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APPLICATION OF THE UNITARY REPRESENTATIONS OF THE LORENTZ GROUP FOR THE DESCRIPTION OF INTERACTION OF A FERMION WITH A SCALAR PARTICLE

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The present work is devoted to the application of the relativistic configurational representation (RCR) [1] for description of the interaction of two particles with spin 1/2 and 0.

The RCR uses instead of the Fourier transformation with three-dimensional exponential (which realizes the unitary irreducible representations (UIR) of the Galilelian group) the expansion in matrix elements of the principle series of UIR of the Lorentz group.

The description in terms of the RCR has a three-dimensional form and is similar in form to quantum mechanics. Due to this fact it is widely used in the relativistic two-particle problem. In particular, the relativistic generalizations of Coulomb and Yukawa potentials of spin-orbital and tensor forces were found /4-6/. The existing three-dimensional apparatus is based on the free Hamiltonian for the scalar particles. In [7] the free Hamiltonians, for functions realizing UIR of the Lorentz group with spins 1/2 and 1 were found. In the present work we use this spin-case Hamiltonian for particles with spin 1/2 and corresponding wave functions for describing the interaction of a fermion with a scalar particle. We shall confine our consideration to the case of the particles with equal masses.

The one-boson exchange model and a variant of the quasipotential equation based on the Hamilton formulation of quanrum field theory [3] developed by Kadyshevsky is chosen as the starting point for taking the particle interaction into account.

In this variant, as in the Logunov - Tavkhelidze approach, the momenta of two particles are on mass-shell

$$p_0^2 - \vec{p}^{\,2} = m^2 . \tag{1.1}$$

In the center of mass system (c.m.s.) the scattering amplitude and wave function equations describing the interaction of scalar and spin 1/2 particles have the form [3]

$$E_p (E_p - E_q) \Psi_{q\sigma}(\vec{p}) =$$

$$= \frac{1}{(2\pi)^3} \sum_{\sigma'=-1/2}^{1/2} \int d\Omega_k \, V_{\sigma\sigma'}(\vec{p}, \vec{k}; E_q) \Psi_{q\sigma'}(\vec{k}) \tag{1.2}$$

$$T_{\sigma\sigma'}(\vec p,\vec q) = V_{\sigma\sigma'}(\vec p,\vec q;E_q) +$$
$$+\frac{1}{(2\pi)^3}\sum_{\sigma''=-1/2}\int d\Omega_k\, \frac{V_{\sigma\sigma''}(\vec p,\vec k;E_q)\, T_{\sigma''\sigma'}(\vec k,\vec q)}{E_k\,(E_k-E_q-i\varepsilon)} \qquad (1.3)$$

where $V_{\sigma\sigma'}(\vec p,\vec k;E_q)$ is the quasipotential, $2E_q=2\sqrt{\vec q^{\,2}+m^2}$ is the total energy of the system, σ, σ' are spin indices and the volume element in the momentum space $d\Omega_k = d\vec k/\sqrt{\vec k^{\,2}+m^2}$ is the invariant measure on the hyperboloid (1.1).

The paper is organized as follows. As it is known, in the one-boson-exchange model the potential is taken to be matrix elements of the relativistic scattering amplitude in the second order in the coupling constant. In /2/ this Born approximation in the form given by quantum field theory is reduced to the three-dimensional covariant form allowing the transition to RCR.

In §3 the properties of the Hamiltonian and momentum operators introduced in /7/ are discussed, and questions concerning the partial wave expansion of the plane waves describing free motion of a particle with spin 1/2 are considered there. In §4 the addition theorem for such waves is used for construction of the local quasi-potential in RCR. It is shown that the expressions for transforms of Yukawa and Coulomb potentials do not contain singularities.

2. The local form of one-boson-exchange potential in Lobachevsky space

Equation (1.1) defines the three-dimensional surface of the hyperboloid the upper sheet of which models the Lobachevsky psace x). The Lorentz group is the motion group of this space.

Pure Lorentz transformations Λ_p ("boosts") i.e. such that

$$\Lambda_p(m,\vec o)=(p^\circ,\vec p)\,;$$
$$\overrightarrow{\Lambda_p^{-1}k} \equiv \vec k(-)\vec p = \vec k - \frac{\vec p}{m}\left(k_o - \frac{\vec k\vec p}{p_o+m}\right)=\vec\Delta \qquad (2.1)$$

$$(\Lambda_p^{-1}k)^\circ \equiv (\vec k(-)\vec p)^\circ = (k^\circ p^\circ - \vec k\vec p)/m =$$
$$= \sqrt{m^2+(\vec k(-)\vec p)^2} = \Delta^\circ \qquad (2.2)$$

in the nonrelativistic limit turn into transformations of the trans-

x) We use the unit system in which $\hbar = c = 1$.

lation in the flat Euclidean space: $\vec k(-)\vec p \to \vec k - \vec p$.
In spherical coordinates

$$p_o = m\,ch\chi_p\,;\quad \vec p = m\,sh\chi_p\cdot\vec n_p\,;\quad \vec n_p = \frac{\vec p}{|\vec p|}\,; \qquad (2.3)$$
$$k_o = m\,ch\chi_k\,;\quad \vec k = m\,sh\chi_k\cdot\vec n_k\,;\quad \vec n_k = \frac{\vec k}{|\vec k|}\,; \qquad (2.4)$$

equality (2.2) takes on the form of the theorem on a cosine of a composite angle in the Lobachevsky trigonometry

$$ch\chi_{pk} = \sqrt{1+(\vec k(-)\vec p)^2/m^2} =$$
$$= ch\chi_p\, ch\chi_k - sh\chi_p\cdot sh\chi_k\cdot \vec n_p\vec n_k \qquad (2.5)$$

