6 & \cdot & \sqrt{6} & \cdot & \cdot \\ \cdot & \sqrt{6} & \cdot & -\sqrt{6} & \cdot \\ \cdot & \cdot & -\sqrt{6} & \cdot & -6 \\ \cdot & \cdot & \cdot & -6 & \cdot \end{pmatrix}$$

$$\frac{1}{2}\{\hat{S}_1\hat{S}_2\hat{S}_2\} = \frac{1}{2}\begin{pmatrix} \cdot & 5 & \cdot & -9 & \cdot \\ 5 & \cdot & 4\sqrt{6} & \cdot & -9 \\ \cdot & 4\sqrt{6} & \cdot & 4\sqrt{6} & \cdot \\ -9 & \cdot & 4\sqrt{6} & \cdot & 5 \\ \cdot & -9 & \cdot & 5 & \cdot \end{pmatrix} \quad \frac{1}{2}\{\hat{S}_1\hat{S}_3\hat{S}_3\} = \frac{1}{2}\begin{pmatrix} \cdot & 14 & \cdot & \cdot & \cdot \\ 14 & \cdot & \sqrt{6} & \cdot & \cdot \\ \cdot & \sqrt{6} & \cdot & \sqrt{6} & \cdot \\ \cdot & \cdot & \sqrt{6} & \cdot & 14 \\ \cdot & \cdot & \cdot & 14 & \cdot \end{pmatrix} \quad \frac{1}{2}\{\hat{S}_2\hat{S}_1\hat{S}_1\} = \frac{i}{2}\begin{pmatrix} \cdot & -5 & \cdot & -9 & \cdot \\ 5 & \cdot & -4\sqrt{6} & \cdot & -9 \\ \cdot & 4\sqrt{6} & \cdot & -4\sqrt{6} & \cdot \\ 9 & \cdot & 4\sqrt{6} & \cdot & -5 \\ \cdot & 9 & \cdot & 5 & \cdot \end{pmatrix}$$

$$\frac{1}{2}\{\hat{S}_2\hat{S}_3\hat{S}_3\} = \frac{i}{2}\begin{pmatrix} \cdot & -14 & \cdot & \cdot & \cdot \\ 14 & \cdot & -\sqrt{6} & \cdot & \cdot \\ \cdot & \sqrt{6} & \cdot & -\sqrt{6} & \cdot \\ \cdot & \cdot & \sqrt{6} & \cdot & -14 \\ \cdot & \cdot & \cdot & 14 & \cdot \end{pmatrix} \quad \frac{1}{2}\{\hat{S}_3\hat{S}_1\hat{S}_1\} = \frac{1}{2}\begin{pmatrix} 10 & \cdot & 3\sqrt{6} & \cdot & \cdot \\ \cdot & 14 & \cdot & \cdot & \cdot \\ 3\sqrt{6} & \cdot & \cdot & \cdot & -3\sqrt{6} \\ \cdot & \cdot & \cdot & -14 & \cdot \\ \cdot & \cdot & -3\sqrt{6} & \cdot & -10 \end{pmatrix} \quad \frac{1}{2}\{\hat{S}_3\hat{S}_2\hat{S}_2\} = \frac{1}{2}\begin{pmatrix} 10 & \cdot & -3\sqrt{6} & \cdot & \cdot \\ \cdot & 14 & \cdot & \cdot & \cdot \\ -3\sqrt{6} & \cdot & \cdot & \cdot & 3\sqrt{6} \\ \cdot & \cdot & \cdot & -14 & \cdot \\ \cdot & \cdot & 3\sqrt{6} & \cdot & -10 \end{pmatrix}$$

$$\{\hat{S}_1\hat{S}_2\hat{S}_3\} = 3i\sqrt{6}\begin{pmatrix} \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot & \cdot \end{pmatrix}$$

$$\frac{1}{4}\{\hat{S}_1\hat{S}_1\hat{S}_2\hat{S}_2\} \quad \begin{pmatrix} 5 & \cdot & \cdot & \cdot & -9 \\ \cdot & 17 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 24 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 17 & \cdot \\ -9 & \cdot & \cdot & \cdot & 5 \end{pmatrix}$$

