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INTRODUCTION OF THE LOBACHEVSKY'S GEOMETRY INTO THE NEWTON'S THEORY OF GRAVITATION

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The first of December 1992 is the 200 th anniversary of birth of Nikolai Ivanovich Lobachevsky. The present article is devoted to this date.

N.I. Lobachevsky upon solving the long-standing problem discovered an extraordinary geometry. He delivered a report on it in 1826 at Kazan University and wrote a paper [1] published in "The Kazan's Bulletin edited by the Imperator's University of Kazan". In this paper, for the first time in the mathematical literature, a geometric scheme was presented based on all Euclidean postulates except for the fifth one which is referred to as the Euclidean postulate of parallels.

It is useful to illustrate the difference between the two geometric schemes, the Lobachevskyan and Euclidean ones, by using a sphere with radius ρ as an example.

According to the third Euclidean postulate "from any center it is possible to draw a circle of arbitrary radius" [2, p. 14]. This means that both in Euclidean and Lobachevskyan geometries the value ρ may be arbitrary. Lobachevsky has shown that the internal geometry of a sphere is independent of the fifth Euclidean postulate. Therefore, both in the Euclidean and Lobachevskyan geometries we can draw on the sphere "parallels and meridians" and we may introduce polar coordinates θ and φ. We denote the length of the "equator" by $2\pi r$. The distance along the meridian from the "north pole" to the point (θ,φ) on the sphere will be put to be equal to θr; while the distance along the parallel $\theta = \text{const}$ from the zero meridian to the same point, $\varphi r \sin\theta$. In the usual way we get the metric form $r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$ of the sphere and surface element $r^2 \sin\theta\, d\theta\, d\varphi$ of the sphere. The surface of the whole sphere is equal to $4\pi r^2$. The difference between the two geometric schemes consists in the dependence of r on ρ : in the Euclidean geometry $r = \rho$ while in the Lobachevskyan one

$$r = k\ sh\ \frac{\rho}{k} \tag{1}$$

Here k is some characteristic constant called the Lobachevsky constant. For $k \longrightarrow \infty$ the Lobachevskyan geometry transforms into the Euclidean geometry. If $k < \infty$, then in the sphere of radius ρ for small values of ρ / k we may approximately use the Euclidean geometry.

In both the geometrical schemes the radius is perpendicular to the sphere. Therefore the metric form of the space is equal to

$$d\,s^2 = d\,\rho^2 + r^2 (d\,\theta^2 + \sin^2\theta\ d\,\varphi^2) . \tag{2}$$

The volume element is equal to

$$d\,V = r^2 \sin\theta\ d\,\rho\ d\,\theta\ d\,\varphi . \tag{3}$$

In the equatorial plane $\theta = \pi / 2$ and the metric form equals

$$d\,s^2 = d\,\rho^2 + r^2\,d\,\varphi^2 \tag{4}$$

and the surface element is given by

$$d\,\Sigma = r\ d\,\rho\ d\,\varphi . \tag{5}$$

The constant k enters into all the formulae of the Lobachevskyan geometry in that part where it differs from the Euclidean geometry. For example, the sum of angles A, B, C of an arbitrary triangle in the Euclidean planimetry is equal to π, while in the Lobachevsky planimetry it is smaller than π. As Lobachevsky has shown, the area of the triangle is equal to

$$F = k^2 (\pi - A - B - C) . \tag{6}$$

Another example. Let us draw a perpendicular from the origin of a straight line beam to a parallel straight line. The angle Π between the beam and the perpendicular in the Euclidean planimetry is equal to $\pi / 2$. In the Lobachevskyan planimetry this angle is smaller than $\pi / 2$ and depends on the height p from which the perpendicular is drown. The quantity