The vector $\vec k(-)\vec p$ can be considered as a relativistic geometric generalization of the momentum transfer vector $\vec k-\vec p$. The square of the four-dimensional vector of the momentum transfer is expressed through the vector $\vec\Delta = \vec k(-)\vec p$ in the following way

$$t=(k-p)^2=2m^2-2p k = 2m^2-2m\sqrt{m^2+(\vec k(-)\vec p)^2} \qquad (2.6)$$

We consider the case when the interaction is realized through the exchange of a scalar particle with mass m which is described by the Feynman diagram. The scattering amplitude in the second order in the coupling constant is given by the expression

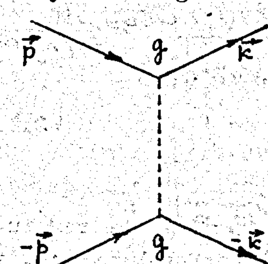

Fig.1

$$\langle \vec p\sigma|T^{(2)}|\vec k\sigma'\rangle =$$
$$= -g^2\,\frac{j_{\sigma\sigma'}(\vec p,\vec k)}{\mu^2-(p-k)^2} \qquad (2.7)$$

The current-matrix elements in (2.7) can be transformed by passing to bispinors defined in the rest frame of references

$$j^{\sigma\sigma'}(\vec p,\vec k) = \bar u^\sigma(\vec p)\,u^{\sigma'}(\vec k) = \bar u^\sigma(0)\,S_p^{-1} S_k\, u^{\sigma'}(0). \qquad (2.8)$$

Four-dimensional matrices of the bispinor transformation $u^\sigma(\vec p) = S_p\, u^\sigma(0)$, corresponding to the pure Lorentz transformations are defined in the following way

$$S_p = \sqrt{(p_0+m)/2m}\cdot\left(1+\vec{\alpha}\vec{p}/(p_0+m)\right) \tag{2.9}$$

where $\vec{\alpha} = \gamma_0\vec{\gamma}$. (2.I0)

According to [8]

$$S_p^{-1}S_k = S_{(\Lambda^{-1}_p k)}\begin{pmatrix} I & 0 \\ 0 & I\end{pmatrix}\cdot \mathcal{D}^{1/2}\{V^{-1}(\Lambda_p, k)\} \tag{2.11}$$

and the rotation matrix (Wigner rotation) is written in the form of [9]

$$\mathcal{D}^{1/2}\{V^{-1}(\Lambda_p,k)\} \equiv \mathcal{D}^W(\vec{k},\vec{p}) = \tag{2.12}$$

$$= \frac{(p_0+m)(k_0+m) - \vec{k}\vec{p} - i\vec{\sigma}[\vec{k}\cdot\vec{p}]}{\sqrt{2(p_0+m)(k_0+m)(k_0p_0-\vec{k}\vec{p}+m^2)}}$$

As a result we have for (2.8)

$$j^{\sigma\sigma'}(\vec{p},\vec{k}) = \sum_{\sigma_p=\pm 1/2} j^{\sigma\sigma_p}(\vec{k}(-)\vec{p})\mathcal{D}^W_{\sigma_p\sigma'}(\vec{k},\vec{p}) \tag{2.13}$$

where

$$j^{\sigma\sigma_p}(\vec{k}(-)\vec{p}) = \sqrt{2m(\Delta_0+m)}\cdot\delta_{\sigma\sigma_p} \tag{2.14}$$

Accordingly the expression for the quasipotential is written in the form

$$V^{(2)\sigma\sigma_p}(\vec{p},\vec{k};E_q) = -g^2\frac{\sqrt{2m(\Delta_0+m)}\,\delta_{\sigma\sigma_p}}{\mu^2-2m^2+2m\Delta_0}\mathcal{D}^W(\vec{k},\vec{p}) \equiv \tag{2.15}$$

$$\equiv V(\vec{\Delta};E_q)\cdot\mathcal{D}^W(\vec{k},\vec{p}).$$

Upon separating the Wigner rotation matrix $\mathcal{D}^W(\vec{k},\vec{p})$, the remaining part of the quasipotential is local in the Lobachevsky space, i.e. depends only on $(\vec{p}(-)\vec{k})^2$.

The rotation matrix $\mathcal{D}(\vec{k},\vec{p})$ is of a pure kinematical nature and according to the terminology of the authors of [9] it transfers the spin indices from one momentum onto another.