$$\frac{1}{4}\{\hat{S}_1\hat{S}_1\hat{S}_3\hat{S}_3\} \quad \frac{1}{2}\begin{pmatrix} 34 & \cdot & 7\sqrt{6} & \cdot & \cdot \\ \cdot & 31 & \cdot & 3 & \cdot \\ 7\sqrt{6} & \cdot & 6 & \cdot & 7\sqrt{6} \\ \cdot & 3 & \cdot & 31 & \cdot \\ \cdot & \cdot & 7\sqrt{6} & \cdot & 34 \end{pmatrix}$$

$$\frac{1}{4}\{\hat{S}_2\hat{S}_2\hat{S}_3\hat{S}_3\} \quad \frac{1}{2}\begin{pmatrix} 34 & \cdot & -7\sqrt{6} & \cdot & \cdot \\ \cdot & 31 & \cdot & -3 & \cdot \\ -7\sqrt{6} & \cdot & 6 & \cdot & -7\sqrt{6} \\ \cdot & -3 & \cdot & 31 & \cdot \\ \cdot & \cdot & -7\sqrt{6} & \cdot & 34 \end{pmatrix}$$

$$\frac{1}{6}\{\hat{S}_1\hat{S}_2\hat{S}_2\hat{S}_2\} \quad i\begin{pmatrix} \cdot & \cdot & -4\sqrt{6} & \cdot & 6 \\ \cdot & \cdot & \cdot & -15 & \cdot \\ 4\sqrt{6} & \cdot & \cdot & \cdot & -4\sqrt{6} \\ \cdot & 15 & \cdot & \cdot & \cdot \\ -6 & \cdot & 4\sqrt{6} & \cdot & \cdot \end{pmatrix}$$

$$\frac{1}{6}\{\hat{S}_1\hat{S}_3\hat{S}_3\hat{S}_3\} \quad \frac{1}{2}\begin{pmatrix} \cdot & 30 & \cdot & \cdot & \cdot \\ 30 & \cdot & \sqrt{6} & \cdot & \cdot \\ \cdot & \sqrt{6} & \cdot & -\sqrt{6} & \cdot \\ \cdot & \cdot & -\sqrt{6} & \cdot & -30 \\ \cdot & \cdot & \cdot & -30 & \cdot \end{pmatrix}$$

$$\frac{1}{6}\{\hat{S}_2\hat{S}_1\hat{S}_1\hat{S}_1\} \quad i\begin{pmatrix} \cdot & \cdot & -4\sqrt{6} & \cdot & -6 \\ \cdot & \cdot & \cdot & -15 & \cdot \\ 4\sqrt{6} & \cdot & \cdot & \cdot & -4\sqrt{6} \\ \cdot & 15 & \cdot & \cdot & \cdot \\ 6 & \cdot & 4\sqrt{6} & \cdot & \cdot \end{pmatrix}$$

$$\frac{1}{6}\{\hat{S}_2\hat{S}_3\hat{S}_3\hat{S}_3\} \quad \frac{i}{2}\begin{pmatrix} \cdot & -30 & \cdot & \cdot & \cdot \\ 30 & \cdot & -\sqrt{6} & \cdot & \cdot \\ \cdot & \sqrt{6} & \cdot & \sqrt{6} & \cdot \\ \cdot & \cdot & -\sqrt{6} & \cdot & 30 \\ \cdot & \cdot & \cdot & -30 & \cdot \end{pmatrix}$$

$$\frac{1}{6}\{\hat{S}_3\hat{S}_1\hat{S}_1\hat{S}_1\} \quad \frac{1}{2}\begin{pmatrix} \cdot & 24 & \cdot & 6 & \cdot \\ 24 & \cdot & 7\sqrt{6} & \cdot & -6 \\ \cdot & 7\sqrt{6} & \cdot & -7\sqrt{6} & \cdot \\ 6 & \cdot & -7\sqrt{6} & \cdot & -24 \\ \cdot & -6 & \cdot & -24 & \cdot \end{pmatrix}$$