$\Pi = \Pi(p / k)$ has been called by Lobachevsky the angle of parallelism. Lobachevsky has shown that

$$tg\ \frac{1}{2}\,\Pi(x) = e^{-x} . \tag{7}$$

The third example. For an arbitrary triangle Lobachevsky has obtained the following four formulae (in which a, b, c are sides of a triangle and A, B, C are the corresponding angles):

$$ch\ \frac{c}{k} = ch\ \frac{a}{k}\ ch\ \frac{b}{k} - sh\ \frac{a}{k}\ sh\ \frac{b}{k}\ \cos C$$

$$sh\ \frac{b}{k}\ \sin A = sh\ \frac{a}{k}\ \sin B$$

$$ctg\ A\ \sin C + ch\ \frac{b}{k}\ \cos C = cth\ \frac{a}{k}\ sh\ \frac{b}{k}$$

$$\cos C + \cos A\ \cos B = \sin A\ \sin B\ ch\ \frac{c}{k}$$

and for a rectangular triangle (for which $C = \pi / 2$) he obtained the following six formulae:

$$ch\ \frac{c}{k} = ch\ \frac{a}{k}\ ch\ \frac{b}{k} \qquad th\ \frac{b}{k} = th\ \frac{c}{k}\ \cos A$$

$$sh\ \frac{a}{k} = sh\ \frac{c}{k}\ \sin A \qquad \cos A = ch\ \frac{a}{k}\ \sin B$$

$$th\ \frac{a}{k} = sh\ \frac{b}{k}\ tg\ A \qquad ch\ \frac{c}{k}\ tg\ A\ tg\ B = 1$$

On the basis of these formulae an arbitrary trigonometric problem may be solved.

The volume of publication restricts our review, but mention is to be done of the surfaces orthogonal to a bundle of parallel lines. Lobachevsky called them limiting spheres (orispheres) and showed that the internal geometry of an orisphere coincides with the Euclidean planimetry. Here we should emphasize: rejecting the fifth Euclidean postulate for the plane, Lobachevsky has proved it for the orisphere!

It should also be stressed that " the new Geometry, the foundation of which is done … opens a new wide field for the interplay of Geometry and Analytics " [1, p. 209]. Lobachevsky himself reached great results in this field:

" A wide circle of mathematicians, physicists and engineers do not know that the handbooks and tables of definite integrals used by them contain formulae obtained by Lobachevsky using the methods of his "imaginary geometry" [3, p. 413].

We also mention that in this field H. Poincare created in 1882 the theory of automorphic functions. He obtained a great profit from his idea of using the conformal map of the Lobachevskyan plane onto Euclidean semi-plane [4, p. 306].

A final mention is to be made that Lobachevsky formulated two completely new problems: on the astronomical verification of the geometry of our visible world and on the " kinds of changes which will occur in Mechanics after introducing in it the imaginary Geometry " [1, p. 261].

Both these problems are actual now. In paper [5] published in 1835 – 38 in the "Scientific Notes of the Imperator University of Kazan" Lobachevsky wrote:

" ... there should be no contradiction in our brain when we admit that some forces in Nature follow a certain Geometry while other forces follow their particular Geometry. To clarify this idea, we assume, many others believe in this also, that the attractive forces weaken due to the propagation of their action on the sphere. In the used Geometry the surface of the sphere is $4 \pi r^2$ and therefore the decrease of the force is quadratic in distance. In the imaginary Geometry I found the area of the sphere

$$\pi (e^{r} - e^{-r})^2$$

and perhaps the molecular forces follow this Geometry and their manifold should then depend on the number e always being very large. By the way let this be a pure hypothesis which should be proved by more convenient arguments; but we should not doubt that the forces produce all by themselves: the motion, velocity, time, mass and even distances and angles" [5, p. 159].

It is possible to get agreement of the formula presented in this part of the text with the previous one describing the area of the sphere of radius ρ, if we abandon notation (1) and put

$$k = 1 , \quad r = \rho , \quad e = \lim_{n \to \infty} (1 + \frac{1}{n})^n = \sum_{k=0}^{\infty} \frac{1}{k!} .$$

However in the subsequent consideration it is more convenient to keep notation (1).

