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CHARACTER OF PHASE TRANSITION IN TWO- AND THREE-DIMENSIONAL SCALAR $g\phi^4$ THEORY

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1 Introduction

The scalar ϕ^4 theory in two- and three-dimensions was intensively investigated [1-9] as a simple, but nontrivial example, on which the problem of spontaneous symmetry breaking or, in other words, the phase structure of quantum field models is studied. It has been found [1] that high order quantum corrections can give rise to the instability of classical symmetric vacuum.There are two phases in this system and phase transition phenomena take place at certain coupling strengths. The problem is to determine which order phase transition occurs here.

This problem in two-dimensions was studied in a series of publications [2-9]. There exist mathematical theorems [2,3] proving that a second-order phase transition takes place. On the other hand, nonperturbative Gaussian approximation [4,5] and canonical transformation scheme [6] predict the existence of a first-order one. However, post-Gaussian approximation [7] and non-Gaussian variational approach [8] reveal the existence of a second-order phase transition at certain coupling constant.

To our mind, these results are doubtful because first of all, attempts have been made in these works to investigate the behaviour of the theory in the critical region where the coupling strength is considerable and any reliable methods of calculations do not exist. Therefore, we assume that the question under consideration is still open.

In this paper, we study this problem utilizing the method of Effective Potential. The absolute minimum of the Effective Potential $V(\varphi_o)$ at the point $\varphi_o = \varphi_c$ determines the true ground state (vacuum) of the theory. If a phase transition takes place at certain coupling $g = g_c$, then for $g < g_c$ the system is still in the original symmetry unbroken phase with $\varphi_c = 0$. At reaching $g = g_c$ the origin $\varphi_o = 0$ is not more the absolute minimum of $V(\varphi_o)$ and the system goes to the new lowest energetical state with $\varphi_c \neq 0$. The first-order phase transition means that the point $\varphi_o = 0$ remains to be a local, but not absolute minimum of $V(\varphi_o)$. In other words, the first derivative of $V(\varphi_o)$ is zero and the second one is positive at the origin $\varphi_o = 0$. In the case of the second-order transition, the point $\varphi_o = 0$ is a local maximum of Effective Potential at $g = g_c$. The second derivative of $V(\varphi_o)$ at $\varphi_o = 0$ becomes negative. Thus, the coefficient $\alpha(g)$ in the representation of $V(\varphi_o)$ for small φ_o

$$V(\varphi_o) = E(g) + \alpha(g) \cdot {\varphi_o}^2 + O(\varphi_o^4) \tag{1.1}$$

plays an important role in determination of the character of a phase transition. If $\alpha(g)$ is zero at certain $g = g_c$ and negative for $g > g_c$ up to $g \to \infty$, one can say that a second-order phase transition appears here. On the contrary, the positiveness of $\alpha(g)$ for any g excludes the second-order transition. Rigorous calculation of $\alpha(g)$ at an arbitrary coupling constant is a complicated problem. However, we know that at large g, the coefficient $\alpha(g)$ remains to be negative in case of a second-order phase transition and is positive if the transition is of a first-order.

Our idea is to investigate $\alpha(g)$ as $g \to \infty$ using the method [10] proposed for the calculation of functional integrals in the strong coupling regime. This method has recently been successfully applied to some problems [11,12] of quantum theory. Utilizing this to the problem under consideration we have found that

$$\alpha(G) = 3 \cdot G \cdot ln(G) + O(G) \tag{1.2}$$

in two-dimensions and

$$\alpha(G) = \frac{3}{2} \cdot G^2 \cdot ln(G) + O(\frac{G^2}{\sqrt{lnG}}) \tag{1.3}$$

for three-dimensional case as the dimensionless coupling constant $G = g/2\pi m^{4-n}$ tends to infinity. These results can be accepted as arguments in favour of the existence of the first-order phase transitions in scalar ϕ_2^4 and ϕ_3^4 models since it excludes the second-order ones.