$$\frac{1}{6}\{\hat{S}_3\hat{S}_2\hat{S}_2\hat{S}_2\} \quad \frac{i}{2}\begin{pmatrix} \cdot & -24 & \cdot & 6 & \cdot \\ 24 & \cdot & -7\sqrt{6} & \cdot & -6 \\ \cdot & 7\sqrt{6} & \cdot & 7\sqrt{6} & \cdot \\ -6 & \cdot & -7\sqrt{6} & \cdot & 24 \\ \cdot & 6 & \cdot & -24 & \cdot \end{pmatrix}$$

$$\frac{1}{2}\{\hat{S}_1\hat{S}_2\hat{S}_3\hat{S}_3\} \quad i\begin{pmatrix} \cdot & \cdot & -7\sqrt{6} & \cdot & \cdot \\ \cdot & \cdot & \cdot & -3 & \cdot \\ 7\sqrt{6} & \cdot & \cdot & \cdot & -7\sqrt{6} \\ \cdot & 3 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 7\sqrt{6} & \cdot & \cdot \end{pmatrix}$$

$$\frac{1}{2}\{\hat{S}_1\hat{S}_3\hat{S}_2\hat{S}_2\} \quad \frac{1}{2}\begin{pmatrix} \cdot & 24 & \cdot & -18 & \cdot \\ 24 & \cdot & 7\sqrt{6} & \cdot & 18 \\ \cdot & 7\sqrt{6} & \cdot & -7\sqrt{6} & \cdot \\ -18 & \cdot & -7\sqrt{6} & \cdot & -24 \\ \cdot & 18 & \cdot & -24 & \cdot \end{pmatrix}$$

$$\frac{1}{2}\{\hat{S}_2\hat{S}_3\hat{S}_1\hat{S}_1\} \quad \frac{i}{2}\begin{pmatrix} \cdot & -24 & \cdot & -18 & \cdot \\ 24 & \cdot & -7\sqrt{6} & \cdot & 18 \\ \cdot & 7\sqrt{6} & \cdot & 7\sqrt{6} & \cdot \\ 18 & \cdot & -7\sqrt{6} & \cdot & 24 \\ \cdot & -18 & \cdot & -24 & \cdot \end{pmatrix}$$

Symmetrized products are expressed via ordinary ones as follows

$$\{\hat{S}_i\hat{S}_i\} = 2\hat{S}_i^2,$$
$$\{\hat{S}_i\hat{S}_i\hat{S}_i\} = 6\hat{S}_i^3,$$
$$\{\hat{S}_i\hat{S}_i\hat{S}_i\hat{S}_i\} = 24\hat{S}_i^4,$$
$$\{\hat{S}_i\hat{S}_j\hat{S}_j\} = 2\left(\hat{S}_i\hat{S}_j^2 + \hat{S}_j^2\hat{S}_i + \hat{S}_j\hat{S}_i\hat{S}_j\right),$$
$$\{\hat{S}_i\hat{S}_i\hat{S}_j\hat{S}_j\} = 4\left(\hat{S}_i^2\hat{S}_j^2 + \hat{S}_j^2\hat{S}_i^2 + \hat{S}_i\hat{S}_j^2\hat{S}_i + \hat{S}_j\hat{S}_i^2\hat{S}_j + \hat{S}_i\hat{S}_j\hat{S}_i\hat{S}_j + \hat{S}_j\hat{S}_i\hat{S}_j\hat{S}_i\right),$$
$$\{\hat{S}_i\hat{S}_j\hat{S}_j\hat{S}_j\} = 6\left(\hat{S}_i\hat{S}_j^3 + \hat{S}_j^3\hat{S}_i + \hat{S}_j\hat{S}_i\hat{S}_j^2 + \hat{S}_j^2\hat{S}_i\hat{S}_j\right),$$
$$\{\hat{S}_i\hat{S}_j\hat{S}_k\hat{S}_k\} = 2\Big((\hat{S}_i\hat{S}_j + \hat{S}_j\hat{S}_i)\hat{S}_k^2 + \hat{S}_k^2(\hat{S}_i\hat{S}_j + \hat{S}_j\hat{S}_i) + \hat{S}_k(\hat{S}_i\hat{S}_j + \hat{S}_j\hat{S}_i)\hat{S}_k +$$
$$+ \hat{S}_i\hat{S}_k^2\hat{S}_j + \hat{S}_j\hat{S}_k^2\hat{S}_i + \hat{S}_i\hat{S}_k\hat{S}_j\hat{S}_k + \hat{S}_j\hat{S}_k\hat{S}_i\hat{S}_k + \hat{S}_k\hat{S}_i\hat{S}_k\hat{S}_j + \hat{S}_k\hat{S}_j\hat{S}_k\hat{S}_i\Big)$$

