сообщения
объединенного
института
ядерных
исследований
дубна

E2-91-448

I. Lukáč

A COVARIANT FORM OF THE MAXWELL'S EQUATIONS IN FOUR-DIMENSIONAL SPACES WITH AN ARBITRARY SIGNATURE

1991

1 INTRODUCTION

Different four-dimensional metric spaces are of great importance in theoretical physics. The real physical processes occur in four-dimensional space-time and therefore, naturally, the most important physical theories - special and general theory of relativity and electromagnetic field theory - use Minskowski's space as a fundamental four-dimensional manifold for the description of physical phenomena. The four-dimensional Euclidean space has an appropriate place in theoretical physics, too, since e. g. the hidden symmetry of three-dimensional hydrogen atom is connected with this space [1].

It is impossible to imagine a description of the physical reality without the introduction of a coordinate system in a space of certain dimensionality. An invariant introduction of a coordinate system into a manifold is a really very essential problem. We know that a description of physical reality cannot (and must not) depend on the chosen coordinate system but we do not always realize an elementary fact that each introduction of a coordinate system within given space means the introduction of a symmetry into a problem although the connection of different coordinate systems with an appropriate symmetry is well-known and quite a number of investigations devoted to this topic is carried out at present (e. g. [2]). Among the different coordinate systems an exceptional role belongs to the orthogonal systems of coordinates the skillful application of which substantially simplifies the solution of most particular physical problems. The orthogonalization (generally diagonalization) of different quantities in physics has not only advantages but at least also one essential drawback. Each diagonalization means simultaneously a transition into particular coordinate system (or basis) and, thus, the operation contradicts a (widely understood) principle of relativity in a manifold. We are going to show in this paper an advantage of an approach when we do not diagonalize a constant metric tensor in the four-dimensional space from the very beginning but when we carry out its diagonalization in the final formulae. If the symmetry of a physical problem is obvious and known or can be obtained on the basis of an analysis of this problem by means of exact mathematical methods and techniques it is the first step to its successful solution. If the symmetry of the problem under consideration is unknown we still use a coordinate system for description of the different physical items. In this case we very often solve a problem which has one type of symmetry by means of a coordinate system with quite a different kind of symmetry, and naturally, we meet insuperable difficulties. Therefore it is extremely important to find out a symmetry of the problem before an introduction of any coordinate system within a space. From this point of view it is till more important to discover all the possible symmetries of a space in which we have entered no coordinate systems (perhaps except an affine coordinate system). Such symmetries are inherent to the space itself and we have to take account of them from the very beginning, i. e. before the consideration of the symmetry connected with the transformation of the chosen coordinate system. Such symmetries exist at least for certain spaces. A separate paper will be soon devoted to an invariant introduction of a coordinate system in three- and four-dimensional metric spaces with arbitrary signature. In this paper we would like to pay attention to the symmetry of four-dimensional metric spaces which exists independent of the presence of a coordinate system.

All the real even-dimensional metric spaces possess an additional symmetry for the multivectors a rank of which equals to half of the dimensionality of the given space. Here we shall concern such a symmetry for antisymmetrical doubly contravariant or doubly covariant tensors (so-called bivectors) in real four-dimensional spaces. This symmetry is known in physics as the dual symmetry or, simply, the duality. The term "duality" can be only seldom met in theories formulated in a four-dimensional space and it should be noted that many physicists do not give proper consideration to this kind of symmetry. By the way, the most famous monographs on field theory do not even mention a concept of duality in space-time (e. g. [3]).

All the four-dimensional metric spaces have a specific feature connected with the possibility to introduce the concept of duality in these spaces. Note should be taken the fact that the concept of duality introduced in the articles and monographs is rather formal and different (e. g. [4]). It is connected with specific aims which a particular physical problem brings in the process of its solution as well as with not fully understanding the duality in four-dimensional spaces. Here we will give no attempt at comparing different determinations of the duality in physics. At the end of this article it will be clear why it has no sense to do so. Maybe from the methodological point of view the best way to start an investigation of duality in the even-dimensional metric spaces is to consider the symmetry in two-dimensional spaces. However, the two-dimensional spaces seem to be rather simple and no theory of real physical meaning can be constructed there. We would like only to note that the existence of spinors in two-dimensional spaces is directly due to the dual symmetry of these spaces. We prefer to begin the study of duality in four-dimensional spaces first of all in connection with an existence of a suitable physical theory based on a skew-symmetric tensor of the second rank to which the dual symmetry can be applied. Such theory is a well-known theory of electromagnetic field in real space-time which uses the skew-symmetric tensor of electromagnetic field as a fundamental mathematical object for the description of electromagnetic phenomena in nature.

The modern theoretical physics cannot exist without the system of Maxwell's equations for electromagnetic field which was formulated by their author more then a hundred years ago (exactly in 1861, [5]). These equations originated as a result of the careful analysis and generalization of many experimental facts. They first of all generalized the Faraday's investigations. We shall use here the Maxwell's equations for a three-dimensional electrical field $\vec{\mathcal{E}}(\vec{x},t)$ and a three-dimensional magnetic field $\vec{\mathcal{H}}(\vec{x},t)$ in the presence of electrical currents $\vec{j}(\vec{x},t)$ and electrical charges with the density $\rho_0(\vec{x},t)$ in the form:

$$\begin{aligned} \mathrm{rot}\,\vec{\mathcal{H}}(\vec{x},t) - c_0^{-1}\partial_t\vec{\mathcal{E}}(\vec{x},t) &= \vec{j}(\vec{x},t), \\ \mathrm{div}\,\vec{\mathcal{E}}(\vec{x},t) &= \rho_0(\vec{x},t), \\ \mathrm{rot}\,\vec{\mathcal{E}}(\vec{x},t) + c_0^{-1}\partial_t\vec{\mathcal{H}}(\vec{x},t) &= 0, \\ \mathrm{div}\,\vec{\mathcal{H}}(\vec{x},t) &= 0, \end{aligned} \tag{1}$$

where c_0 is velocity of light in vacuum and ∂_t is derivative with respect to time coordinate. Equations (1) are assumed to be always true in physics.

Besides a number of applications of the Maxwell's equations in physics and technology the unique distinction of them is their rich symmetry to which has been paid attention

by many mathematicians and theoretical physicists a long time ago. Already Heaviside observed a symmetry of the Maxwell's equations in vacuum ($\vec{j}(\vec{x},t) = 0$, $\rho_0(\vec{x},t) = 0$) under a substitution [6]:

$$\vec{\mathcal{E}}(\vec{x},t) \to \pm\vec{\mathcal{H}}(\vec{x},t), \quad \vec{\mathcal{H}}(\vec{x},t) \to \mp\vec{\mathcal{E}}(\vec{x},t).$$

The generalization of this symmetry to one-parametrical transformations of the type:

$$\begin{aligned} \vec{\mathcal{E}}(\vec{x},t) &\to \cos\theta\,\vec{\mathcal{E}}(\vec{x},t) + \sin\theta\,\vec{\mathcal{H}}(\vec{x},t), \\ \vec{\mathcal{H}}(\vec{x},t) &\to -\sin\theta\,\vec{\mathcal{E}}(\vec{x},t) + \cos\theta\,\vec{\mathcal{H}}(\vec{x},t), \end{aligned}$$

has been discovered by Rainich and Larmor [7]. However the most interesting symmetry of the Maxwell's equations is their invariance under the Lorentz transformations if we consider time t and three-dimensional Euclidean space in which a point is determined by the vector $\vec{x}$ as one four-dimensional manifold [8]. The Lorentz invariance of the Maxwell's equations can be demonstrated in a better way if we rewrite the Maxwell's equations (1) through a skew-symmetric tensor of the electromagnetic field $\mathcal{F}^{ik}(\vec{x},t) = -\mathcal{F}^{ki}(\vec{x},t)$ the components of which can be expressed by means of a three-dimensional (vector) electrical field $\vec{\mathcal{E}}(\vec{x},t)$ and a three-dimensional (pseudovector) magnetic field $\vec{\mathcal{H}}(\vec{x},t)$ on the base of a correspondence ($i,k = 1,2,3,4$; $x^4 = c_0\,t$, diagonalized metric of space-time $g_{ik} \equiv (-1,-1,-1,+1)$):

$$\mathcal{F}^{ik}(\vec{x},t) = \begin{pmatrix} 0 & -\mathcal{H}^3(\vec{x},t) & \mathcal{H}^2(\vec{x},t) & \mathcal{E}_1(\vec{x},t) \\ \mathcal{H}^3(\vec{x},t) & 0 & -\mathcal{H}^1(\vec{x},t) & \mathcal{E}^2(\vec{x},t) \\ -\mathcal{H}^2(\vec{x},t) & \mathcal{H}^1(\vec{x},t) & 0 & \mathcal{E}^3(\vec{x},t) \\ -\mathcal{E}^1(\vec{x},t) & -\mathcal{E}^2(\vec{x},t) & -\mathcal{E}^3(\vec{x},t) & 0 \end{pmatrix}, \tag{2}$$

i. e. if we define six components of the tensor of electromagnetic field $\mathcal{F}^{ik}(\vec{x},t)$ in the following way ($g_{\alpha\beta} \equiv (-1,-1,-1)$, $\alpha,\beta,\gamma,\delta = 1,2,3$):

$$\mathcal{F}^{\alpha\beta}(\vec{x},t) = \epsilon^{\alpha\beta\gamma}\,g_{\gamma\delta}\,\mathcal{H}^\delta(\vec{x},t), \quad \mathcal{F}^{\alpha 4}(\vec{x},t) = \mathcal{E}^\alpha(\vec{x},t),$$

where $\epsilon^{\alpha\beta\gamma}$ is three-dimensional completely antisymmetric contravariant tensor with components equal to 1 or -1 depending on the parity of the permutation of its indices or to 0 if its two or more indices are coincided. In the previous formulae we have accepted the convention about summation through the identical upper (contravariant) and lower (covariant) greek indices from 1 to 3. In further formulae the summation through identical latin indices will be always implied from 1 to 4. Note that the operation of correspondence (2) between a bivector in four-dimensional space-time and two three-dimensional vectors of electrical and magnetic fields is a strongly non-covariant transformation.