We shall come to the Newton theory of gravitation in the Lobachevskyan space provided we can find in the quoted text the fundamental solution of the Poisson equation

$$\Delta \Phi = 4 \pi \alpha \, \delta(x) \, \delta(y) \, \delta(z) \tag{8}$$

in the space with metric (2). This equation contains the following ingredients:

The gravitation potential Φ acting on a test body and produced by the point mass m;

The constant $\alpha = \gamma m$, where γ is the Newton constant;

The Laplace operator in the Lobachevskyan space equals

$$\Delta = \frac{1}{r^2} \frac{\partial}{\partial \rho} r^2 \frac{\partial}{\partial \rho} + \frac{1}{r^2} (\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2});$$

The Dirac δ-function has arguments

$$x = r \sin \theta \cos \varphi , \quad y = r \sin \theta \cos \varphi , \quad z = r \cos \theta . \tag{9}$$

According to the remark of Lobachevsky we assume that the "attractive force" (a covector of the force) has the following components:

$$F_1 = - \frac{\partial \Phi}{\partial \rho} = - \frac{\alpha}{r^2} , \quad F_2 = - \frac{\partial \Phi}{\partial \theta} = 0 , \quad F_3 = - \frac{\partial \Phi}{\partial \varphi} = 0 .$$

Consequently

$$\Phi = - \frac{\alpha}{k \, tg \frac{\rho}{k}} . \tag{10}$$

Let us consider the motion of the test body. Denoting the time by t, we obtain from (2) and (10) the Lagrangian of the test body

$$L = \frac{1}{2} (\dot{\rho}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \, \dot{\varphi}^2) - \Phi . \tag{11}$$

as well as the equations of motion

$$\frac{d}{dt}\frac{\partial L}{\partial \dot\rho} - \frac{\partial L}{\partial \rho} = 0 \ ,$$
$$\frac{d}{dt}\frac{\partial L}{\partial \dot\theta} - \frac{\partial L}{\partial \theta} = 0 \ , \qquad (12)$$
$$\frac{d}{dt}\frac{\partial L}{\partial \dot\varphi} - \frac{\partial L}{\partial \varphi} = 0 \ .$$

We have

$$\frac{\partial L}{\partial \dot\rho} = \dot\rho \qquad \frac{\partial L}{\partial \dot\theta} = r^2\,\dot\theta \qquad \frac{\partial L}{\partial \dot\varphi} = r^2 \sin^2\theta\ \dot\varphi$$
$$\frac{\partial L}{\partial \rho} = r\, r'\,(\,\dot\theta^2 + \sin^2\theta\ \dot\varphi^2\,) - \Phi'$$
$$\frac{\partial L}{\partial \theta} = r^2 \sin\theta \cos\theta\ \dot\varphi^2 \qquad \frac{\partial L}{\partial \varphi} = 0$$

where

$$\Phi' = \frac{\alpha}{r^2} \ , \qquad r' = ch\,\frac{\rho}{k} \ . \qquad (13)$$

Substituting this into (12) we get the following system of equations:

$$\frac{d}{dt}\,\dot\rho - r\,r'\,(\,\dot\theta^2 + \sin^2\theta\ \dot\varphi^2\,) + \frac{\alpha}{r^2} = 0$$
$$\frac{d}{dt}\,(r^2\,\dot\theta) - r^2 \sin\theta\cos\theta\ \dot\varphi^2 = 0$$
$$\frac{d}{dt}\,(r^2 \sin^2\theta\ \dot\varphi) = 0 , \qquad (14)$$

where the dot denotes the time derivative.