(no summation everywhere), $i,j,k = 1,2,3$.

Appendix B. The general trace of interest is

$$\mathrm{tr}\Big[\left(\alpha\cdot\mathbb{1} + \beta(\vec{a}\hat{\vec{S}}) + \gamma(\vec{a}\hat{\vec{S}})^2 + \delta(\vec{a}\hat{\vec{S}})^3 + \varepsilon(\vec{a}\hat{\vec{S}})^4\right)\cdot$$
$$\cdot\left(\alpha'\cdot\mathbb{1} + \beta'(\vec{b}\hat{\vec{S}}) + \gamma'(\vec{b}\hat{\vec{S}})^2 + \delta'(\vec{b}\hat{\vec{S}})^3 + \varepsilon'(\vec{b}\hat{\vec{S}})^4\right)\Big] \tag{B.1}$$

for spins up to 2. The calculation can be essentially simplified in the special frame of reference with $Oz \parallel \vec{b}$, where the second matrix takes form

$$\left(\alpha'\cdot\mathbb{1} + \beta'\hat{S}_3 + \gamma'\hat{S}_3^2 + \delta'\hat{S}_3^3 + \varepsilon'\hat{S}_3^4\right) \tag{B.2}$$

and is purely diagonal (see Appendix A for spin 2). First of all we calculate the traces $\mathrm{tr}[(\vec{a}\hat{\vec{S}})^n\hat{\rho}(m,\vec{b})]$ with n = 0,1,2,3,4. We need simply to find only one diagonal matrix element of $(\vec{a}\hat{\vec{S}})^n$ for each m, since

$$\hat{\rho}(2,\vec{b}) = \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}, \quad \hat{\rho}(1,\vec{b}) = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}, \quad \hat{\rho}(0,\vec{b}) = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix},$$
$$\hat{\rho}(-1,\vec{b}) = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}, \quad \hat{\rho}(-2,\vec{b}) = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 \end{bmatrix}. \tag{B.3}$$

Using the following decompositions in the symmetrized products of the spin 2 matrices

$$(\vec{a}\hat{\vec{s}})^2=a_1^2\hat{s}_1^2+a_2^2\hat{s}_2^2+a_3^2\hat{s}_3^2+a_1a_2\{\hat{s}_1\hat{s}_2\}+a_2a_3\{\hat{s}_2\hat{s}_3\}+a_3a_1\{\hat{s}_3\hat{s}_1\},$$

$$(\vec{a}\hat{\vec{s}})^3=a_1^3\hat{s}_1^3+a_2^3\hat{s}_2^3+a_3^3\hat{s}_3^3+a_1a_2a_3\{\hat{s}_1\hat{s}_2\hat{s}_3\}+$$
$$+a_1a_2^2\tfrac{1}{2}\{\hat{s}_1\hat{s}_2\hat{s}_2\}+a_1a_3^2\tfrac{1}{2}\{\hat{s}_1\hat{s}_3\hat{s}_3\}+a_2a_1^2\tfrac{1}{2}\{\hat{s}_2\hat{s}_1\hat{s}_1\}+a_2a_3^2\tfrac{1}{2}\{\hat{s}_2\hat{s}_3\hat{s}_3\}+$$
$$+a_3a_1^2\tfrac{1}{2}\{\hat{s}_3\hat{s}_1\hat{s}_1\}+a_3a_2^2\tfrac{1}{2}\{\hat{s}_3\hat{s}_2\hat{s}_2\},$$