It is a generally known fact that by means of the correspondence (2) the Maxwell's equations (1) can be rewritten in an equivalent and covariant form (e. g. [9]):

$$\partial_i\mathcal{F}^{ik}(\vec{x},t) = j^k(\vec{x},t), \quad \partial_i\tilde{\mathcal{F}}^{ik}(\vec{x},t) = 0, \tag{3}$$

where $\tilde{\mathcal{F}}^{ij}(\vec{x},t)$ is so-called "dual" tensor of the electromagnetic field [1] which is usually defined by means of a completely antisymetric tensor of the fourth rank ϵ^{ijkl} (which is defined analogously to the completely antisymmetrical tensor in three-dimensional space) in the following way:

$$\tilde{\mathcal{F}}^{ij}(\vec{x},t) = \tfrac{1}{2}\,\epsilon^{ijkl}\,g_{km}\,g_{ln}\,\mathcal{F}^{mn}(\vec{x},t)\,. \tag{4}$$

There is another type of symmetry of the Maxwell's equations which is connected with the conformal transformations in space-time [10]. We are not going to consider here all the known kinds of symmetry of the Maxwell's equations . There are monographs especially devoted to this problem (e. g. [11]). Nevertheless, in this paper we shall regard a kind of symmetry of Maxwell's equations which is connected with the concept of duality in four-dimensional metric spaces. Strictly speaking, based on the exact definition of duality as a constant operator we shall derive the Maxwell's equations in a covariant form in the four-dimensional space with an arbitrary signature.

2 A CONCEPT OF DUALITY IN FOUR-DIMENSIONAL METRIC SPACES

In this section we shall try to exclude any arbitrariness in the definition of duality insofar as we shall define the duality by means of an operator which always exists in each four-dimensional metric space. In order to use the tensor calculus widely we can not restrict our considerations to a particular four-dimensional space like the Minkowski's space which is extremely interesting from the point of many physical applications. Generally, we shall regard a real four-dimensional space $\mathcal{R}^4$ in which an arbitrary constant covariant symmetrical metric tensor $g_{ik} = g_{ki}$ is given [2] and where a point in an affine coordinate system is determined by a contravariant vector x^i $(i = 1,2,3,4)$. At this stage we shall not be interested in the presence of time in the set of four coordinates x^i. The time coordinate can be introduced in the final formulae putting e. g. $x^4 = c_0\,t$. We shall not even suppose that the metric tensor g_{ik} has a diagonalized form. It can always be done since the diagonalization of a real symmetrical matrix is a well-known procedure in algebra. We would like to stress that it is extremely effective if the operation of diagonalization is carried out in the final formulae. We only suppose that the metric tensor g_{ik} is not singular, i. e. the determinant g_0 of the metric matrix g_{ik} is not equal to zero ($g_0 = \det \mid g_{ik} \mid \neq 0$). We shall strictly distinguish the upper (contravariant) and lower (covariant) indices of all the quantities which will appear in the following formulae. In addition, throughout this paper we shall successively mark any invariant or constant quantity by means of an index "naught" which can be placed up or down depending on the free spot at a tensor quantity (e. g. n_0^i means a constant contravariant vector, n_i^0 is a covariant constant vector). If we have two (or more) invariants of the same order we shall number them by numerals 1, 2, ... as a contravariant vector in an abstract space (in a manifold) and an order of invariants will be placed into parenthesis next to the index "naught"(e. g. $\mathcal{I}^1_{0(2)}(f)$ and $\mathcal{I}^2_{0(2)}(f)$ are two different invariants of the second order which can be constructed from a tensor quantity "f"). Furthermore, the invariant expressions will be often located in parenthesis or square brackets. Without any misunderstanding the (scalar) quantities with the index "naught" can be raised to power. Thus, all the formulae of this article will have strictly tensor form and they will consist of different invariants and tensors of different ranks. We suppose a reader is acquainted with the tensor calculus and theory of invariants even though in the scope of the presently already classical monographs [12].

[1]We have written the word "dual" in this sentence in quotation marks. It was especially done and we call a tensor of the type determined in (4) as a so-called "dual" tensor of the electromagnetic field throughout the whole this paper. It is due to a possibility to introduce the duality in four-dimensional spaces as a strictly defined operator what will be carried out in the next section. After such new definition of duality we shall be able to construct the skew-symmetric tensors with certain value of the duality.

[2]As a matter of fact the restriction of metric tensor to be constant is not obligatory.

The square of distance $l_0^2(x,y)$ between two points with the affine coordinates x^i and y^i in $\mathcal{R}^4$ is determined by means of a usual way:

$$l_0^2(x,y) = g_{ik}\,(x^i - y^i)(x^k - y^k)\,. \tag{5}$$

If we consider an arbitrary centered affine transformation (an affinor) $\mathcal{A}^i_k$ in $\mathcal{R}^4$ which changes the coordinates of that space in the following way:

$$x'^i = \mathcal{A}^i_k\,x^k\,, \quad x^k = \bar{\mathcal{A}}^k_l\,x'^l\,, \quad \mathcal{A}^i_k\,\bar{\mathcal{A}}^k_l = \delta^i_l\,, \quad \mathcal{A}_0 = \det \mid \mathcal{A}^i_k \mid \neq 0 \tag{6}$$

where $\bar{\mathcal{A}}^k_l$ is an inverse affinor to the affinor $\mathcal{A}^i_k$ then the metric g_{ik} changes as a doubly covariant tensor according to the formulae:

$$g'_{ik} = \bar{\mathcal{A}}^p_i\,\bar{\mathcal{A}}^q_k\,g_{pq}\,, \quad g_{pq} = \mathcal{A}^i_p\,\mathcal{A}^k_q\,g'_{ik}, \quad g_0 = \mathcal{A}_0^2\,g'_0. \tag{7}$$

It is obvious that the square of distance between two points in $\mathcal{R}^4$ written in the form (5) is an (absolute) invariant. The determinant of a matrix is always an (relative) invariant as well. Moreover, the quantity ε_0 which equals to the sign of the determinant of metric tensor g_{ik} conserves under the transformations (6) since the following relation takes place:

$$\varepsilon_0 = g_0 \mid g_0 \mid^{-1} = g'_0\,\mathcal{A}_0^2 \mid g'_0\,\mathcal{A}_0^2 \mid^{-1} = g'_0 \mid g'_0 \mid^{-1} = \varepsilon'_0. \tag{8}$$

Thus, any transformation in $\mathcal{R}^4$ (including a diagonalization of g_{ik}) cannot change the sign of determinant of a metric of given space and ε_0 represents a discrete invariant. This discrete invariant is of great importance for the correct introduction of the concept of duality in $\mathcal{R}^4$.

Let a real skew-symmetric covariant (or contravariant) tensor of the second rank (so-called bivector) $f_{ij}(x)$ be given in the considered real four-dimensional metric space $\mathcal{R}^4$. As we regard the metric space the constant metric tensor (more precisely the inverse metric tensor g^{jk} which fulfills the relation of orthogonality: $g_{ik}\,g^{jk} = \delta^j_i$) allows to carry out the operation of raising of any covariant indices for a tensor determined in $\mathcal{R}^4$. Thus, in $\mathcal{R}^4$ a contravariant bivector $f^{ij}(x)$ can be built up from the given covariant bivector $f_{ij}(x)$:

$$f^{ij}(x) = g^{ik}\,g^{jl}\,f_{kl}(x)\,, \quad f_{ij}(x) = g_{ik}\,g_{jl}\,f^{kl}(x) \tag{9}$$

what is a well-known fact. The six-component covariant bivector $f_{ik}(x)$ can be considered an (covariant) antisymmetrical matrix in $\mathcal{R}^4$. Each matrix in $\mathcal{R}^4$ fulfills an appropriate Hamilton-Cayley's equation [12]. This equation for a bivector $f_{ik}(x)$ in $\mathcal{R}^4$ has the form:

$$f_{ip}(x)\, g^{pq}\, f_{qr}(x)\, g^{rs}\, f_{st}(x)\, g^{tu}\, f_{uk}(x) + \mathcal{I}^1_{0(2)}(f)\, f_{ip}(x)\, g^{pq}\, f_{qk}(x) + g_0^{-1}\, f_0(x)\, g_{ik} = 0 \qquad (10)$$

where the following three invariants were introduced:

$$\begin{aligned} \mathcal{I}^1_{0(2)}(f) &= \tfrac{1}{2}\, g^{pr}\, g^{qs} f_{pq}(x)\, f_{rs}(x)\,, \\ \mathcal{I}^2_{0(2)}(f) &= \tfrac{1}{4}\, e^{pqrs}\, f_{pq}(x)\, f_{rs}(x)\,, \\ f_0(x) &= (4!)^{-1}\, \epsilon^{abcd}\, \epsilon^{pqrs}\, f_{ap}(x)\, f_{bq}(x)\, f_{cr}(x)\, f_{ds}(x) \equiv \\ &\equiv \left[\tfrac{1}{8}\, \epsilon^{ijkl}\, f_{ij}(x)\, f_{kl}(x)\right]^2 \equiv \tfrac{1}{4}\, \varepsilon_0\, g_0 \left[\mathcal{I}^2_{0(2)}(f)\right]^2 \geq 0, \\ e^{ijkl} &= |\, g_0\,|^{-\frac{1}{2}}\, \epsilon^{ijkl}\,, \quad e_{ijkl} = |\, g_0\,|^{\frac{1}{2}}\, \epsilon_{ijkl}\,. \end{aligned} \qquad (11)$$

The invariant $f_0(x)$ in matrix equation (10) represents a determinant of antisymmetrical matrix $f_{ik}(x)$. From the set of invariants in (11) an important invariant of the fourth order $\mathcal{I}_{0(4)}(f)$ can be constructed, namely:

$$\mathcal{I}_{0(4)}(f) = \tfrac{1}{4}\, \{\, [\mathcal{I}^1_{0(2)}(f)\,]^2 - \varepsilon_0\, [\mathcal{I}^2_{0(2)}(f)\,]^2\, \}. \qquad (12)$$

The meaning of this invariant will be discussed later.