Since the Lagrangian (11) does not depend on time, the energy

$$E = \frac{1}{2}\,(\,\dot\rho^2 + r^2\,\dot\theta^2 + r^2 \sin^2\theta\ \dot\varphi^2\,) + \Phi \qquad (15)$$

is conserved. Due to the spherical symmetry of the Lagrangian all the components of the angular momentum

$$M_1 = y\dot z - z\dot y = -\,r^2\,(\,\sin\theta\cos\theta\cos\varphi\ \dot\varphi + \sin\varphi\ \dot\theta\,)$$
$$M_2 = z\dot x - x\dot z = -\,r^2\,(\,\sin\theta\cos\theta\sin\varphi\ \dot\varphi - \cos\varphi\ \dot\theta\,)$$
$$M_3 = x\dot y - y\dot x = r^2 \sin^2\theta\ \dot\varphi \qquad (16)$$

are also conserved; by the way, it is easy to check this directly by differentiating Eqs. (15) and (16).

The conservation of the angular momentum means in this case

that the test body is moving on the Lobachevskyan plane that passes through the origin of attraction. Without loss of generality we can take this plane as the equatorial with $\theta = \pi / 2$. For this choice $M_1 = 0$, $M_2 = 0$, $M_3 = M$, where

$$M = r^2\,\dot\varphi \ , \qquad (17)$$

and the energy integral takes the form

$$E = \frac{1}{2}\,(\,\dot\rho^2 + r^2\,\dot\varphi^2\,) + \Phi \ . \qquad (18)$$

Substituting (17) into (18) we get the differential equation for the trajectory $\rho = \rho(\varphi)$ in the form

$$E = \frac{M^2}{2r^4}\left[\,\left(\frac{d\rho}{d\varphi}\right)^2 + r^2\,\right] + \Phi \ . \qquad (19)$$

Now, taking into account the form of Φ given by (10) we introduce the notation

$$k\ th\,\frac{\rho}{k} = \frac{1}{u} \ . \qquad (20)$$

Since

$$du = -\,\frac{1}{r^2}\,d\rho \ , \qquad u^2 = \frac{1}{r^2} + \frac{1}{k^2} , \qquad (21)$$

equation (19) takes the form

$$E = \frac{M^2}{2}\left[\,\left(\frac{du}{d\varphi}\right)^2 + u^2 - \frac{1}{k^2}\,\right] - \alpha\,u \ . \qquad (22)$$

The solution of this equation is

$$u = \frac{\alpha}{M^2} + \sqrt{\frac{\alpha^2}{M^4} + \frac{2E}{M^2} + \frac{1}{k^2}}\ \cos\varphi \ . \qquad (23)$$

The integration constant is chosen here so that the maximal value of u corresponds to $\varphi = 0$. If we denote

$$p = \frac{M^2}{2}, \qquad \varepsilon = \sqrt{1 + \frac{2EM^2}{\alpha^2} + \frac{M^4}{\alpha^2 k^2}} , \qquad (24)$$

the equation for the trajectory of the test body my be written in the form

$$k\ th\,\frac{\rho}{k} = \frac{p}{1 + \varepsilon\cos\varphi} \ . \qquad (25)$$

In accordance with our choice of the integration constant for equation (22) the nearest point of the orbit (25) to the attraction center corresponds to the angle $\varphi = 0$. The distance from the attraction center to the nearest point of the orbit equals ρ_1 given by

$$k\, th\, \frac{\rho_1}{k} = \frac{p}{1+\varepsilon} \quad . \tag{26}$$

The most remote point of the orbit exists only for a finite motion. In this case the following condition

$$k > \frac{p}{1-\varepsilon} \quad , \text{ i. e. } \quad \varepsilon < 1 - \frac{p}{k} \quad , \tag{27}$$

should be satisfied which is equivalent to the condition

$$E < - \frac{\alpha}{k} \quad . \tag{28}$$

We shall restrict ourselves to this case. We shall then get the solution of the Kepler problem of motion of a planet around the Sun in the Lobachevsky space.