$$(\vec{a}\hat{\vec{s}})^4=a_1^4\hat{s}_1^4+a_2^4\hat{s}_2^4+a_3^4\hat{s}_3^4+a_1a_2^3\tfrac{1}{6}\{\hat{s}_1\hat{s}_2\hat{s}_2\hat{s}_2\}+a_1a_3^3\tfrac{1}{6}\{\hat{s}_1\hat{s}_3\hat{s}_3\hat{s}_3\}+$$
$$+a_2a_1^3\tfrac{1}{6}\{\hat{s}_2\hat{s}_1\hat{s}_1\hat{s}_1\}+a_2a_3^3\tfrac{1}{6}\{\hat{s}_2\hat{s}_3\hat{s}_3\hat{s}_3\}+a_3a_1^3\tfrac{1}{6}\{\hat{s}_3\hat{s}_1\hat{s}_1\hat{s}_1\}+a_3a_2^3\tfrac{1}{6}\{s_3s_2s_2s_2\}$$
$$+a_1^2a_2^2\tfrac{1}{4}\{\hat{s}_1\hat{s}_1\hat{s}_2\hat{s}_2\}+a_1^2a_3^2\tfrac{1}{4}\{\hat{s}_1\hat{s}_1\hat{s}_3\hat{s}_3\}+a_2^2a_3^2\tfrac{1}{4}\{\hat{s}_2\hat{s}_2\hat{s}_3\hat{s}_3\}+$$
$$+a_1a_2a_3^2\tfrac{1}{2}\{\hat{s}_1\hat{s}_2\hat{s}_3\hat{s}_3\}+a_1a_3a_2^2\tfrac{1}{2}\{\hat{s}_1\hat{s}_3\hat{s}_2\hat{s}_2\}+a_2a_3a_1^2\tfrac{1}{2}\{\hat{s}_2\hat{s}_3\hat{s}_1\hat{s}_1\} \tag{B.4}$$

and taking these products in the canonical representation (Appendix A) we get

$$\mathrm{tr}\,[\hat{\rho}(2,\vec{b})\mathbb{1}]=1,$$
$$\mathrm{tr}\,[\hat{\rho}(2,\vec{b})(\vec{a}\hat{\vec{s}})]=2a_3,$$
$$\mathrm{tr}\,[\hat{\rho}(2,\vec{b})(\vec{a}\hat{\vec{s}})^2]=1+3a_3^2,$$
$$\mathrm{tr}\,[\hat{\rho}(2,\vec{b})(\vec{a}\hat{\vec{s}})^3]=5a_3+3a_3^3,$$
$$\mathrm{tr}\,[\hat{\rho}(2,\vec{b})(\vec{a}\hat{\vec{s}})^4]=\tfrac{5}{2}+12a_3^2+\tfrac{3}{2}a_3^4, \tag{B.5}$$

$$\mathrm{tr}\,[\hat{\rho}(1,\vec{b})\mathbb{1}]=1,$$
$$\mathrm{tr}\,[\hat{\rho}(1,\vec{b})(\vec{a}\hat{\vec{s}})]=a_3,$$
$$\mathrm{tr}\,[\hat{\rho}(1,\vec{b})(\vec{a}\hat{\vec{s}})^2]=\tfrac{5}{2}-\tfrac{3}{2}a_3^2,$$
$$\mathrm{tr}\,[\hat{\rho}(1,\vec{b})(\vec{a}\hat{\vec{s}})^3]=7a_3-6a_3^3,$$
$$\mathrm{tr}\,[\hat{\rho}(1,\vec{b})(\vec{a}\hat{\vec{s}})^4]=\tfrac{17}{2}-\tfrac{3}{2}a_3^2-6a_3^4, \tag{B.6}$$

$$\mathrm{tr}\,[\hat{\rho}(0,\vec{b})\mathbb{1}]=1,$$
$$\mathrm{tr}\,[\hat{\rho}(0,\vec{b})(\vec{a}\hat{\vec{s}})]=0,$$
$$\mathrm{tr}\,[\hat{\rho}(0,\vec{b})(\vec{a}\hat{\vec{s}})^2]=3-3a_3^2,$$
$$\mathrm{tr}\,[\hat{\rho}(0,\vec{b})(\vec{a}\hat{\vec{s}})^3]=0,$$
$$\mathrm{tr}\,[\hat{\rho}(0,\vec{b})(\vec{a}\hat{\vec{s}})^4]=12-21a_3^2+9a_3^4. \tag{B.7}$$