It is generally known that especially in $\mathcal{R}^4$ one can raise (or lower) the indices of a skew-symmetric covariant (or contravariant) tensor of the second rank by means of the completely antisymmetrical contravariant (or covariant) tensor of the fourth rank ϵ^{ijkl} (or ϵ_{ijkl}). The second way of raising (or lowering) the indices of bivectors in $\mathcal{R}^4$ leads to so-called "dual" contravariant (or covariant) antisymmetrical tensors of the second rank $\tilde{f}^{ij}(x)$ (or $\tilde{f}_{ij}(x)$): [3]

$$\begin{aligned} \tilde{f}^{ij}(x) &= \tfrac{1}{2}\, e^{ijkl}\, f_{kl}(x) \equiv \tfrac{1}{2}\, e^{ijkl}\, g_{km}\, g_{jl}\, f^{mn}(x) \equiv \tfrac{1}{2}\, \varepsilon_0\, g^{ik}\, g^{jl}\, e_{klmn}\, f^{mn}(x)\,, \\ \tilde{f}_{ij}(x) &= \tfrac{1}{2}\, e_{ijkl}\, f^{kl}(x) \equiv \tfrac{1}{2}\, e_{ijkl}\, g^{km}\, g^{jl}\, f_{mn}(x) \equiv \tfrac{1}{2}\, \varepsilon_0\, g_{ik}\, g_{jl}\, e^{klmn}\, f_{mn}(x)\,. \end{aligned} \qquad (13)$$

In the formulae (13) we have used the relations from the appendix of the report [13] between the components of a covariant metric tensor g_{ij} and inverse (contravariant) metric tensor g^{kl}.

[3] There is a third way of raising the indices of a covariant bivector $f_{ij}(x)$ in $\mathcal{R}^4$ by means of two completely skew-symmetric contravariant tensors, namely:

$$\bar{f}^{ij}(x) = [3!\, f_0(x)]^{-1}\, \epsilon^{iabc}\, \epsilon^{jpqr}\, f_{ap}(x)\, f_{bq}(x)\, f_{cr}(x)\,,$$

where $f_0(x)$ is the determinant of the covariant bivector $f_{ij}(x)$ defined in (11). The contravariant bivector $\bar{f}^{jk}(x)$ represents the so-called inverse bivector to the covariant bivector $f_{ik}(x)$ which fulfills a relation of orthogonality: $f_{ik}(x)\, \bar{f}^{jk}(x) = \delta^j_i$. There is a simple direct coupling between an inverse bivector $\bar{f}^{ij}(x)$ as well as a so-called "dual" bivector $\tilde{f}^{ij}(x)$ introduced in (13):

$$\tilde{f}^{ij}(x) = \sqrt{f_0(x)}\, \bar{f}^{ij}(x)\,.$$

We bring here four such relations without any deduction:

$$\begin{aligned} \epsilon_{abcd}\, g_0 &= \epsilon^{ijkl}\, g_{ai}\, g_{bj}\, g_{ck}\, g_{dl}\,, \\ g_0 &= (4!)^{-1}\, \epsilon^{abcd}\, \epsilon^{ijkl}\, g_{ai}\, g_{bj}\, g_{ck}\, g_{dl}\,, \\ g^{ij} &= (3!\, g_0)^{-1}\, \epsilon^{iabc}\, \epsilon^{jpqr}\, g_{ap}\, g_{bq}\, g_{cr}\,, \\ \pm\sqrt{\varepsilon_0}\, e_{abcd}\, g^{ic}\, g^{kd} &= \pm\frac{1}{\sqrt{\varepsilon_0}}\, e^{ijkl}\, g_{ak}\, g_{bl}\,. \end{aligned} \qquad (14)$$

In $\mathcal{R}^4$ one can always define two constant nilpotent operators $\mathcal{N}^{pq}_{ij}(+)$ and $\mathcal{N}^{pq}_{ij}(-)$ which represent (constant) mixed tensors of the fourth rank with the antisymmetrical doubly covariant and doubly contravariant indices (so-called biaffinors) by the following way:

$$\mathcal{N}^{pq}_{ij}(\pm) = \mathcal{N}^{qp}_{ji}(\pm) = -\mathcal{N}^{qp}_{ij}(\pm) = -\mathcal{N}^{pq}_{ji}(\pm) = \tfrac{1}{2}\, \left(\delta^{pq}_{ij} \pm \sqrt{\varepsilon_0}\, e_{ijkl}\, g^{kp}\, g^{lq}\,\right), \qquad (15)$$

where $\delta^{pq}_{ij} = \delta^p_i\, \delta^q_j - \delta^p_j\, \delta^q_i$ is so-called generalized Kronecker's symbol of the second order. Note the operators $\mathcal{N}^{pq}_{ij}(+)$ and $\mathcal{N}^{pq}_{ij}(-)$ become complexly conjugated ones for $\varepsilon_0 = -1$ (e. g. for the Minkowski's space with the signature $(-1,-1,-1,+1)$). Using the formulae (14) one can easily check that the tensors $\mathcal{N}^{pq}_{ij}(\pm)$ fulfil the relations:

$$\tfrac{1}{2}\, \mathcal{N}^{pq}_{ij}(\pm)\, \mathcal{N}^{kl}_{pq}(\pm) = \mathcal{N}^{kl}_{ij}(\pm)\,, \quad \tfrac{1}{2}\, \mathcal{N}^{pq}_{ij}(\pm)\, \mathcal{N}^{kl}_{pq}(\mp) \equiv 0. \qquad (16)$$

From the first formula in (16) il follows that the operators $\mathcal{N}^{pq}_{ij}(\pm)$ are really nilpotent quantities.

Now for any real covariant bivector $f_{ij}(x)$ in $\mathcal{R}^4$ one can define two new covariant bivectors $f^{\pm}_{ij}(x)$ by means of just introduced biaffinors $\mathcal{N}^{pq}_{ij}(\pm)$:

$$f^{\pm}_{ij}(x) = \tfrac{1}{2}\, \mathcal{N}^{pq}_{ij}(\pm)\, f_{pq}(x)\,, \quad f^{ij}_{\pm}(x) = g^{ik}\, g^{jl}\, f^{\pm}_{kl}(x)\,. \qquad (17)$$

From the explicit form of biaffinors $\mathcal{N}^{pq}_{ij}(\pm)$ in (15) the following relations can be found:

$$\begin{aligned} f^{\pm}_{ij}(x) + f^{\mp}_{ij}(x) &= f_{ij}(x)\,, \\ f^{\pm}_{ij}(x) - f^{\mp}_{ij}(x) &= \pm\sqrt{\varepsilon_0}\, e_{ijkl}\, g^{km}\, g^{ln}\, f_{mn}(x) \equiv \pm\frac{1}{\sqrt{\varepsilon_0}}\, g_{ik}\, g_{jl}\, \tilde{f}^{kl}(x)\,, \end{aligned} \qquad (18)$$

and, thus, the so-called "dual" bivector $\tilde{f}^{ij}(x)$ introduced before can be expressed in terms of the bivectors $f^{+}_{ij}(x)$ and $f^{-}_{ij}(x)$. Two nilpotent operators $\mathcal{N}^{pq}_{ij}(\pm)$ allow to build up a (constant) tensor operator $\mathcal{D}^{pq}_{ij}$ in each real four-dimensional metric space $\mathcal{R}^4$ which has the form:

$$\begin{aligned} \mathcal{D}^{pq}_{ij} &= \mathcal{D}^{qp}_{ji} = -\mathcal{D}^{qp}_{ij} = -\mathcal{D}^{pq}_{ji} = \mathcal{N}^{pq}_{ij}(+) - \mathcal{N}^{pq}_{ij}(-) \equiv \\ &\equiv \sqrt{\varepsilon_0}\, e_{ijkl}\, g^{km}\, g^{lq} \equiv \frac{1}{\sqrt{\varepsilon_0}}\, e^{pqrs}\, g_{ir}\, g_{js}\,. \end{aligned} \qquad (19)$$

We shall call the operator $\mathcal{D}^{pq}_{ij}$ defined in (19) an operator of duality in $\mathcal{R}^4$. As it is seen from the definition (19) the operator of duality in real $\mathcal{R}^4$ can be real or pure imaginary quantity depending on the sign of the determinant of the (real) metric tensor g_{ik}. We make sure that a discrete affine invariant ε_0 in $\mathcal{R}^4$ plays an exceptional role. Moreover, the double application of the operator of duality $\mathcal{D}^{pq}_{ij}$ gives:

$$\tfrac{1}{2}\, \mathcal{D}^{pq}_{ij}\, \mathcal{D}^{kl}_{pq} = \tfrac{1}{2}\, \left[\mathcal{N}^{pq}_{ij}(+) - \mathcal{N}^{pq}_{ij}(-)\right] \left[\mathcal{N}^{kl}_{pq}(+) - \mathcal{N}^{kl}_{pq}(-)\right] = \delta^{kl}_{ij}\,,$$

and, consequently, the eigenvalues of this operator can be equal to ± 1 only. Indeed, if we act on the bivectors $f^{\pm}_{ij}(x)$ by means of the operator of duality $\mathcal{D}^{pq}_{ij}$ (19) we are convinced by virtue of the relations (16) that an equation can be written as:

$$\tfrac{1}{2}\,\mathcal{D}^{pq}_{ij}\, f^{\pm}_{pq}(x) = \pm\, f^{\pm}_{ij}(x)\,. \tag{20}$$

The equation (20) can be regarded as an eigenvalue problem for the operator of duality on the space of skew-symmetric functions in $\mathcal{R}^4$. We see the bivectors $f^{\pm}_{ij}(x)$ constructed before by means of two biaffinors $\mathcal{N}^{pq}_{ij}(\pm)$ from a given bivector $f_{ij}(x)$ are just the "eigenfunctions" of the operator of duality $\mathcal{D}^{pq}_{ij}$ with the eigenvalues equal to ± 1 respectively. Therefore, strictly speaking, we will be right if we call bivectors $f^{\pm}_{ij}(x)$ as "genuine" dual bivectors with the certain values of duality in $\mathcal{R}^4$. Other definitions of the "dual" bivectors in $\mathcal{R}^4$ which can be met in the literature (like the relation (13)) are fully arbitrary inasmuch they use no strict definition of an operator of duality and such "dual" bivectors are not the eigenfunctions of an operator in $\mathcal{R}^4$. Nevertheless, we give an expression of the dual bivectors $f^{\pm}_{ij}(x)$ here as a linear combination of the initial bivector $f_{ij}(x)$ and the so-called "dual" bivector $\tilde{f}_{ij}(x)$ in $\mathcal{R}^4$ which has the form:

$$f^{\pm}_{ij}(x) = \tfrac{1}{2}\left[f_{ij}(x) \pm \frac{1}{\sqrt{\varepsilon_0}}\,\tilde{f}_{ij}(x)\right]. \tag{21}$$