2 Corrections to GEP

We consider the $g\phi^4$ scalar field model in two- and three-dimensions. This theory contains ultraviolet divergences, but it is super-renormalizable, i.e., it has only a finite number of the divergent Feynman diagrams. In order to remove these divergences we should introduce in the Lagrangian appropriate counter-terms. In this section we consider the super-renormalized scalar field theory with the Lagrangian:

$$\mathcal{L} = \frac{1}{2}\varphi(\mathbf{x})\cdot[\partial^2 - m^2]\cdot\varphi(\mathbf{x}) - \frac{g}{4}\cdot N_m\{\varphi^4(\mathbf{x})\} - R_m\,, \tag{2.1}$$

where

$$N_m\{\varphi^4(\mathbf{x})\} \equiv \varphi^4(\mathbf{x}) - 6\cdot\varphi^2(\mathbf{x})\cdot D_m(0) + 3\cdot D_m^2(0);$$
$$D_m(\mathbf{x}) \equiv \int \frac{d\mathbf{k}}{(2\pi)^n}\frac{\exp\{i\mathbf{k}\mathbf{x}\}}{m^2+\mathbf{k}^2}. \tag{2.2}$$

Here $\mathbf{x} \in \Omega$, Ω is a finite volume in $\mathbf{R}^n$ (n=2,3) and m and g are the mass and self-coupling constant, respectively.In two-dimensions ($n = 2$) all divergences are only of the "tadpole"-type and are readily removed by introducing in (2.1) the normal product N_m of fields $\varphi(\mathbf{x})$. In this case $R_m = 0$. In the three-dimensional theory there arise additional divergences which are cancelled by counter-terms [12,13]

$$R_m \equiv \frac{1}{2}A_m\,N_m\{\varphi^2(\mathbf{x})\} + \delta E_m\,, \tag{2.3}$$

where

$$A_m \equiv 6\,g^2 \cdot \int_\Omega d^3\mathrm{x}\, D_m^3(\mathrm{x}),$$

$$\delta E_m \equiv \frac{3}{4} g^2 \cdot \int_\Omega d^3\mathrm{x}\, D_m^4(\mathrm{x}) - \frac{3}{2} g^3 \cdot \int \frac{d^3\mathrm{k}}{(2\pi)^3} \left\{ \int_\Omega d^3\mathrm{x}\, e^{i\mathrm{kx}} \cdot D_m^2(\mathrm{x}) \right\}^3 . \tag{2.4}$$

The Effective Potential is defined as

$$V(\varphi_o) = -\lim_{\Omega\to\infty} \frac{1}{\Omega} \ln I_\Omega(\varphi_o),$$

$$I_\Omega(\varphi_o) \equiv C_m \int \delta\varphi \cdot \delta\{\varphi_o - \frac{1}{\Omega}\int_\Omega d^n\mathrm{x}\varphi(\mathrm{x})\} \exp \int_\Omega d^n\mathrm{x} \cdot \mathcal{L}[\varphi(\mathrm{x})], \tag{2.5}$$

$$C_m \equiv \sqrt{det\{-\partial^2 + m^2\}}.$$

All integrations are performed in the Euclidean metrics.

According to the method [10] we introduce a transformation of field variable:

$$\varphi(\mathrm{x}) = \phi_o + b(\mathrm{x}) + \phi(\mathrm{x}), \tag{2.6}$$

where the new field variable $\phi(\mathrm{x})$ corresponding to the new mass μ and a function $b(\mathrm{x})$ satisfy the conditions:

$$\int_\Omega d^n\mathrm{x}\, \phi(\mathrm{x}) = 0, \quad \int_\Omega d^n\mathrm{x}\, b(\mathrm{x}) = 0, \quad \text{and} \quad b^2(\mathrm{x}) = b^2. \tag{2.7}$$

Let us go over to the normal ordering in the new fields $\phi(\mathrm{x})$ using the well-known [1] formula as

$$N_m\left\{\exp\{\beta\varphi(\mathrm{x})\}\right\} = N_\mu\left\{\exp\{\beta(\phi_o + b(\mathrm{x}) + \phi(\mathrm{x})) + \frac{\beta^2}{2}\Delta(m,\mu)\}\right\},$$

$$\Delta(m,\mu) \equiv D_m(0) - D_\mu(0), \tag{2.8}$$

$$D_\mu(\mathrm{x}) \equiv \int \frac{d\mathrm{k}}{(2\pi)^n} \frac{\exp\{i\mathrm{kx}\}}{\mu^2 + \mathrm{k}^2} - \frac{1}{\mu^2\Omega}.$$