3. Relativistic configurational representation for particles with spin 1/2

In spin-zero case the transition to the relativistic configurational representation [1] is performed with the help of functions that realize the infinite dimetial UIR of the Lorentz group [I0,11] (with spin = 0)

$$\xi^0(\vec{p};\vec{n},r) = \left(\frac{p_0-\vec{p}\vec{n}}{m}\right)^{-1-irm}, \tag{3.1}$$

$$\vec{n} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta);\ \vec{n}^2=1$$

and compose the complete system in the Lobachevsky space. The parameter r in (3.1) is connected with the eigenvalues of the Casimir operator of the Lorentz group $\hat{C}_1(p) = \vec{N}^2(\vec{p}) - \vec{L}^2(\vec{p})$ in the following way

$$\hat{C}_1(p)\cdot\left(\frac{p_0-\vec{p}\vec{n}}{m}\right)^{-1-irm} = \left(\frac{1}{m^2}+r^2\right)\left(\frac{p_0-\vec{p}\vec{n}}{m}\right)^{-1-irm} \tag{3.2}$$

In (3.2) $(0\le r<\infty\ ;\ i,j,k=1,2,3)$:

$$\vec{N}_1(\vec{p}) = ip_0\frac{\partial}{\partial\vec{p}}\ ;\quad \mathcal{L}_i(\vec{p}) = i\varepsilon_{ijk}p_k\frac{\partial}{\partial p_j} \tag{3.3}$$

In the nonrelativistic limit when the Lobachevsky space turns into the Euclidean space we have [1]

$$\xi^{*0}(\vec{p};\vec{n},r) \to \exp(-i\vec{p}\vec{r});\ \vec{r} = \vec{n}r$$

and the Casimir operator of the Lorentz group $C_1(\vec{p})$ turns into the Casimir operator of the group of motions of the Euclidean space

$$\hat{C}_1(\vec{p}) \Longrightarrow \left(i\frac{\partial}{\partial\vec{p}}\right)^2 = \hat{C}_э(\vec{p}),$$

and

$$\hat{C}_э(\vec{p})\exp(-i\vec{p}\vec{r}) = r^2\exp(-i\vec{p}\vec{r}) \tag{3.5}$$

The expansion takes place

$$\psi(r,\vec{n}) = \frac{1}{(2\pi)^3}\int d\Omega_p\,\xi^0(\vec{p};\vec{n},r)\,\psi(\vec{p}) \tag{3.6}$$

$$\Psi(\vec{p}) = \frac{1}{(2\pi)^3}\int d\omega_{\vec{n}}\,r^2dr\,\xi^{*0}(\vec{p};\vec{n},r)\,\psi(r,\vec{n}) \tag{3.7}$$

$$d\omega_{\vec{n}} = \sin\theta\,d\theta\,d\varphi$$

In ref. [1] the form of the Hamiltonian and the momentum operators for the plane waves (3.1) was found as follows:

$$\hat{H}^{o} = m\, ch\left(\frac{i}{m}\frac{\partial}{\partial r}\right) + \frac{i}{r}\, sh\left(\frac{i}{m}\frac{\partial}{\partial r}\right) + \frac{\vec{\mathcal{L}}^{2}(\theta,\varphi)}{2mr^{2}}\exp\left(\frac{i}{m}\frac{\partial}{\partial r}\right) \tag{3.8}$$

$$\hat{\vec{p}}^{o} = -\vec{n}\left(m\exp\left(\frac{i}{m}\frac{\partial}{\partial r}\right) - \hat{H}^{o}\right) - \frac{[\vec{n}\cdot\vec{\mathcal{L}}(\theta,\varphi)]}{r}\cdot\exp\left(\frac{i}{m}\frac{\partial}{\partial r}\right) \tag{3.9}$$

The partial expansion for (3.1) has the form

$$\xi^{(0)}(\vec{p};\vec{n},r) = \sum_{l=0}^{\infty}(2l+1)\, i^{l}\, P_{l}^{(0)}(ch\chi_{p}, r)\, P_{l}(\vec{n}_{p}\cdot\vec{n}) \tag{3.10}$$

with the radial functions

$$P_{l}^{(0)}(ch\chi_{p}, r) = (-i)^{l}\sqrt{\frac{\pi}{2\, sh\chi_{p}}}\cdot\frac{\Gamma(irm+l+1)}{\Gamma(irm+1)}\, P^{-1/2-l}_{-1/2+irm}(ch\chi_{p}), \tag{3.11}$$

$$P_{0}^{(0)}(ch\chi_{p}, r) = \frac{\sin(rm\chi_{p})}{rm\, sh\chi_{p}}. \tag{3.12}$$

The addition theorem for the plane waves has the form

$$\int d\omega_{\vec{n}}\, \xi^{*(0)}(\vec{p};\vec{n},r)\, \xi^{(0)}(\vec{k};\vec{n},r) = \int d\omega_{\vec{n}}\, \xi^{*(0)}(\vec{\Delta};\vec{n},r) \tag{3.13}$$

where

$$\vec{\Delta} = \vec{p}\,(-)\,\vec{k}\ ; \quad d\omega_{\vec{n}_{k}} = (k_{0}-\vec{k}\vec{n})^{-2}\, d\omega_{\vec{n}}$$

$$\vec{n}_{k} = \frac{m\vec{n} - \vec{k}\left(1 - \frac{\vec{n}\vec{k}}{k_{0}+m}\right)}{k_{0} - \vec{k}\vec{n}} \tag{3.14}$$

Let us consider now the formalism for the particles with spin 1/2. In what follows it will be more convenient to use in part the dimensionless notation introduced by the substitution

The generalization of expansions (3.6), (3.7) for the spin case was found in [12,13].