The initial bivector $f_{ij}(x)$ was introduced as real quantity and it follows from the definition of so-called "dual" bivector $\tilde{f}_{ij}(x)$ that it is a real quantity too. As seen from the formulae (17) and (21) the dual bivectors $f^{\pm}_{ij}(x)$ are dual quantities [4] with dual unit equal to $\sqrt{\varepsilon_0}$. The dual bivectors $f^{\pm}_{ij}(x)$ have real and dual parts which are equal to:

$$Re\, f^{\pm}_{ij}(x) = \tfrac{1}{2}\, f_{ij}(x)\,, \quad Du\, f^{\pm}_{ij}(x) = \pm\tfrac{1}{2}\,\varepsilon_0\, \tilde{f}_{ij}(x)\,.$$

It is useful to bring up all the possible invariants which can be built up from dual bivectors $f^{\pm}_{ij}(x)$. It follows from the relations

$$\tfrac{1}{2}\, g^{ik}\, g^{jl}\, \mathcal{N}^{mn}_{ij}(\pm)\, \mathcal{N}^{pq}_{kl}(\mp) = \tfrac{1}{4}\, e^{ijkl}\, \mathcal{N}^{mn}_{ij}(\pm)\, \mathcal{N}^{pq}_{kl}(\mp) \equiv 0$$

that the invariants constructed from bivectors of different duality are identically equal to zero:

$$\tfrac{1}{2}\, g^{ik}\, g^{jl}\, f^{\pm}_{ij}(x)\, f^{\mp}_{kl}(x) = \tfrac{1}{4}\, e^{ijkl}\, f^{\pm}_{ij}(x)\, f^{\mp}_{kl}(x) \equiv 0.$$

[4] We understand a dual number z as a number of the type $z = a + e_0\, b$ where a pair of numbers a and b belongs to the field of real numbers and e_0 represents a dual unit a square of which is a real number. The dual conjugated number z^* to a dual number z is a dual number $z^* = a - e_0\, b$, $(z^*)^* = z$. The product of a dual number z and a dual conjugated number z^* is a real number $z\,z^* = a^2 - e_0^2\, b^2$ which is a positive real number for $e_0^2 < 0$ or any real number for $e_0^2 > 0$. The inverse dual number $\bar{z}$ to a dual number z is the dual number $\bar{z} = (z\,z^*)^{-1} z^*$ which fulfills the relation $\bar{z}\,z = 1$, $a^2 - e_0^2\, b^2 \neq 0$. The special case of dual numbers with $e_0 = i$ is field of complex numbers. If we accept $\sqrt{\varepsilon_0}$ as a dual unit in $\mathcal{R}^4$ and decide to use the field of dual numbers in the theory we should omit to mark the dual quantities by the symbols "$\pm$" realizing that for each dual quantity (i. e. for a quantity marked by a symbol "+") exists the dual conjugated quantity (i. e. a quantity marked by a symbol "−". In connection with an absence of wide application of the field of dual numbers in mathematics and physics (in contrast to the field of complex numbers) we conserve the symbols "$\pm$" in all the dual quantities.

The invariants $\mathcal{I}^1_{0(2)}(f)$ and $\mathcal{I}^2_{0(2)}(f)$ from (11) can be simply expressed in terms of new invariants $\mathcal{I}^1_{0(2)}(f^{\pm})$ and $\mathcal{I}^2_{0(2)}(f^{\pm})$ which are defined in an analogous way:

$$\mathcal{I}^1_{0(2)}(f^{\pm}) = \tfrac{1}{2}\, g^{ik}\, g^{jl}\, f^{\pm}_{ij}(x)\, f^{\pm}_{kl}(x)\,, \quad \mathcal{I}^2_{0(2)}(f^{\pm}) = \tfrac{1}{4}\, e^{ijkl}\, f^{\pm}_{ij}(x)\, f^{\pm}_{kl}(x)\,. \tag{22}$$

As a result of non-complicated calculations we have:

$$\begin{aligned} \mathcal{I}^1_{0(2)}(f^{\pm}) &= \tfrac{1}{2}\left[\mathcal{I}^1_{0(2)}(f) \pm \frac{1}{\sqrt{\varepsilon_0}}\,\mathcal{I}^2_{0(2)}(f)\right], \\ \mathcal{I}^2_{0(2)}(f^{\pm}) &= \pm\tfrac{1}{2}\sqrt{\varepsilon_0}\left[\mathcal{I}^1_{0(2)}(f) \pm \frac{1}{\sqrt{\varepsilon_0}}\,\mathcal{I}^2_{0(2)}(f)\right]. \end{aligned} \tag{23}$$

Note the invariants $\mathcal{I}^1_{0(2)}(f^{\pm})$ and $\mathcal{I}^2_{0(2)}(f^{\pm})$ are linearly dependent as seeing from the formula (23) and the relation takes place:

$$\mathcal{I}^1_{0(2)}(f^{\pm}) = \pm\frac{1}{\sqrt{\varepsilon_0}}\,\mathcal{I}^2_{0(2)}(f^{\pm}) \tag{24}$$

and therefore only two invariants from the set (23) are independent (e. g. invariant $\mathcal{I}^1_{0(2)}(f^{+})$ and invariant $\mathcal{I}^1_{0(2)}(f^{-})$). In addition, an invariant of the fourth order $\mathcal{I}_{0(4)}(f^{\pm})$ built up from invariants $\mathcal{I}^1_{0(2)}(f^{\pm})$ and $\mathcal{I}^2_{0(2)}(f^{\pm})$ is identically equal to zero because of the relation (24). The invariant of the fourth order $\mathcal{I}_{0(4)}(f)$ defined in (12) can be written by means of invariants (22) as well:

$$\mathcal{I}_{0(4)}(f) = \mathcal{I}^1_{0(2)}(f^{\pm})\,\mathcal{I}^1_{0(2)}(f^{\mp}) \equiv \pm\varepsilon_0\,\mathcal{I}^2_{0(2)}(f^{\pm})\,\mathcal{I}^2_{0(2)}(f^{\mp})\,. \tag{25}$$

If we regard the invariants $\mathcal{I}^1_{0(2)}(f)$ and $\mathcal{I}^2_{0(2)}(f)$ in the definition of invariant $\mathcal{I}_{0(4)}(f)$ in (12) as doubled coordinates in a two-dimensional space we can observe that $\mathcal{I}_{0(4)}(f)$ represents a quadric in that space with a diagonal metric tensor g_{AB} $(A, B = 1, 2)$ in the form:

$$g_{AB} = \begin{pmatrix} 1 & 0 \\ 0 & -\varepsilon_0 \end{pmatrix}.$$

The use of the dual bivectors $f^{\pm}_{ij}(x)$ instead of the initial bivector $f_{ij}(x)$ in $\mathcal{R}^4$ induces a transformation in an abstract two-dimensional space built on two coordinates $\tfrac{1}{2}\,\mathcal{I}^1_{0(2)}(f)$ and $\tfrac{1}{2}\,\mathcal{I}^2_{0(2)}(f)$. A higher symmetry in the space of bivectors in $\mathcal{R}^4$ can be studied which is connected with an invariance of invariant $\mathcal{I}_{0(4)}(f)$ under transformations in two-dimensional space mentioned above but such investigation is fully out of frame of this paper.

Thereby we have shown that in any real four-dimensional metric space where a covariant (or contravariant) bivector is given one can build up two different bivectors each of which possesses a certain value of duality. The duality defined in the way above is based on a possibility to introduce special operators with certain transformation properties in any real four-dimensional metric space and, thus, the duality can be regarded as an additional symmetry for the bivectors which already exists in each $\mathcal{R}^4$. From the physical point of view the concept of duality allows to introduce two sets of quantities which differ in the discrete (quantum) number connected with the operator of duality in any theory formulated in $\mathcal{R}^4$

and based on a skew-symmetric field (like the electromagnetic field). The duality in four-dimensional spaces is a kind of "parity" for skew-symmetric tensors. The dual symmetry represents an additional symmetry of any four-dimensional metric space and we should not ignore this symmetry in the physical theories formulated in $\mathcal{R}^4$. In the next sections we apply the results obtained above to the formulation of the covariant Maxwell's equations in $\mathcal{R}^4$ with arbitrary non-diagonalized constant metric (not only in Minkowski's space).

3 A COVARIANT THREE-VECTOR MODEL FOR BIVECTORS IN $\mathcal{R}^4$

As we have shown in the previous section the concept of duality can be strictly introduced in any four-dimensional metric space. Let a skew-symmetric tensor of electromagnetic field [5] exist in $\mathcal{R}^4$. Besides this let an arbitrary constant contravariant vector n_0^i be chosen in the considered real four-dimensional space ($n_0^2 = g_{ik}\, n_0^i\, n_0^k \equiv n_i^0\, n_0^i$). [6] Now using the constant vector n_0^i, the constant metric tensor g_{ik} and the completely skew-symmetric tensor e_{ijkl} one can form a (real) vector $E^i(x)$ and a (real) pseudovector $H^i(x)$ from each (real) bivector $f_{ij}(x)$. The vectors $E^i(x)$ and $H^i(x)$ have the following explicit form: [7]

$$E^i(x) = n_0^{-1}\, g^{ij}\, f_{jk}(x)\, n_0^k\,, \quad E_i(x) = g_{ij}\, E^j(x) = n_0^{-1}\, f_{ij}(x)\, n_0^j\,, \tag{26}$$

$$H^i(x) = \tfrac{1}{2}\, n_0^{-1}\, e^{ijkl}\, f_{jk}(x)\, n_l^0\,, \quad H_i(x) = g_{ij}\, H^j(x) = \tfrac{1}{2}\, \varepsilon_0\, n_0^{-1}\, e_{ijkl}\, f^{jk}(x)\, n_0^l\,. \tag{27}$$

The definition of vectors $E^i(x)$ and $H^i(x)$ is made in such a way that both these vectors are perpendicular to the arbitrary constant vector n_0^i in $\mathcal{R}^4$:

$$E_i(x)\, n_0^i = H_i(x)\, n_0^i \equiv 0. \tag{28}$$

Two relations in (28) represent two couplings to which the four-dimensional vectors $E^i(x)$ and $H^i(x)$ automatically obey, and, thus, each of four-component vectors $E^i(x)$ and $H^i(x)$ has really three independent components only. As a result of the transformations (26) and (27) six independent components of a bivector $f_{ij}(x)$ in $\mathcal{R}^4$ can be substituted by three independent components of a four-dimensional vector $E^i(x)$ and three independent components of a four-dimensional pseudovector $H^i(x)$.