In a general frame of reference a_3 converts into $(\vec{a}\vec{b})$. Then the traces with $\hat{\rho}(m,\vec{b})$, m = -1,-2 follow from eqs. (B.6) and (B.5),

respectively, simply substituting $-\vec{b}$ for $\vec{b}$, or $-a_3$ for a_3 in the r.h.s!s of eqs. (B.6) and (B.5). Now we can easily obtain both the traces given in Table 1 and the general trace (B.1)

$$\mathrm{tr}\Big[\big(\alpha\cdot\mathbb{1}+\beta(\vec{a}\hat{\vec{s}})+\gamma(\vec{a}\hat{\vec{s}})^2+\delta(\vec{a}\hat{\vec{s}})^3+\varepsilon(\vec{a}\hat{\vec{s}})^4\big)\cdot$$
$$\cdot\big(\alpha'\cdot\mathbb{1}+\beta'(\vec{b}\hat{\vec{s}})+\gamma'(\vec{b}\hat{\vec{s}})^2+\delta'(\vec{b}\hat{\vec{s}})^3+\varepsilon'(\vec{b}\hat{\vec{s}})^4\big)\Big]=$$
$$=(\alpha'+2\beta'+4\gamma'+8\delta'+16\varepsilon')\cdot$$
$$\cdot\Big[\alpha+\beta\cdot 2(\vec{a}\vec{b})+\gamma(1+3(\vec{a}\vec{b})^2)+\delta(5(\vec{a}\vec{b})+3(\vec{a}\vec{b})^3)+\varepsilon(\tfrac{5}{2}+12(\vec{a}\vec{b})^2+\tfrac{3}{2}(\vec{a}\vec{b})^4)\Big]+$$
$$+(\alpha'+\beta'+\gamma'+\delta'+\varepsilon')\cdot$$
$$\cdot\Big[\alpha+\beta(\vec{a}\vec{b})+\gamma(\tfrac{5}{2}-\tfrac{3}{2}(\vec{a}\vec{b})^2)+\delta(7(\vec{a}\vec{b})-6(\vec{a}\vec{b})^3)+\varepsilon(\tfrac{17}{2}-\tfrac{3}{2}(\vec{a}\vec{b})^2-6(\vec{a}\vec{b})^4)\Big]+$$
$$+\alpha'\Big[\alpha+\gamma(3-3(\vec{a}\vec{b})^2)+\varepsilon(12-21(\vec{a}\vec{b})^2+9(\vec{a}\vec{b})^4\Big]+$$
$$+(\alpha'-\beta'+\gamma'-\delta'+\varepsilon')\cdot$$
$$\cdot\Big[\alpha-\beta(\vec{a}\vec{b})+\gamma(\tfrac{5}{2}-\tfrac{3}{2}(\vec{a}\vec{b})^2)-\delta(7(\vec{a}\vec{b})-6(\vec{a}\vec{b})^3)+\varepsilon(\tfrac{17}{2}-\tfrac{3}{2}(\vec{a}\vec{b})^2-6(\vec{a}\vec{b})^4)\Big]+$$
$$+(\alpha'-2\beta'+4\gamma'-8\delta'+16\varepsilon')\cdot$$
$$\cdot\Big[\alpha-\beta\cdot 2(\vec{a}\vec{b})+\gamma(1+3(\vec{a}\vec{b})^2)-\delta(5(\vec{a}\vec{b})+3(\vec{a}\vec{b})^3)+\varepsilon(\tfrac{5}{2}+12(\vec{a}\vec{b})^2+\tfrac{3}{2}(\vec{a}\vec{b})^4)\Big]. \tag{B.8}$$

This formula covers all the traces of interest, such as in Table 1 and others.