For the spin 1/2 these expansions look as follows:

$$\Psi^{(1/2)}(\alpha,\vec{n}) = \frac{1}{(2\pi)^{3}}\int d\Omega_{p}\, \mathcal{D}^{(1/2)}(R_{W})\left(\frac{p_{0}-\vec{p}\vec{n}}{m}\right)^{-1-irm}\Psi^{(1/2)}(\vec{p}) \tag{3.15}$$

$$\Psi^{(1/2)}(\vec{p}) = \frac{1}{(2\pi)^{3}}\int(\alpha^{2}+1/4)\, d\alpha\, d\omega_{\vec{n}}\, \mathcal{D}^{(1/2)}(R_{W})\left(\frac{p_{0}-\vec{p}\vec{n}}{m}\right)^{-1+irm}\Psi^{(1/2)}(\alpha,\vec{n}) \tag{3.16}$$

The functions $\mathcal{D}^{(1/2)}(R_{W}) \equiv \mathcal{D}^{1/2}(\vec{p},\vec{n})$ in (3.15), (3.16) describe the rotation from the direction of the $\vec{p}$ vector to the direction of the n-vector. In [7] it has been shown by means of the direct solution of the problem on proper functions for the Casimir operator of the Lorentz group

$$C_{1}^{(1/2)}(\vec{p}) = \vec{N}^{2}(\vec{p}) - \vec{J}^{2}(\vec{p}) \tag{3.17}$$

that the $\mathcal{D}^{(1/2)}(R_{W})$-function has the form of the Wigner rotation matrix

$$\hat{C}_{1}^{(1/2)}(\vec{p})\cdot\mathcal{D}^{*(1/2)}(\vec{p},\vec{n})\cdot\xi^{*(0)}(\vec{p};\vec{n},\alpha) = \left(1+\alpha^{2}+\frac{1}{4}\right)\cdot\mathcal{D}^{*(1/2)}(\vec{p},\vec{n})\cdot\xi^{*(0)}(\vec{p};\vec{n},\alpha) \tag{3.18}$$

where

$$\mathcal{D}^{*(1/2)}(\vec{p},\vec{n}) = \frac{p_{0}-\vec{p}\vec{n} - i\vec{\sigma}[\vec{p}\cdot\vec{n}]}{\sqrt{2(p_{0}+1)(p_{0}-\vec{p}\vec{n})}} \tag{3.19}$$

In the spin 1/2 case

$$\vec{N}(\vec{p}) = ip_{0}\frac{\partial}{\partial\vec{p}} - \frac{[\vec{\sigma}\vec{p}]}{2(p_{0}+1)}\ ; \quad \vec{J}(\vec{p}) = \vec{\mathcal{L}}(\vec{p}) + \frac{\vec{\sigma}}{2} \tag{3.20}$$

The second Casimir operator of the Lorentz group with the eigenvalues $\nu\alpha$

$$\hat{C}_{2}^{(1/2)}(\vec{p}) = \vec{N}(\vec{p})\,\vec{J}(\vec{p}) \Rightarrow \nu\alpha$$

is nondiagonal on the functions

$$\xi^{*(1/2)}(\vec p;\vec n,\alpha)=\xi^{*(0)}(\vec p;\vec n,\alpha)\cdot \mathcal{D}^{1/2}(\vec p;\vec n) \qquad (3.21)$$

The helicity basis functions

$$\xi^{*(1/2)}(\vec p;\vec n,\alpha)=\xi^{*(1/2)}(\vec p;\vec n,\alpha)\cdot\mathcal{D}^{*1/2}(\vec n) \qquad (3.22)$$

where $\mathcal{D}^*(\vec n)$ are eigenfunctions of the operator $(\vec\sigma\vec n)$ form a common basis for $C_1^{1/2}(\vec p)$ and $C_2^{1/2}(\vec p)$.

Further we shall use the spin 1/2 plane waves (3.21), the relations of completeness and orthogonality for which have the following form

$$\frac{1}{(2\pi)^3}\int d\Omega_p\, \xi^{(1/2)}(\vec p;\vec n,\alpha)\,\xi^{*(1/2)}(\vec p;\vec n_1,\alpha_1)=\delta^{(3)}(\vec n-\vec n_1)\frac{\delta(\alpha-\alpha_1)}{\alpha^2+1/4} \qquad (3.23)$$

$$\frac{1}{(2\pi)^3}\int d\alpha\left(\alpha^2+\frac14\right)d\omega_{\vec n}\,\xi^{*(1/2)}(\vec p;\vec n,\alpha)\cdot\xi^{(1/2)}(\vec k;\vec n,\alpha)= k_0\cdot\delta^{(3)}(\vec p-\vec n) \qquad (3.24)$$

Using the equalities

$$\mathcal{D}^{1/2}(\vec p,\vec n)\,\mathcal{D}^{*1/2}(\vec k,\vec n)=\mathcal{D}^{*1/2}(\vec k,\vec p)\cdot\mathcal{D}^{*1/2}(\vec\Delta,\vec n_k) \qquad (3.25)$$

and (3.13), (3.14) we get the addition theorem in the chosen basis (see also [14])

$$\int d\omega_{\vec n}\,\xi^{(1/2)}(\vec p;\vec n,\alpha)\cdot\xi^{*(1/2)}(\vec k;\vec n,\alpha)= \mathcal{D}^{*(1/2)}(\vec k,\vec p)\int d\omega_{\vec n_k}\cdot\xi^{*1/2}(\vec\Delta,\vec n_k,\alpha)= \qquad (3.26)$$

$$= \mathcal{D}^{*1/2}(\vec k,\vec p)\int d\omega_{\vec n}\,\xi^{*(1/2)}(\vec\Delta;\vec n,\alpha).$$

The operators of the free Hamiltonian $\hat H^{(1/2)}$, and the momentum $\hat{\vec P}^{(1/2)}$ for the particles with spin 1/2 were found in [7], by the following method.