The explicit form of the bivector $f_{pq}(x)$ expressed through the vectors $E^i(x)$ and $H^i(x)$ can be found easily by calculating the expression:

$$e_{ijkl}\, H^k(x)\, n_0^l = \tfrac{1}{2}\, \delta_{ijl}^{pqr}\, f_{pq}(x)\, n_r^0\, n_0^l = n_0\, f_{ij}(x) + n_i^0\, E_j(x) - n_j^0\, E_i(x)$$

[5] In this section and further on we shall use the terminology from electromagnetic field theory for the quantities connected with a bivector in $\mathcal{R}^4$ in spite of the fact that we do not consider electromagnetic field in real space-time only.

[6] Without loss of any generality the vector n_0^i can be chosen as an unit vector putting $n_0^2 = 1$. We do not use such simplification in order to conserve the homogeneity of all the formulae.

[7] It would be more correct to introduce the notation $E^i(x,n)$ for electrical field and the notation $H^i(x,n)$ for magnetic field in order to stress the dependence of electrical and magnetic fields on an arbitrary constant four-dimensional vector n_0^i.

where $\delta_{ijl}^{pqr} = \delta_i^p\, \delta_{jl}^{qr} + \delta_j^p\, \delta_{li}^{qr} + \delta_l^p\, \delta_{ijl}^{qr}$ is the generalized Kronecker's symbol of the third order. It is clear that the following (exactly covariant) three-vector model of a covariant bivector $f_{ij}(x)$ in $\mathcal{R}^4$ exists:

$$f_{ij}(x) = n_0^{-1}\, \left[E_i(x)\, n_j^0 - E_j(x)\, n_i^0 + e_{ijkl}\, H^k(x)\, n_0^l\right]. \tag{29}$$

Note the fact of the linear dependence of the bivector $f_{ij}(x)$ on the vectors $E^i(x)$, $H^i(x)$ and $n_0^{-1} n_0^i$ in the model (29) as well as the strictly covariant form of this relation. Thus, if we want to construct a (vector) model for a bivector in $\mathcal{R}^4$ correctly based on a set of four-dimensional vectors conserving the covariant form of all the formulae we have to use three vectors in $\mathcal{R}^4$: one vector is arbitrary constant and two variable vectors are perpendicular to it.

As we have mentioned before besides the covariant bivector $f_{ij}(x)$ in $\mathcal{R}^4$ one can consider the contravariant bivector $f^{ij}(x)$ as well as the so-called "dual" contravariant bivector $\tilde{f}^{ij}(x)$ and the so-called "dual" covariant bivector $\tilde{f}_{ij}(x)$. These bivectors can be expressed in terms of the vectors $E^i(x)$, $H^i(x)$ and n_0^i too. In the end the following relations can be derived:

$$\begin{aligned} f^{ij}(x) &= n_0^{-1}\, \left[E^i(x)\, n_0^j - E^j(x)\, n_0^i + \varepsilon_0\, e^{ijkl}\, H_k(x)\, n_l^0\right], \\ \tilde{f}^{ij}(x) &= n_0^{-1}\, \left[H^i(x)\, n_0^j - H^j(x)\, n_0^i + \quad e^{ijkl}\, E_k(x)\, n_l^0\right], \\ \tilde{f}_{ij}(x) &= n_0^{-1}\, \left[H_i(x)\, n_j^0 - H_j(x)\, n_i^0 + \varepsilon_0\, e_{ijkl}\, E^k(x)\, n_0^l\right]. \end{aligned} \tag{30}$$

From the formulae (30) it is obvious that the transition from the bivector $f^{ij}(x)$ to the so-called "dual" bivector $\tilde{f}^{ij}(x)$ in terms of the fields $E^i(x)$ and $H^i(x)$ means the transformation:

$$E^i(x) \to H^i(x)\,, \quad H^i(x) \to \varepsilon_0\, E^i(x)\,. \tag{31}$$

The repeated application of the operation (31) to the formulae (30) leads to the transformation of the initial bivectors:

$$f^{ij}(x) \to \varepsilon_0\, f^{ij}(x)\,, \quad \tilde{f}^{ij}(x) \to \varepsilon_0\, \tilde{f}^{ij}(x)\,.$$

To reach the initial bivectors we have to apply the transformations (31) twice more. Such transformations of the bivectors in $\mathcal{R}^4$ seem to be very complicated and obviously do not reflect a "genuine" symmetry in a four-dimensional space.

Having the explicit form of a bivector $f^{ij}(x)$ and so-called "dual" bivector $\tilde{f}^{ij}(x)$ in terms of the fields $E^i(x)$ and $H^i(x)$ one can write the explicit form of the "genuine" dual bivectors $f_\pm^{ij}(x)$. They have the form:

$$\begin{aligned} f_\pm^{ij}(x) &= n_0^{-1}\, \left[F_\pm^i(x)\, n_0^j - F_\pm^j(x)\, n_0^i \pm \frac{1}{\sqrt{\varepsilon_0}}\, e^{ijkl}\, F_k^\pm(x)\, n_l^0\right] \equiv \\ &\equiv n_0^{-1}\, \mathcal{N}_{mn}^{ij}(\pm)\, \left[F_\pm^m(x)\, n_0^n - F_\pm^n(x)\, n_0^m\right] \equiv 2\, n_0^{-1}\, \mathcal{N}_{mn}^{ij}(\pm)\, F_\pm^m(x)\, n_0^n\,, \end{aligned} \tag{32}$$

where we introduced two mutually dual (vector) electromagnetic fields $F_\pm^i(x)$ instead of the (real) electrical field $E^i(x)$ and the (real) magnetic field $H^i(x)$ in the following way:

$$F_\pm^i(x) = \tfrac{1}{2}\, \left[E^i(x) \pm \frac{1}{\sqrt{\varepsilon_0}}\, H^i(x)\right] = \tfrac{1}{2}\, \mathcal{N}_{pq}^{ij}(\pm)\, f^{pq}(x)\, n_j^0 \equiv f_\pm^{ij}(x)\, n_j^0\,. \tag{33}$$

As seen from the formulae (32) and (33) the transition $f_{\pm}^{ij}(x) \to f_{\mp}^{ij}(x)$ means the transformation $F_{\pm}^{i}(x) \to F_{\mp}^{i}(x)$ and *vice versa*, i. e. the change of duality of primary dual bivectors $f_{\pm}^{ij}(x)$ induces the dual conjugation in dual vector electromagnetic fields $F_{\pm}^{i}(x)$. Once more we would like to emphasize the exceptional role of the square root of the discrete invariant ε_0. As a matter of fact the square root of sign of the determinant of metric in $\mathcal{R}^4$ represents a dual unit in the theory. Thereby we have found an application of dual numbers (more precisely dual quantities) in a theory formulated in $\mathcal{R}^4$. If we consider a non-diagonalized metric in $\mathcal{R}^4$ the general approach to this space demonstrated here automatically leads to the dual quantities.

In conclusion of this section we bring the explicit form of the invariants $\mathcal{I}^1_{0(2)}(f)$, $\mathcal{I}^2_{0(2)}(f)$ and $\mathcal{I}_{0(4)}(f)$ in terms of just introduced electrical field $E^i(x)$ and magnetic field $H^i(x)$ as well as the dual electromagnetic fields $F_{\pm}^{i}(x)$. These invariants can be calculated without a big effort:

$$\begin{aligned} \mathcal{I}^1_{0(2)}(f) &= E^i(x)\,E_i(x) + \varepsilon_0\,H^i(x)\,H_i(x) \equiv 2\left[F_{\pm}^{i}(x)\,F_i^{\pm}(x) + F_{\mp}^{i}(x)\,F_i^{\mp}(x)\right], \\ \mathcal{I}^2_{0(2)}(f) &= 2E^i(x)\,H_i(x) \equiv \pm 2\sqrt{\varepsilon_0}\,\left[F_{\pm}^{i}(x)\,F_i^{\pm}(x) - F_{\mp}^{i}(x)\,F_i^{\mp}(x)\right], \\ \mathcal{I}_{0(4)}(f) &= \tfrac{1}{4}\,\left[E^i(x)\,E_i(x) + \varepsilon_0\,H^i(x)\,H_i(x)\right]^2 - \varepsilon_0\,\left[E^i(x)\,H_i(x)\right]^2 \equiv \\ &\equiv 4\left[F_{\pm}^{i}(x)\,F_i^{\pm}(x)\right]\left[F_{\mp}^{k}(x)\,F_k^{\mp}(x)\right]. \end{aligned} \tag{34}$$

Other invariants which can be constructed from the vectors $E^i(x)$, $H^i(x)$ and $F_{\pm}^{i}(x)$ are model dependent (i. e. they depend on an arbitrary constant vector n_0^i in $\mathcal{R}^4$) and therefore they have not a real physical meaning.

Thus, the six-component bivector field in $\mathcal{R}^4$ can be covariantly described either by means of two mutually dual vector electromagnetic fields or by means of a real vector electrical field and a real pseudovector magnetic field which are perpendicular to an arbitrary chosen constant vector. For the real space-time it means that we are always able to substitute a six-component skew-symmetric field $f_{ij}(x)$ by means of a complex vector electromagnetic field $F_i^{\pm}(x)$ with the doubled real part equal to vector electrical field and the doubled imaginary part equal to pseudovector magnetic field which have to fulfil a condition of orthogonality to an arbitrary chosen constant four-dimensional vector n_0^i.

4 DUAL SYMMETRY IN $\mathcal{R}^4$ AND MAXWELL'S EQUATIONS

Let us imagine we know (almost) nothing about electromagnetic field in real four-dimensional space-time. We shall use only the dual symmetry in $\mathcal{R}^4$ and some theoretical reasons in order to postulate a system of equations for a bivector field which then will be analyzed and interpreted with regard to the real circumstances in space-time with an electromagnetic field.

Let a real contravariant bivector $f^{ik}(x) = -f^{ki}(x)$ be given in $\mathcal{R}^4$. This bivector will be called the skew-symmetric tensor of electromagnetic field in given four-dimensional space.