Appendix C. A typical integral we need is

$$I=(2s+1)\int d\mu(\vec{a})\left[\alpha'\mathbb{1}+\beta'(\vec{a}\hat{\vec{s}})+\gamma'(\vec{a}\hat{\vec{s}})^2+\delta'(\vec{a}\hat{\vec{s}})^3+\varepsilon'(\vec{a}\hat{\vec{s}})^4\right]\otimes$$
$$\otimes\left[\alpha\mathbb{1}+\beta(\vec{a}\hat{\vec{s}})+\gamma(\vec{a}\hat{\vec{s}})^2+\delta(\vec{a}\hat{\vec{s}})^3+\varepsilon(\vec{a}\hat{\vec{s}})^4\right] \tag{C.1}$$

for spins up to 2. It equals

$$I=A_0\cdot\mathbb{1}\otimes\mathbb{1}+A_1\hat{s}_i\otimes\hat{s}_i+A_2\{\hat{s}_i\hat{s}_j\}\otimes\{\hat{s}_i\hat{s}_j\}+A_3\{\hat{s}_i\hat{s}_j\hat{s}_k\}\otimes\{\hat{s}_i\hat{s}_j\hat{s}_k\}+$$
$$+A_4\{\hat{s}_i\hat{s}_j\hat{s}_k\hat{s}_\ell\}\otimes\{\hat{s}_i\hat{s}_j\hat{s}_k\hat{s}_\ell\}, \tag{C.2}$$

where

$$A_0=(2s+1)\Big[\alpha'\alpha+\tfrac{1}{3}s(s+1)(\alpha'\gamma+\gamma'\alpha)+\tfrac{1}{3\cdot 5}s(s+1)(3s(s+1)-1)(\alpha'\varepsilon+\varepsilon'\alpha)+$$
$$+\tfrac{1}{3\cdot 5\cdot 7}(s(s+1))^2(3s(s+1)+1)(\gamma'\varepsilon+\varepsilon'\gamma)+\tfrac{1}{3\cdot 5}(s(s+1))^2\gamma'\gamma+$$
$$+\tfrac{1}{3\cdot 5\cdot 7\cdot 9}(s(s+1))^2(9(s(s+1))^2+18s(s+1)-13)\varepsilon'\varepsilon\Big],$$

$$A_1=(2s+1)\left[\frac{1}{3}\beta'\beta+\frac{1}{3\cdot 5}(3s(s+1)-1)(\beta'\delta+\delta'\beta)+\frac{1}{3\cdot 5\cdot 7}(3s(s+1)-1)^2\delta'\delta\right],$$

$$A_2=(2s+1)\left[\frac{1}{2\cdot 3\cdot 5\cdot 7}(6s(s+1)-5)(\gamma'\varepsilon+\varepsilon'\gamma)+\frac{1}{2\cdot 3\cdot 5}\gamma'\gamma+\right.$$
$$\left.+\frac{1}{2\cdot 3\cdot 5\cdot 7\cdot 9}(6s(s+1)-5)^2\varepsilon'\varepsilon\right],$$

$$A_3=(2s+1)\frac{1}{3\cdot 5\cdot 7\cdot 6}\delta'\delta,$$

$$A_4=(2s+1)\frac{1}{3\cdot 5\cdot 7\cdot 9\cdot 24}\varepsilon'\varepsilon. \tag{C.3}$$

Now we can calculate the integrals entering eq. (16)

$$(2s+1)\int d\mu(\vec{\alpha})\,\hat{\rho}(m,\vec{\alpha})\otimes\hat{\rho}(m,\vec{\alpha})=$$
$$=A_0(m)\,\mathbb{1}\otimes\mathbb{1}+A_1(m)\hat{s}_i\otimes\hat{s}_i+A_2(m)\{\hat{s}_i\hat{s}_j\}\otimes\{\hat{s}_i\hat{s}_j\}+$$
$$+A_3(m)\{\hat{s}_i\hat{s}_j\hat{s}_k\}\otimes\{\hat{s}_i\hat{s}_j\hat{s}_k\}+A_4(m)\{\hat{s}_i\hat{s}_j\hat{s}_k\hat{s}_l\}\otimes\{\hat{s}_i\hat{s}_j\hat{s}_k\hat{s}_l\}, \tag{C.4}$$