Basis functions

$$\xi^{(1/2)}(\vec p;\vec n,\alpha)=\xi^{(0)}(\vec p;\vec n,\alpha)\cdot\mathcal{D}^{1/2}(\vec p,\vec n) \qquad (3.27)$$

with the help of the operators $\hat H^{(0)}$, $\hat{\vec P}^{(0)}$ can be represented in the form

$$\xi^{(1/2)}(\vec p;\vec n,\alpha)=\left(2\,\mathrm{ch}\frac{i}{2}\frac{\partial}{\partial\alpha}-\frac{i\vec\sigma\vec{\mathcal L}(\vec n)}{\alpha-i/2}\exp\frac{i}{2}\frac{\partial}{\partial\alpha}\right)\cdot \qquad (3.28)$$

$$\cdot\frac{\xi^{(0)}(\vec p;\vec n,\alpha)}{\sqrt{2(p_0+1)}}\equiv\hat B\cdot\frac{\xi^{(0)}(\vec p;\vec n,\alpha)}{\sqrt{2(p_0+1)}}.$$

We can find the operator $\hat k$ for which the equation

$$\hat H^{(0)}=\frac12\hat k\hat B-1 \qquad (3.29)$$

takes place and it is of the form

$$\hat k=2\,\mathrm{ch}\left(\frac{i}{2}\frac{\partial}{\partial\alpha}\right)+\frac{2i}{\alpha}\,\mathrm{sh}\left(\frac i2\frac{\partial}{\partial\alpha}\right)+\frac{i\vec\sigma\vec{\mathcal L}(\vec n)}{\alpha}\exp\left(\frac{i}{2}\frac{\partial}{\partial\alpha}\right) \qquad (3.30)$$

Then from the relation

$$\hat B\left(\frac12\hat k\hat B-1\right)\frac{\xi^{(0)}(\vec p;\vec n,\alpha)}{\sqrt{2(p_0+1)}}=p_0\,\xi^{(1/2)}(\vec p;\vec n,\alpha) \qquad (3.31)$$

we find, that the operator

$$\frac12\hat B\hat k-1 \qquad (3.32)$$

is diagonal on functions (3.27) with the proper eigenvalues which are equal to the energy of the particle $-p_0$. This is the looked for operator $\hat H^{(1/2)}$ in RCR in the dimensional notation

$$\hat H^{(1/2)}=mc^2\,\mathrm{ch}\left(i\lambda\frac{\partial}{\partial r}\right)+\frac{i\hbar c}{2(r+\frac{i\lambda}{2})}\exp\left(i\lambda\frac{\partial}{\partial r}\right)-$$

$$-\frac{i\hbar c}{2(r-\frac{i\lambda}{2})}\exp\left(-i\lambda\frac{\partial}{\partial r}\right)+\frac{\hat{\mathcal L}^2(\theta,\varphi)+\hbar\vec\sigma\vec{\mathcal L}(\theta,\varphi)}{2m(r^2+\lambda^2/4)}\exp\left(i\lambda\frac{\partial}{\partial r}\right)$$

$$-\frac{\hbar\vec\sigma\vec{\mathcal L}(\theta,\varphi)+\hbar^2}{2m(r^2+\lambda^2/4)}\quad;\quad \lambda=\hbar/mc \qquad (3.33)$$

The momentum operators $\hat{\vec{P}}^{(1/2)}$ can be found from the expression

$$\hat{\vec{P}}^{(1/2)} \cdot \hat{B} \cdot \frac{\xi^{(0)}(\vec{p};\vec{n},\alpha)}{\sqrt{2(p_0+1)}} = \hat{B} \cdot \hat{\vec{P}}^{(0)} \cdot \frac{\xi^{(0)}(\vec{p};\vec{n},\alpha)}{\sqrt{2(p_0+1)}} \qquad (3.34)$$

$$\hat{\vec{P}}^{(1/2)}(\alpha,\vec{n}) = -\vec{n}\left(\exp\left(i\frac{\partial}{\partial\alpha}\right) - \hat{H}^{(1/2)}(\alpha,\vec{n})\right) + \frac{[\vec{n}\,\vec{\sigma}]}{2(\alpha+i/2)} -$$

$$- \frac{2\alpha[\vec{n}\,\vec{\mathcal{L}}(\theta,\varphi)] + (\alpha-i/2)[\vec{n}\,\vec{\sigma}] + [\vec{n}\,\vec{\sigma}]\vec{\mathcal{L}}(\theta,\varphi)}{2(\alpha^2+1/4)} \exp\left(i\frac{\partial}{\partial\alpha}\right). \qquad (3.35)$$