We know that in any four-dimensional space with the metric g_{ik} the so-called "dual" contravariant bivector $\tilde{f}^{ik}(x)$ and two dual contravariant bivectors $f_{\pm}^{ik}(x)$ can be strictly determined. If we differentiate a bivector $f^{ik}(x)$ in $\mathcal{R}^4$ with respect to x^i we shall always obtain a vector. Consequently, the differentiation of two dual bivectors $f_{\pm}^{ik}(x)$ with respect to x^i leads to two different vectors $j_{\pm}^{k}(x)$ which we shall call dual currents:

$$\partial_i f_{\pm}^{ik}(x) = j_{\pm}^{k}(x) \tag{35}$$

where the notation ∂_i means the differentiation with respect to a coordinate x^i in $\mathcal{R}^4$. Now we postulate two equations (35) as the fundamental equations for a six-component skew-symmetric tensor field in $\mathcal{R}^4$. Note that further differentiation of the equations (35) with respect to x^k gives zero identically inasmuch the differential operator $\partial_i \partial_k$ and bivectors $f_{\pm}^{ik}(x)$ have the different permutation symmetry. Therefore we have two local conservation laws for the dual currents:

$$\partial_k j_{\pm}^{k}(x) \equiv 0 \tag{36}$$

which means that the dual currents $j_{\pm}^{k}(x)$ fulfil the equations of continuity automatically. It is known a bivector in $\mathcal{R}^4$ has six independent components and, thus, for a determination of such bivector we need six independent equations. The system of eight equations (35) and two couplings (36) represents a set of completely consistent conditions for determination of a bivector in $\mathcal{R}^4$.

If we now apply the covariant model for the bivectors $f_{\pm}^{ik}(x)$ introduced earlier which operates with the two dual vector electromagnetic fields $F_{\pm}^{i}(x)$ the system of equations (35) can be rewritten in the form:

$$n_0^{-1}\,n_0^i\,\partial_p F_{\pm}^{p}(x) - \partial_0 F_{\pm}^{i}(x) \mp \frac{1}{\sqrt{\varepsilon_0}}\,n_0^{-1}\,e^{ipqr}\,n_p^0\,\partial_q F_r^{\pm}(x) = j_{\pm}^{i}(x) \tag{37}$$

where we introduced a notation ∂_0 for differentiation with respect to x^i along the direction n_0^i: $\partial_0 = n_0^{-1}\,(n_0^p\,\partial_p)$. Multiplying both sides of equations (37) by the covariant vector n_i^0 and taking account of the relation of orthogonality of vectors n_0^i and $F_{\pm}^{i}(x)$ we immediately find two equations:

$$\partial_p F_{\pm}^{p}(x) = \rho_0^{\pm}(x) \tag{38}$$

where we introduced two invariant quantities $\rho_0^{\pm}(x) = n_0^{-1}\,j_{\pm}^{i}(x)\,n_i^0$ which can be regarded as the (scalar) densities of dual charges. Note two equations (38) are a direct consequence of the equations (37) and they express the fact that the 4-divergence of the dual vector electromagnetic fields $F_{\pm}^{i}(x)$ equals to the density of dual charges. Using the equations (38) we can rewrite the equations (37) in a slightly different form:

$$\partial_0 F_{\pm}^{i}(x) \pm \sqrt{\varepsilon_0}\,n_0^{-1}\,g^{ip}\,e_{pqrs}\,n_0^q\,\partial^r F_{\pm}^{s}(x) = -j_{\pm}^{i}(x) + n_0^{-1}\,\rho_0^{\pm}(x)\,n_0^i\,. \tag{39}$$

Now a contraction of both sides of equations (39) with the covariant vector n_i^0 gives an identity (i. e. it defines densities of dual charges). We can consider eight equations (39) together with two equations (couplings) (38) a covariant system of Maxwell's equations for two dual vector electromagnetic fields $F_{\pm}^{i}(x)$ based on the dual symmetry in $\mathcal{R}^4$.

Taking half of the sum and difference of two equations (39) as well as two equations (38) we can write Maxwell's equations as a system of eight covariant equations for a real four-dimensional electrical field $E^i(x)$ and a real four-dimensional magnetic field $H^i(x)$ with two couplings:

$$\begin{aligned} \partial_0 E^i(x) + n_0^{-1}\, e^{ipqr}\, n_p^0\, \partial_q H_r(x) &= \\ = -\tfrac{1}{2}\left[j^i_+(x) + j^i_-(x)\right] &+ \tfrac{1}{2} n_0^{-1}\left[\rho_0^+(x) + \rho_0^-(x)\right] n_0^i, \\ \sqrt{\varepsilon_0}\left[\partial_0 H^i(x) + \varepsilon_0\, n_0^{-1}\, e^{ipqr}\, n_p^0\, \partial_q E_r(x)\right] &= \\ = \tfrac{1}{2}\left[j^i_+(x) - j^i_-(x)\right] &+ \tfrac{1}{2} n_0^{-1}\left[\rho_0^+(x) - \rho_0^-(x)\right] n_0^i, \\ \partial_i E^i(x) &= \tfrac{1}{2}\left[\rho_0^+(x) + \rho_0^-(x)\right], \\ \sqrt{\varepsilon_0}\, \partial_i H^i(x) &= \tfrac{1}{2}\left[\rho_0^+(x) - \rho_0^-(x)\right]. \end{aligned} \tag{40}$$

If in lieu of dual currents $j^i_\pm(x)$ and densities of dual charges $\rho_0^\pm(x)$ we introduce (real) electrical current $j^i_{(e)}(x)$, (real) magnetic current $j^i_{(m)}(x)$, (real) density of electrical charges $\rho_0^{(e)}(x)$ and (real) density of magnetic charges $\rho_0^{(m)}(x)$ in the system of equations (40) by means of the following simple relations:

$$\begin{aligned} j^i_{(e)}(x) &= \tfrac{1}{2}\left[j^i_+(x) + j^i_-(x)\right], & \rho_0^{(e)}(x) &= \tfrac{1}{2}\left[\rho_0^+(x) + \rho_0^-(x)\right], \\ \sqrt{\varepsilon_0}\, j^i_{(m)}(x) &= \tfrac{1}{2}\left[j^i_+(x) - j^i_-(x)\right], & \sqrt{\varepsilon_0}\, \rho_0^{(m)}(x) &= \tfrac{1}{2}\left[\rho_0^+(x) - \rho_0^-(x)\right], \end{aligned}$$

we obtain Maxwell's equations in covariant form with the real quantities only:

$$\begin{aligned} \partial_0 E^i(x) + n_0^{-1}\, e^{ipqr}\, n_p^0\, \partial_q H_r(x) &= -j^i_{(e)}(x) + n_0^{-1}\, \rho_0^{(e)}(x)\, n_0^i, \\ \partial_i E^i(x) &= \rho_0^{(e)}(x), \\ \partial_0 H^i(x) + \varepsilon_0\, n_0^{-1}\, e^{ipqr}\, n_p^0\, \partial_q E_r(x) &= j^i_{(m)}(x) + n_0^{-1}\, \rho_0^{(m)}(x)\, n_0^i, \\ \partial_i H^i(x) &= \rho_0^{(m)}(x). \end{aligned} \tag{41}$$

Thus, the set of covariant Maxwell's equations (35) and (36) for two dual bivector fields $f^{ik}_\pm(x)$ in $\mathcal{R}^4$ is equivalent to eight covariant equations (41) for a real four-dimensional vector electrical field $E^i(x)$ which 4-divergence is equal to density of electrical charges and a real four-dimensional pseudovector magnetic field $H^i(x)$ which 4-divergence is equal to density of magnetic charges. We would like to note three important features of the Maxwell's equations in the form (41). First, it is a completely covariant form of these equations obviously invariant under transformations conserving the constant metric tensor g_{ik} in $\mathcal{R}^4$, i. e. under the six-parametrical transformations with transformation matrix $t^i_k(u)$ which fulfills the conditions:

$$g_{pq} = t^i_p(u)\, t^k_q(u)\, g_{ik}, \quad t_0(u) = \det \mid t^i_k(u) \mid = \pm 1,$$

where an argument u stands for a set of six group parameters of proper continuous group. Second, the covariance of the Maxwell's equations in the dual vector form (39) and in real two-vector form (41) can be reached by a cost of the introduction of an arbitrary constant vector n_0^i which determines a fixed direction in $\mathcal{R}^4$. In general, dual currents $j^i_\pm(x)$ are the four-dimensional vectors and the densities of dual charges represent a scalar product

of dual currents and an arbitrary constant vector in $\mathcal{R}^4$. These densities are the invariant quantities which represent no fourth components of appropriate dual currents. Third, on the right side of the Maxwell's equations in a covariant form (41) besides the electrical currents and charges the magnetic currents and charges present. The presence of magnetic currents and charges in the equations (41) is directly due to a possibility to introduce the dual tensors of electromagnetic field with different dualities in $\mathcal{R}^4$.