In the case of finite motion the remote point of the orbit corresponds to the angle $\varphi = \pi$ while the maximal distance ρ_2 is given by

$$k\, th\, \frac{\rho_2}{k} = \frac{p}{1-\varepsilon} \quad . \tag{29}$$

Let us denote the size of the orbit by $2\,a$. Clearly it is equal to $\rho_1 + \rho_2$. From (26) and (29) we find

$$k\, th\, \frac{2\,a}{k} = \frac{2\,p}{1-\varepsilon^2+p^2/k^2} = -\frac{\alpha}{E} \quad . \tag{30}$$

Now we find the period T of the motion of the planet. According to (17) we have

$$M\,T = \int_0^{2\pi} r^2(\varphi)\, d\,\varphi \quad . \tag{31}$$

The integrand may be found from (1) and (25)

$$r^2(\varphi) = \frac{p^2}{(1+\varepsilon\cos\varphi)^2 - p^2/k^2} = \tag{32}$$

$$= \frac{p\,k}{2} \left[\frac{1}{1+\varepsilon\cos\varphi - p/k} - \frac{1}{1+\varepsilon\cos\varphi + p/k} \right] .$$

The integral is evaluated by the substitution $\xi = tg\, \frac{\varphi}{2}$. Then we get

$$\frac{d\,\varphi}{m + n\cos\varphi} = \frac{2}{\sqrt{m^2-n^2}}\, d\, arctg \left(\sqrt{\frac{m-n}{m+n}}\; tg\, \frac{\varphi}{2} \right) ,$$

$$\int_0^{2\pi} \frac{d\,\varphi}{m+n\cos\varphi} = \frac{2\,\pi}{\sqrt{m^2-n^2}}$$

for $m > |n|$. Therefore

$$M\,T = \pi\, p\, k \left(\frac{1}{\sqrt{(1-p/k)^2-\varepsilon^2}} - \frac{1}{\sqrt{(1+p/k)^2-\varepsilon^2}} \right). \tag{33}$$

Substituting (24) we get

$$T = \frac{\pi\, k}{\sqrt{2}} \left(\frac{1}{\sqrt{-E-\alpha/k}} - \frac{1}{\sqrt{-E+\alpha/k}} \right). \tag{34}$$

Then the period T depends on the energy E, but it does not depend on the angular momentum M .

The substitution of (30) into (34) gives the following expression for the square of the period :

$$T^2 = \frac{4\,\pi^2}{\alpha} \left(k\, sh\, \frac{a}{k} \right)^3 ch\, \frac{a}{k} \quad . \tag{35}$$

Lobachevsky himself, on the basis of observations of the parallaxes of stars, states that the constant k is larger than the distance from the Earth to these stars. He explained that this result

" ... justifies the accuracy of all calculations of the customary Geometry and admits to take its fundamentals as if proved".

On the other hand, it is impossible to disregard the opinion by Laplace that the visible stars and the Milky Way belong only to one set of celestial objects similar to those which we observe as faintly twinkling spots in constellations of Orion, Andromeda, Capricorn, and others. Thus, even without saying that the space may be indefinitely extended in our mind, the Nature itself indicates such distances in comparison with which even the distances from the Earth to fixed stars are negligible.

After this it is impossible any more to state that the assumption that the measure of lines is independent of angles, i. e. the assumption which was considered by most geometers as a rigorous truth to be taken for granted, may turn to be wrong before we go over the border or our visible world" [1, p. 209].

It is very interesting that it is possible to obtain the solved problem in actual tensor theory of gravitation [6] choosing the background connection in the form of the Kristoffel symbols for the metric

$$c^2 \, d\, t^2 \; - \; d\, \rho^2 \; - \; (k\, sh\, \frac{\rho}{k})^2 \, (d\, \theta^2 \; + \; sin^2\, \theta \;\; d\, \varphi^2)\, .$$

To the best of my belief this problem is important from the point of view of the relativistic theory of gravitation [7].

By the way, the velocity of light c plays the role of the Lobachevsky constant in the velocity space. In the invisible world of velocities the role of the distance ρ plays the rapidity s, and the role of the quantity represented by the left-hand side of (25) is taken by the velocity v so that

$$\frac{v}{c} = th\, \frac{s}{c}\, .$$

The rapidities which are much larger than the light velocity are observed in the cosmic rays and produced in the modern accelerators. Therefore, in high energy physics the Lobachevskyan geometry is essential.

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