The Casimir operators of the Lorentz group $\hat{C}_1^{1/2}$, $\hat{C}_2^{1/2}$ in the variables $\alpha, \vec{n}$ have the form ($\vec{S} = \vec{\sigma}/2$)

$$\hat{C}_1^{1/2} = 1+\alpha^2-(\vec{S}\vec{n})^2 \; ; \qquad \hat{C}_2^{1/2} = \alpha(\vec{S}\vec{n}).$$

To find the radial functions for the spin 1/2, we use (3.27) where we shall represent $\xi^{(0)}(\vec{p};\vec{n},\alpha)$ in the form

$$\xi^{(0)}(\vec{p};\vec{n},\alpha) = 4\pi\sum_{j,l,m} i^l \mathcal{P}_l^{(0)}(ch\chi_p,\alpha)\, \Omega^{*(1/2)}_{jlm}(\vec{n}_p)\, \Omega^{(1/2)}_{jlm}(\vec{n}) \qquad (3.36)$$

with $\vec{n}_p = \vec{P}/|\vec{P}|$, and spherical spinors $\Omega^{*(1/2)}_{jlm}(\vec{n}_p)$, $\Omega^{(1/2)}_{jlm}(\vec{n})$ being the proper functions of the operators $\vec{\sigma}\vec{\mathcal{L}}(\vec{p})$, and $\vec{\sigma}\vec{\mathcal{L}}(\vec{n})$ (they have the same form as in the nonrelativistic formalism).

Let us integrate the expression

$$\xi^{(1/2)}(\vec{p};\vec{n},\alpha)\,\Omega^{(1/2)}_{jlm}(\vec{n}_p) = \hat{B}\,\frac{\xi^{(0)}(\vec{p};\vec{n},\alpha)}{\sqrt{2(p_0+1)}}\,\Omega^{(1/2)}_{jlm}(\vec{n}_p) \qquad (3.37)$$

over the angular variables of the $\vec{n}_p$ vector. The matrix elements obtained in this way can be written in the form

$$\left(2ch\frac{i}{2}\frac{\partial}{\partial\alpha} - \frac{i\vec{\sigma}\vec{\mathcal{L}}(\vec{n})}{\alpha-i/2}\exp\left(\frac{i}{2}\frac{\partial}{\partial\alpha}\right)\right)\frac{4\pi\cdot i^l}{\sqrt{2(p_0+1)}}\cdot \mathcal{P}_l^{(0)}(ch\chi_p,\alpha)\cdot\Omega^{(1/2)}_{jlm}(\vec{n}) \qquad (3.38)$$

with

$$\vec{\sigma}\vec{\mathcal{L}}(\vec{n})\,\Omega^{(1/2)}_{jlm}(\vec{n}) = \beta\,\Omega^{(1/2)}_{jlm}(\vec{n}) \qquad (3.39)$$

$$\beta = j(j+1) - l(l+1) - 3/4$$

taken into account we define the radial functions for the particles with spin 1/2 $\mathcal{P}^{(1/2)}_{j,l}(ch\chi_p,\alpha)$ as coefficients that stand before $4\pi i^l\,\Omega^{(1/2)}_{jlm}(\vec{n})$ in the expression (3.38) where $\vec{\sigma}\vec{\mathcal{L}}(\vec{n}) \to \beta$. These functions can be expressed through $\mathcal{P}_l^{(0)}(ch\chi_p,\alpha)$

$$\mathcal{P}^{(1/2)}_{j,l}(ch\chi_p,\alpha) = \frac{1}{2\,ch\left(\frac{\chi_p}{2}\right)}\left\{\left(1-\frac{i\beta}{\alpha-i/2}\right)\mathcal{P}_l^{(0)}(ch\chi_p,\alpha+i/2) + \mathcal{P}_l^{(0)}(ch\chi_p,\alpha-i/2)\right\}; \quad \beta = l, -l-1. \qquad (3.40)$$

For the s-state it follows from (3.12) that

$$\mathcal{P}^{(1/2)}_{1/2,0}(ch\chi_p,r) = \frac{2mr\cdot\sin(mr\chi_p)\cdot ch\left(\frac{\chi_p}{2}\right) + \cos(mr\chi_p)\,sh\left(\frac{\chi_p}{2}\right)}{2m^2 sh\chi_p\cdot ch\left(\frac{\chi_p}{2}\right)\cdot\left(r^2+\frac{1}{4m^2}\right)} \qquad (3.41)$$

In (3.41) the dimensional notations of the RCR are used.

The radial Hamiltonian operator can be obtained by the following replacement in (3.33)

$$\hat{\mathcal{L}}^2 \to \hbar^2 l(l+1) \; ; \quad \hbar\vec{\sigma}\cdot\vec{\mathcal{L}}(\vec{n}) \to \hbar^2\beta .$$