If now we consider real space-time [8] the Maxwell's equations in the form (41) with $\varepsilon_0 = -1$ represent a covariant form of Maxwell's equations based on the dual symmetry in Minkowski's space. In addition, if we even fix the arbitrary constant vector n_0^i along the time axis, i. e. we choose it as a unit vector in space-time in the form $(0,0,0,1)$, we lose 4-covariance of these equations since we pass in $\mathcal{R}^4$ to three-dimensional quantities and receive the Maxwell's equations which are close to usual Maxwell's equations . In the end we find for $i = \alpha$ ($\alpha,\beta,\gamma = 1,2,3$; $e^{\alpha\beta\gamma} \equiv e^{\alpha\beta\gamma 4}$; $e_{\alpha\beta\gamma} = -e^{\alpha\beta\gamma}$; $E^\alpha(x) = \mathcal{E}^\alpha(\vec{x},t) = -\mathcal{E}_\alpha(\vec{x},t)$; $H^\alpha(x) = \mathcal{H}^\alpha(\vec{x},t) = -\mathcal{H}_\alpha(\vec{x},t)$) a system of (non-covariant) equations:

$$\begin{aligned} c_0^{-1}\partial_t \mathcal{E}^\alpha(\vec{x},t) - e^{\alpha\beta\gamma}\, \partial_\beta \mathcal{H}_\gamma(\vec{x},t) &= -j^\alpha_{(e)}(\vec{x},t), \\ \partial_\alpha \mathcal{E}^\alpha(\vec{x},t) &= \rho_0^{(e)}(\vec{x},t), \\ c_0^{-1}\partial_t \mathcal{H}^\alpha(\vec{x},t) + e^{\alpha\beta\gamma}\, \partial_\beta \mathcal{E}_\gamma(\vec{x},t) &= j^\alpha_{(m)}(\vec{x},t), \\ \partial_\alpha \mathcal{H}^\alpha(\vec{x},t) &= \rho_0^{(m)}(\vec{x},t), \end{aligned} \tag{42}$$

and, for $i = 4$, we have two additional relations which determine the densities of electrical and magnetic charges as the forth components of appropriate currents:

$$j^4_{(e)}(\vec{x},t) \equiv \rho_0^{(e)}(\vec{x},t), \quad j^4_{(m)}(\vec{x},t) \equiv \rho_0^{(m)}(\vec{x},t).$$

From the formulae (42) we see that we have derived slightly different system of Maxwell's equations in comparison with the usual system of Maxwell's equations (1). This difference consists in the presence of magnetic current $j^\alpha_{(m)}(\vec{x},t)$ and density of magnetic charge $\rho_0^{(m)}(\vec{x},t)$ if the dual currents $j^i_+(\vec{x},t)$ and $j^i_-(\vec{x},t)$ are not identically equal. The Maxwell's equations with magnetic currents and charges in the form (42) can be met in literature devoted to the problem of symmetry of Maxwell's equations under the change of electrical and magnetic items (fields, currents and charges) what leads to a hypothesis of magnetic monopole [14]. Deriving the Maxwell's equations in $\mathcal{R}^4$ on the base of dual symmetry we have transparently shown that if a magnetic monopole exists in the nature then its existence is due to dual symmetry of real space-time understood in the sense of this paper.

We return again to the system of general Maxwell's equations (41) in order to derive still few known formulae in a covariant form. As a matter of principle having the Maxwell's equations in symmetrical covariant form (41) one can take a handbook of electromagnetic field theory and try to rewrite large majority of formulae where electrical and magnetic fields are presented in the more "symmetrical" form. Here we show only an explicit form of the energy-momentum tensor of electromagnetic field expressed in terms of electrical and

[8] **We suppose the diagonalization of the metric tensor g_{ik} for space-time is carried out in such a way that the signature of four-dimensional space is $(-1,-1,-1,+1)$, $\varepsilon_0 = -1$, $x^4 = c_0 t$.**

magnetic fields. Multiplying the first relation in (41) by $E_i(x)$ and the third one by $\varepsilon_0\, H_i(x)$ a conservation law can be found:

$$\partial_0 \mathcal{E}_0(x) - \varepsilon_0\, \partial_i \mathcal{P}^i(x) = -j^i_{(e)}(x)\, E_i(x) + j^i_{(m)}(x)\, H_i(x) \tag{43}$$

where the so-called density of energy of electromagnetic field $\mathcal{E}_0(x)$ as an invariant was introduced:

$$\mathcal{E}_0(x) = \tfrac{1}{2}\, [E^i(x)\, E_i(x) - \varepsilon_0\, H^i(x)\, H_i(x)] \equiv 2F^i_{\pm}(x)\, F^{\mp}_i(x) \tag{44}$$

as well as Poynting's vector of electromagnetic field $\mathcal{P}^i(x)$ as a four-dimensional vector in $\mathcal{R}^4$ which is perpendicular to vectors n^i_0, $E^i(x)$ and $H^i(x)$ was defined:

$$\mathcal{P}^i(x) = n_0^{-1}\, e^{ipqr}\, n^0_p\, E_q(x)\, H_r(x) \equiv \pm\tfrac{1}{2}\, \varepsilon_0 \sqrt{\varepsilon_0}\, n_0^{-1}\, e^{ipqr}\, n^0_p\, F^{\mp}_q(x)\, F^{\pm}_r(x)\,. \tag{45}$$

The square of Poynting's vector $\mathcal{P}^2_0(x)$ is an invariant too and it can be easily calculated. The following result can be obtained:

$$\mathcal{P}^2_0(x) = g_{ik}\, \mathcal{P}^i(x)\, \mathcal{P}^k(x) = \varepsilon_0 \{ [E^i(x)\, E_i(x)]\, [H^i(x)\, H_i(x)] - [E^i(x)\, H_i(x)]^2 \}.$$

By means of just introduced density of energy of electromagnetic field (44) and four-dimensional Poynting's vector (45) the invariant of the fourth order $\mathcal{I}_{0(4)}(f)$ can be rewritten in an attractive form:

$$\mathcal{I}_{0(4)}(f) = \mathcal{E}^2_0(x) + \mathcal{P}^i(x)\, \mathcal{P}_i(x)\,. \tag{46}$$

We have clarified that the vectors of electrical and magnetic fields are model dependent quantities and thus the invariant (44) being the density of energy of electromagnetic field is model dependent, too, since it can not be expressed in terms of the invariants of the second rank $\mathcal{I}^1_{0(2)}(f)$ and $\mathcal{I}^2_{0(2)}(f)$. On the other hand, the dimensionality of the invariant of the fourth order (46) is square of the density of energy of electromagnetic field and it is model independent. Therefore we can accept a quantity $e_0(x)$ which is equal square root of invariant $\mathcal{I}_{0(4)}(f)$:

$$e_0(x) = \sqrt{\mathcal{E}^2_0(x) + \mathcal{P}^i(x)\, \mathcal{P}_i(x)} \tag{47}$$

in the capacity of suitable density of energy of electromagnetic field which can really be measured. We would like note that in the case $\mathcal{E}^2_0(x) \gg \mathcal{P}^2_0(x)$ the relation (47) is reduced to the usually accepted expression for the density of energy of electromagnetic field (44).

As it is known the traceless symmetrical contravariant energy-momentum tensor of electromagnetic field $\mathcal{T}^{ik}(f)$ which can be expressed in terms of a skew-symmetric tensor of electromagnetic field $f^{pq}(x)$ has the form [9]:

$$\mathcal{T}^{ik}(f) = -\left[f^{ip}(x)\, g_{pq}\, f^{qk}(x) + \tfrac{1}{2}\, \mathcal{I}^1_{0(2)}(f)\, g^{ik}\right], \quad g_{pq}\, \mathcal{T}^{pq}(f) \equiv 0. \tag{48}$$

Using three-vector model for the tensor electromagnetic field $f^{ij}(x)$ from previous sections the energy-momentum tensor $\mathcal{T}^{ik}(f)$ can be easily rewritten in a covariant form in terms of electrical and magnetic fields:

$$\begin{aligned} \mathcal{T}^{ik}(f) &= E^i(x)\, E^k(x) - \varepsilon_0\, H^i(x)\, H^k(x) - \\ &- \varepsilon_0\, n_0^{-1}\left[n^i_0\, \mathcal{P}^k(x) + n^k_0\, \mathcal{P}^i(x)\right] - \mathcal{E}_0(x)\, (g^{ik} - 2n_0^{-2} n^i_0\, n^k_0\,). \end{aligned} \tag{49}$$

If we calculate the determinant of the contravariant matrix (49) we find the simple relation:

$$\mathcal{T}_0(f) = \det |\, \mathcal{T}^{ik}(f)\, | \equiv \mathcal{I}_{0(4)}(f)\,. \tag{50}$$

On the base of the relation (50) we are able to give another interpretation of the invariant of the fourth order introduced in (11). This invariant is equal to the determinant of the energy-momentum tensor of electromagnetic field $\mathcal{T}^{ik}(f)$. The relation (50) gives a warrant to consider the quantity (47) as a density of energy of electromagnetic field. It is interesting to calculate an inverse (covariant) tensor $\bar{\mathcal{T}}_{ik}(f)$ to the tensor of energy-momentum of electromagnetic field $\mathcal{T}^{jk}(f)$. This inverse tensor can be obtained after not very complicated calculations:

$$\begin{aligned} \bar{\mathcal{T}}_{ik}(f) &= \left[\mathcal{I}_{0(4)}(f)\right]^{-1} \{ E_i(x)\, E_k(x) - \varepsilon_0\, H_i(x)\, H_k(x) - \\ &- \varepsilon_0\, n_0^{-1}\, [n^0_i\, \mathcal{P}_k(x) + n^0_k\, \mathcal{P}_i(x)] - \mathcal{E}_0(x)\, (g^{ik} - 2n_0^{-2} n^0_i\, n^0_k\,) \} \equiv \\ &\equiv -\left[\mathcal{I}_{0(4)}(f)\right]^{-1} \left[f_{ip}(x)\, g^{pq}\, f_{qk}(x) + \tfrac{1}{2}\, \mathcal{I}^1_{0(2)}(f)\, g_{ik}\right]. \end{aligned} \tag{51}$$

One can check easily that the inverse tensor $\bar{\mathcal{T}}_{ij}(f)$ really fulfills a relation of orthogonality: $\bar{\mathcal{T}}_{ik}(f)\, \mathcal{T}^{jk}(f) = \delta^j_i$. [9] Note an interesting relation between these tensors:

$$\begin{aligned} \bar{\mathcal{T}}_{ik}(f) &= \left[\mathcal{I}_{0(4)}(f)\right]^{-1} g_{ip}\, g_{kq}\, \mathcal{T}^{pq}(f) = \left[\mathcal{I}_{0(4)}(f)\right]^{-1} \mathcal{T}_{ik}(f)\,, \\ \mathcal{T}^{ik}(f) &= \mathcal{I}_{0(4)}(f)\, g^{ip}\, g^{kq}\, \bar{\mathcal{T}}_{pq}(f) = \mathcal{I}_{0(4)}(f)\, \bar{\mathcal{T}}^{ik}(f)\,, \end{aligned}$$

which is an example of the fact that the raising or lowering the indices of a tensor by means of metric tensor and by means of completely antisymmetrical tensors are not quite equivalent operations.

We will not continue with listing the covariant formulae which can be written on the base of the covariant three-vector model of the bivector of electromagnetic field in $\mathcal{R}^4$. We would only like to stress that a general approach to the four-dimensional spaces allows a new approach to many old and well-known physical problems. We must not reject such an approach only because of its mathematical base which at present does not provide enough physical effects and results. Our real physical world is only one of the manifold of abstract mathematical worlds.