where (spin 2)

for $m=0$ $\alpha'=\alpha=1,\ \beta'=\beta=0,\ \gamma'=\gamma=-\frac{5}{4},\ \delta'=\delta=0,\ \varepsilon'=\varepsilon=\frac{1}{4}$,

$$A_0(0)=\frac{8}{3\cdot 7},\ A_1(0)=0,\ A_2(0)=-\frac{127}{2^4\cdot 3^3\cdot 7},\ A_3(0)=0,$$
$$A_4(0)=\frac{1}{2^7\cdot 3^4\cdot 7}; \tag{C.5}$$

for $m=1$ $\alpha'=\alpha=0,\ \beta'=\beta=\frac{2}{3},\ \gamma'=\gamma=\frac{2}{3},\ \delta'=\delta=-\frac{1}{6},\ \varepsilon'=\varepsilon=-\frac{1}{6}$,

$$A_0(1)=\frac{59}{3\cdot 7\cdot 9},\ A_1(1)=-\frac{103}{2^2\cdot 3^3\cdot 7},\ A_2(1)=-\frac{263}{2^3\cdot 3^5\cdot 7},$$
$$A_3(1)=\frac{1}{2^3\cdot 3^4\cdot 7},\ A_4(1)=\frac{1}{2^5\cdot 3^6\cdot 7}; \tag{C.6}$$

for $m=2$ $\alpha'=\alpha=0,\ \beta'=\beta=-\frac{1}{12},\ \gamma'=\gamma=-\frac{1}{24},\ \delta'=\delta=\frac{1}{12},\ \varepsilon'=\varepsilon=\frac{1}{24}$,

$$A_0(2)=\frac{5}{3\cdot 4\cdot 9},\ A_1(2)=\frac{43}{2^3\cdot 3^3\cdot 7},\ A_2(2)=\frac{233}{2^6\cdot 3^5\cdot 7},$$
$$A_3(2)=\frac{1}{2^5\cdot 3^4\cdot 7},\ A_4(2)=\frac{1}{2^9\cdot 3^6\cdot 7}. \tag{C.7}$$

To obtain the completeness relation in form (16) we solve the set of equations

$$\frac{1}{5}\left[A_0(0)v_0+A_0(1)v_1+A_0(2)v_2\right]=u_0+1,$$
$$\frac{1}{5}\left[A_i(0)v_0+A_i(1)v_1+A_i(2)v_2\right]=u_i\ ;\ i=1,2,3,4 \tag{C.8}$$

for v_0, v_1, v_2 . $u_0, u_1, \ldots, u_4$ are the coefficients of the l.h.s. of eq. (6):

$$u_0=\frac{5}{2\cdot 3},\ u_1=-\frac{47}{2^2\cdot 3^3},\ u_2=-\frac{127}{2^5\cdot 3^3},\ u_3=\frac{1}{2^4\cdot 3^4},\ u_4=\frac{1}{2^8\cdot 3^4}. \tag{C.9}$$

The solution is given by eq. (16.a).

To obtain the completeness relation in form (17) we solve the set of equations

$$A_i(m)=u_i\ ,\qquad i=0,1,2,3,4 \tag{C.10}$$

for $\alpha,\beta,\gamma,\delta,\varepsilon$ either with

$$\alpha'=0,\ \beta'=\frac{2}{3},\ \gamma'=\frac{2}{3},\ \delta'=-\frac{1}{6},\ \varepsilon'=-\frac{1}{6}\quad \text{for } m=1$$

or with

$$\alpha'=0,\ \beta'=-\frac{1}{12},\ \gamma'=-\frac{1}{24},\ \delta'=\frac{1}{12},\ \varepsilon'=\frac{1}{24}\quad \text{for } m=2.$$

The completeness relation in form (18) is obtained by solving the set of equations

$$A_i=u_i\ ,\qquad i=0,1,2,3,4 \tag{C.11}$$

for $\alpha,\beta,\gamma,\delta,\varepsilon$ with $\alpha'=\alpha,\ \beta'=\beta,\ \gamma'=\gamma,\ \delta'=\delta,\ \varepsilon'=\varepsilon$.

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