4. Reduction of the Quasipotential to the local form in RCR

Let us perform a transition to the relativistic configuration representation in (1.2) with the help of the transformation (3.15). We get

$$\int \xi^{(1/2)}(\vec{p};\vec{n},\alpha)\cdot G^{-1}\Psi^{(1/2)}(\vec{p})\,d\Omega_p = \qquad (4.1)$$

$$= \int d\Omega_p\,\xi^{(1/2)}(\vec{p};\vec{n},\alpha)\,\frac{1}{(2\pi)^3}\int d\Omega_k\,V(\vec{p},\vec{k};E_q)\,\Psi^{(1/2)}(\vec{k})$$

where $G^{-1} = E_p(E_p - E_q)$.
Let us then make transition in (4.1) to the new function

$$\Psi^{(1/2)}(\vec{k}) = \frac{1}{(2\pi)^3}\int d\alpha\left(\alpha^2+\frac{1}{4}\right)\xi^{*(1/2)}(\vec{k};\vec{n},\alpha)\,\Psi^{(1/2)}(\alpha,\vec{n})\,d\omega_{\vec{n}} \qquad (4.2)$$

In what follows we shall use the relations

$$\mathcal{D}^{*1/2}(\vec{k},\vec{n}) = \mathcal{D}^{*1/2}(\vec{k},\vec{p})\cdot\mathcal{D}^{1/2}(\vec{k}(-)\vec{p})\cdot\mathcal{D}^{*1/2}(\vec{p},\vec{n}) \qquad (4.3)$$

$$\xi^{*(0)}(\vec{k};\vec{n},\alpha) = \xi^{*(0)}(\vec{k}(-)\vec{p};\vec{n}_{\Lambda_p},\alpha)\cdot\xi^{*(0)}(\vec{p};\vec{n},\alpha) \qquad (4.4)$$

where

$$\vec{n}_{\Lambda_p} = \frac{m\vec{n} - \vec{p}\left[1 - \frac{(\vec{p}\vec{n})}{p_0+m}\right]}{p_0 - \vec{p}\vec{n}}, \qquad (4.5)$$

Using (4.3), (4.4) we have for the spin case

$$\xi^{*(1/2)}(\vec{k};\vec{n},\alpha) = \mathcal{D}^{*(1/2)}(\vec{k},\vec{p})\cdot\xi^{*(1/2)}(\vec{k}(-)\vec{p},\vec{n}_{\Lambda_p},\alpha)\cdot\xi^{*(1/2)}(\vec{p};\vec{n},\alpha) \qquad (4.6)$$

Now we substitute (4.6) into (4.2) with the corresponding indices keeping in mind (2.15) and the equation

$$\mathcal{D}^{1/2}(\vec{k},\vec{p})\cdot\mathcal{D}^{*1/2}(\vec{k},\vec{p}) = 1,$$

we get for the right-hand of (4.1)

$$\int d\Omega_p\,\xi^{(1/2)}(\vec{p};\vec{n},\alpha)\cdot\frac{1}{(2\pi)^3}\int d\alpha_1\left(\alpha_1^2+\frac{1}{4}\right)d\omega_{\vec{n}_1}\cdot \qquad (4.7)$$

$$\cdot\frac{1}{(4\pi)^3}\int d\Omega_k\,V(\vec{\Delta};E_q)\cdot\xi^{*(1/2)}(\vec{\Delta};\vec{n}_{\Lambda_p},\alpha_1)\,\xi^{*(1/2)}(\vec{p};\vec{n}_1,\alpha_1)\cdot\Psi^{(1/2)}(\alpha_1,\vec{n}_1)$$

where we have used (3.16). Then making use of the invariance of the volume element $d\Omega_{\vec{k}} = d\Omega_{\vec{\Delta}}$ and formula (3.15) for the potential form in RCR

$$V(\alpha_1,E_q) = \frac{1}{(2\pi)^3}\int d\Omega_{\vec{\Delta}}\cdot V(\vec{\Delta};E_q)\cdot\xi^{*(1/2)}(\vec{\Delta};\vec{n}_{\Lambda_p},\alpha_1) \qquad (4.8)$$

we get with account of (3.23) for the right-hand side a local form

$$\frac{1}{(2\pi)^3}\int d\alpha_1\left(\alpha_1^2+\tfrac{1}{4}\right)d\omega_{\vec{n}_1}\cdot\int d\Omega_p\,\xi^{(1/2)}(\vec{p};\vec{n},\alpha)\cdot V(\alpha_1;E_q)\cdot$$

$$\cdot\,\xi^{*(1/2)}(\vec{p};\vec{n}_1,\alpha_1)\,\Psi^{(1/2)}(\alpha_1,\vec{n}_1) = V(\alpha;E_q)\,\Psi^{(1/2)}(\alpha,\vec{n}) \qquad (4.9)$$

Thus equation (1.2) in RCR takes the form

$$\hat{H}^{(1/2)}\left(\hat{H}^{(1/2)} - E_q\right)\Psi^{(1/2)}(\alpha,\vec{n}) = V(\alpha;E_q)\,\Psi^{(1/2)}(\alpha,\vec{n}) \qquad (4.10)$$