5 CONCLUDING REMARKS

We will discuss shortly the obtained results. First, we have shown that a special symmetry exists for skew-symmetric tensors in any four-dimensional metric space which is connected with the concept of duality. In each four-dimensional metric space an operator of duality can be strictly mathematically introduced and two different dual bivector fields with the different dualities which are determined as the eigenvalues of the operator of duality can be

[9] The orthogonality of tensors $\mathcal{T}^{jk}(f)$ and $\bar{\mathcal{T}}_{ik}(f)$ expressed by means of tensors of electromagnetic field $f^{ik}(x)$ and $f_{ik}(x)$ can be checked directly, too, if we use the Hamilton-Cayley's equation (10) for a bivector $f_{ik}(x)$ in $\mathcal{R}^4$.

built up from any real bivector field. The duality can be regarded as a kind of "higher" or "internal" symmetry in any $\mathcal{R}^4$. In general, from the theoretical point of view we cannot and should not ignore the dual symmetry in the physical theories formulated in four-dimensional spaces. Proceeding from the dual symmetry in $\mathcal{R}^4$ one can write the Maxwell's equations in a four-dimensional space with arbitrary signature. These Maxwell's equations can be formulated too in another equivalent form connected with the existence of a covariant three-vector model for the six-component bivector in four-dimensional space. For the description of a bivector (electromagnetic) field in $\mathcal{R}^4$ we can use either dual vector electromagnetic fields or a vector electrical field and a pseudovector magnetic field together with an arbitrary constant vector. We have shown that the equations first written by J. C. Maxwell represent only one possibility from a number of covariant equations which can be derived in $\mathcal{R}^4$. Nevertheless, the Maxwell's equations are true as before, however, there are no reasons to fix the four-dimensional constant vector along the time axis. The actual role of an arbitrary constant (unit) vector which was introduced into theory is not quite clear. This vector is constant in respect to electromagnetic field but it possibly can be fixed by means of other physical conditions introduced into a theory formulated in $\mathcal{R}^4$. It should be stressed that from this point of view the electrical and magnetic fields as well as some other quantities (like Poynting's vector) are secondary quantities (concepts) which are model dependent what proves their dependence on an arbitrary constant four-dimensional vector determining a fixed direction in $\mathcal{R}^4$. Thus, the vector electrical and magnetic fields have an auxiliary sense only. Really measured quantities can be only the invariants built up from the primary bivectors (which can be expressed, of course, in terms of the components of the dual bivectors or the electrical and magnetic fields) and not all the invariants which can be built up from electrical and magnetic fields are really measurable. We would like to emphasize a methodological aspect of applied method of consideration of four-dimensional spaces. The unified approach to different four-dimensional spaces allows to use strictly the tensor calculus from the very beginning up to the final results. In addition, such approach gives a possibility to discover new connections between quantities which do not seem to be linked (like the connection between the square root of sign of determinant of a metric in $\mathcal{R}^4$ and a dual unit). The simultaneous consideration of four-dimensional spaces with different signatures leads to the field of dual numbers and allows to introduce the dual quantities into theory in a very natural way. It is quite probable that the dual numbers play more significant role in physics then we suppose nowadays. Perhaps all this article could seem too technical or even formal, which is true in a sense, however without overcoming some technical problems we will never be able to construct a more general theory then current theories. We should not be indifferent to the chosen mathematical method since different mathematical methods are not quite equivalent from the point of view of their relation to the general theory as well as physical reality.

At this point everybody who has read this article will expect a plentiful discussion of physical consequences of the formulation of the Maxwell's equations based on the concept of duality in $\mathcal{R}^4$. Actually enormous number of questions can be raised. Especially, one can look for the reasons why the dual currents $j^i_+(x)$ and $j^i_-(x)$ have to be equal quantities in real space-time in order to obtain the usual Maxwell's equations (without magnetic currents and charges) or what consequences follow for the existence of a magnetic monopole in connection

with the dual symmetry of space-time understood in the sense of this article. However, at the present stage we are not to be able to answer to arisen problems correctly. If we propose that dual currents $j^i_+(x)$ and $j^i_-(x)$ in real space-time are identically equal we have to find an additional symmetry which causes this equality. We are not able to show such additional symmetry at present. If the difference of dual currents $j^i_+(x)$ and $j^i_-(x)$ is small (almost negligible) we should show the order of this difference which is impossible to be done without the consideration of geometry of group space of an appropriate group of transformations where the dual symmetry can be completely realized. Thus, in connection to this we avoid raising of abundant discussion about three-vector covariant form of Maxwell's equations here considered since it is probably still early. First of all we wanted to show a mathematical possibility of a covariant formulation of Maxwell's equations in any four-dimensional metric space. Up to this point we have used no physics in deduction of obtained equations, no experimental facts for the formulation of the Maxwell's equations with the dual symmetry. The primary base of all reasonings was symmetry and tensor calculus and the only criterion of correctness of all the formulae was a "mathematical beauty". We would not like to break this criterion introducing the physical proposals into particular theory since we have equally regarded real space-time (Minkowski's space) with other four-dimensional Euclidean and pseudo-Euclidean spaces which differ by a set of topologically significant invariants. We did not concern another possibility of the use of the dual symmetry in $\mathcal{R}^4$ here. Namely, it is necessary to consider the geometry of group space of a six-parametrical group of transformations of an arbitrary four-dimensional homogeneous quadric as well as the geometry of group space of a ten-parametrical group of invariance of an interval (square of distance between two points) in $\mathcal{R}^4$ (a generalized Poincaré group in $\mathcal{R}^4$ with any signature) where the answers to many key questions are hidden. We would like to pay attention to the fact that the large majority of group spaces is not point-like manifolds where a point is determined by a contravariant vector. The group parameters of most continuous groups stand for tensors which represent different geometrical objects. The simplest generalization of the vector are multivectors and it seems we do not fully realize the role of multivectors (especially bivectors) in physical theories at present. All these questions merit a separate consideration because the problems are essential and rather complicated. Several papers devoted to the geometry of the group spaces of some physically interesting continuous groups operating in the four-dimensional spaces will be published before long elsewhere. On the base of the results concerning geometry of group spaces in $\mathcal{R}^4$ we shall return to the discussion of the system of Maxwell's equations with the dual symmetry in the forms (35), (39) and (41). After this we will be able to predict some physically measurable effects or at least to give a recipe where to search such effects. We are firmly convinced that the duality defined in this paper by means of a strictly determined operator in $\mathcal{R}^4$ will have a rich physical meaning. Duality is an additional symmetry in each four-dimensional metric space and we have to take account of it as a conserving "good quantum number" for appropriate skew-symmetric quantities. At the present level of investigations we should be content with a possibility to rewrite all the handbooks of electromagnetic field theory in a slightly more "symmetrical" (covariant) form.

Acknowledgments

The author thanks to Prof. N. B. Skachkov for stimulating discussions during his stay at Laboratory of Theoretical Physics of JINR in Dubna.

References

[1] Fock V. A.: 1935 *Z. Phys.* **98** 145–54

Bargmann V.: 1936 *Z. Phys.* **99** 576–82

Englefield M. J.: 1972 *Group Theory and The Coulomb Problem* (New York: J. Wiley and Sons)

[2] Miller W. (Jr.): 1977 *Symmetry and Separation of Variables* (Reading: Addison – Wesley)

Kalnins E. G.: 1986 *Separation of Variables for Riemannian Spaces of Constant Curvature* (Harlow: Longman Scientific and Technical)

[3] Bjorken J. D. and Drell S. D.: 1964, 1965 *Relativistic Quantum Mechanics and Relativistic Quantum Fields* (New York: McGraw-Hill Book Comp.)

Bogol'ubov N. N. and Shirkov D. V.: 1973 *Introduction into Theory of Quantum Fields* (Moscow: Nauka)

Akhiezer A. I. and Beresteckii V. B.: 1969 *Quantum Electrodynamics* (Moscow: Nauka) (in Russian)

[4] Fedorov F. I.: 1979 *Lorentz's Group* (Moscow: Nauka) (in Russian)

Nash Ch. and Siddhartha Sen: 1983 *Topology and Geometry for Physicists* (London: Academic Press, Inc.)

[5] Maxwell J. C.: 1904 (1st ed. 1873) *Electricity and Magnetism* (London: Oxford University Press)

[6] Heaviside O.: 1893 *Phil. Trans. Roy. Soc.* **A 183** 423–430

[7] Rainich G. I.: 1925 *Trans. Amer. Math. Soc.* **27** 106

Larmor J.: 1928 *Collected Papers* (London:)

[8] Einstein A.: 1905 *An. d. Phys.* **17** 891–921

Lorentz H. A.: 2nd ed. 1953 *Theory of Electrons* (Moscow: GITTL) (in Russian)

[9] Landau L. D. and Lifshitz E. M.: 7th ed. 1988 *Field Theory* (Moscow: Nauka) (in Russian)

Jackson J. D.:1967 *Classical Electrodynamics* (New York–London–Sydney: J. Wiley and Sons)

[10] Bateman H.: 1909 *Proc. London Math. Soc.* **8** 223–264

Cuningham E.: 1909 *Proc. London Math. Soc.* **8** 77–98

[11] Fushchich V. I. and Nikitin A. G.: 1983 *Symmetry of Maxwell's Equations* (Kiev: Naukova Dumka) (in Russian)

[12] Schouten J. A.: 1954 *Ricci-Calculus* (Berlin: Springer)

Gurevich G. B.: 1948 *Foundations of Invariant Theory* (Moscow – Leningrad: OGIZ–GITTL) (in Russian)

Shirokov P. A.: 1961 *Tensor Calculus* (Kazan: Publishing House of Kazan University) (in Russian)

[13] Lukáč I.: 1991 *The Decomposition of The Algebra of The Group of Transformations of An Arbitrary Homogeneous Four-Dimensional Quadric in The Field of Dual Numbers* (*FÚ SAV Report* 91/01 — OTF: Bratislava) (Submitted to: *J. Math. Phys.*)

[14] Dirac P. A. M.: 1931 *Proc. Roy. Soc.* **A 133** 60

——1946 *Phys. Rev.* **74** 817

Schwinger J.: 1966 *Phys. Rev.* **144** 1087–1093

Strazhev V. I. and Tomilchik L. M.: 1975 *Electrodynamics with Magnetic Charge* (Minsk: Nauka i Tekhnika) (in Russian)

Received by Publishing Department
on October 16, 1991.