In /1/ it has been shown that the boson propagator

$$\frac{1}{\mu^2-(p-k)^2} = \frac{1}{\mu^2-2m^2+2m\Delta^0}, \qquad (4.11)$$

after transition to RCR (with spinless functions (3.1)) has the form

$$V^{(0)rel}_{Yuk}(r) = \begin{cases} \dfrac{1}{4\pi r}\cdot\dfrac{ch(rma_1)}{sh(rm\pi)} & \mu^2 < 4m^2, \quad a_1 = \arccos\dfrac{\mu^2-2m^2}{2m^2} \qquad (4.12a) \\[2ex] \dfrac{1}{4\pi r}\cdot\dfrac{\cos(rma_2)}{sh(rm\pi)} & \mu^2 > 4m^2, \quad a_2 = Arch\,\dfrac{\mu^2-2m^2}{2m^2} \qquad (4.12b) \end{cases}$$

that can be treated as the relativistic generalization of the Yukawa potential. For $\mu^2 < 4m^2$ the expression (4.12a) in the nonrelativistic limit turns into the usual Yukawa potential, $\exp(-\mu r)/4\pi r$
At $\mu = 0$ (4.12a) it looks like the Coulomb potential

$$V^{(0)rel}_{Coul.}(r) = \frac{1}{4\pi r}\cdot cth(rm\pi) \qquad (4.13)$$

Let us now find the form of the local part of the propagator

$$\left(\sqrt{2m(\Delta_0+m)}\right)/\left(\mu^2-2m^2+2m\Delta_0\right) \qquad (4.14)$$

We shall use the following simple method:

Plane waves for the particles with spin 1/2 $\xi^{(1/2)}(\vec{\Delta};\vec{n}_{\Lambda_p};\alpha)$ can be expressed through the plane waves in the spinless case $\xi^{(0)}(\vec{\Delta};\vec{n}_{\Lambda_p},r)$ in the form /7/

$$\xi^{(1/2)}(\vec{\Delta};\vec{n}_{\Lambda_p},r)=$$

$$=\frac{\Delta_0-\vec{\Delta}\vec{n}_{\Lambda_p}+m+i\vec{\sigma}[\vec{\Delta}\,\vec{n}_{\Lambda_p}]}{\sqrt{2m(\Delta_0+m)(\Delta_0-\vec{\Delta}\vec{n}_{\Lambda_p})\cdot m}}\,\xi^{(0)}(\vec{\Delta};\vec{n}_{\Lambda_p},r) \quad (4.15)$$

It has been shown above that a relation of that sort can be obtained with the help of the operator $\hat{B}$ as in (3.28) with the substitution $\vec{n}\to\vec{n}_{\Lambda_p}$.

Due to the spherical symmetry of the propagator (4.11) only the s-wave function $P_0^{(0)}(ch\chi_\Delta,r)$ does contribute to the definition of $V^{(0)rel}_{Yuk}(r)$.

An analogous situation will take place in the spin case, but instead of $P_0^{(0)}(ch\chi_\Delta,r)$ there would be the functions $P^{(1/2)}_{1/2,0}(ch\chi_\Delta,r)$. So far as the expression $\sqrt{2m(\Delta_0+m)}$ in the product of (4.15) and (4.14) cancels out, the remaining connection between the functions is defined by the operator $2\,ch\left(\frac{i}{2m}\frac{\partial}{\partial r}\right)$.

Thus, we have

$$V^{(1/2)rel}_{Yuk}(r)=2\,ch\left(\frac{i}{2m}\frac{\partial}{\partial r}\right)V^{(0)rel}_{Yuk}(r)$$

As a result, instead of (4.12), (4.13) we get

$$V^{(0)rel}_{Yuk}(r)=$$

$$\frac{1}{4\pi}\cdot\frac{2r\cdot\sin\frac{a_1}{2}\cdot sh(a_1mr)-\frac{1}{m}\cos\frac{a_1}{2}\cdot ch(a_1mr)}{(r^2+1/4m^2)\cdot ch(\pi mr)}\;;\quad \mu^2<4m^2,$$

$$\frac{1}{4\pi}\cdot\frac{2r\cdot\sin(a_2mr)sh\frac{a_2}{2}-\frac{1}{m}\cos(a_2mr)\,ch\frac{a_2}{2}}{(r^2+1/4m^2)\cdot ch(\pi mr)}\;;\quad \mu^2>4m^2.$$

$$V^{(1/2)rel}_{Coul}(r)=\frac{2r\,th(\pi mr)}{4\pi(r^2+1/4m^2)}$$

A distinguishing feature of the obtained potentials as compared to $V^{(0)rel}(r)$ consists in its finiteness at r=0:

$$V^{(1/2)rel}_{Yuk}(r)\Big|_{r=0}=\begin{cases}-\frac{m}{\pi}\cdot\cos\frac{a_1}{2} & ;\ \mu^2<4m^2\\ -\frac{m}{\pi}\cdot ch\frac{a_2}{2} & ;\ \mu^2>4m^2\end{cases}$$

$$V^{(1/2)rel}_{Coul}(r=0)=0$$

One must take into account that in the case of a bound system, i.e. when the potential sign in (4.12) is changed, the obtained results implies that not only there is no singularity at r=0 but also there appears a repulsion at the distances shorter than the Compton wave-length $\lambda=\hbar/mc$.

Conclusion

In the present paper the unitary irreducible representations (UIR) of the Lorentz group for the spin 1/2 are applied to describe the interaction of two particles. The free Hamiltonian operator and expression for the plane waves with spin 1/2 in the variables of the UIR of the Lorentz group have been used. The Hamiltonian of interaction of two particles in the three-dimensional momentum Lobachevsky space /8/ in the one-boson exchange approximation is reduced with the help of the addition theorem for plane spin waves to the local form in RCR. The form of the relativistic analogs of the Yukawa and Coulomb potentials are found. They have no singularity at the origin.

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