Substituting (2.6) and (2.8) into (2.5) and performing integration over $d\phi_o$ we obtain

$$I_\Omega(\varphi_o) = e^{-\Omega V_c(\varphi_o)} \cdot \int d\sigma_\mu \cdot \exp\left\{ \int_\Omega d^n\mathrm{x} N_\mu \left\{ L_1 + L_2 - \right.\right.$$

$$\frac{g}{4}\left[\phi^4(\mathrm{x}) + 4\phi^3(\mathrm{x})(\varphi_o + b(\mathrm{x})) + 12\varphi_o b(\mathrm{x})\phi^2(\mathrm{x})\right] -$$

$$\left.\left.\left[\frac{1}{2}A_\mu\phi^2(\mathrm{x}) + A_\mu b(\mathrm{x})\phi(\mathrm{x}) + \delta E_\mu + \frac{1}{2}(b^2 + \varphi_o^2) A_\mu\right]\right\}\right\},$$

$$\int d\sigma_\mu \equiv C_\mu \cdot \int \delta\phi \cdot \exp\left\{ -\frac{1}{2} \int_\Omega d^n\mathrm{x}\phi(\mathrm{x})(-\partial^2 + \mu^2)\phi(\mathrm{x}) \right\} = 1, \tag{2.9}$$

where the new counter-terms concentrated in the second square brackets in (2.9) coincide with (2.4) if we substitute $m \to \mu$. The "cactus"-type part $V_c(\varphi_o)$ of the effective potential is:

$$V_c(\varphi_o) = -\frac{1}{2}\int \frac{d\mathrm{k}}{(2\pi)^n}\left[\ln(1 + \frac{m^2-\mu^2}{\mu^2+\mathrm{k}^2}) - \frac{m^2-\mu^2}{\mu^2+\mathrm{k}^2}\right] + \frac{m^2}{2}(\varphi_o^2 + b^2) +$$

$$\frac{g}{4}(\varphi_o^4 + 6\varphi_o^2 b^2 + b^4 - 6\Delta(\varphi_o^2 + b^2) + 3\Delta^2) + \tag{2.10}$$

$$\frac{\varphi_o^2 + b^2}{2}(A_m - A_\mu) + (\delta E_m - \delta E_\mu - \frac{1}{2}A_m\Delta).$$

The linear and quadratic field configurations in (2.9) are concentrated in

$$L_1 = b(\mathrm{x}) \cdot [-m^2 + 3g(\Delta - \varphi_o^2) - g b^2 - A_m + A_\mu] \cdot N_\mu\{\phi(\mathrm{x})\},$$

$$L_2 = \frac{1}{2} \cdot [\mu^2 - m^2 + 3g(\Delta - \varphi_o^2 - b^2) - A_m + A_\mu] \cdot N_\mu\{\phi^2(\mathrm{x})\}. \tag{2.11}$$

We emphasize again that the specific counter-terms A_m, A_μ, δE_m and δE_μ, defined by Eq.(2.4) arise only in three-dimensions and these are absent in the two-dimensional theory.

According to our method, we require that the linear term $N_\mu\{\phi\}$ must not arise in the interaction and the quadratical field configurations be concentrated in the Gaussian measure $d\sigma_\mu$. The requirement leads to the following constraint equations for the parameters $b(\mathrm{x})$ and μ:

$$\begin{aligned} b(\mathrm{x})\cdot[-m^2+3g(\Delta-\varphi_o^2)-gb^2-A_m+A_\mu] &= 0\,, \\ \mu^2-m^2+3g(\Delta-\varphi_o^2-b^2)-A_m+A_\mu &= 0\,. \end{aligned} \tag{2.12}$$

Thus, we finally obtain the formula for the effective potential

$$\begin{aligned} V(\varphi_o) &= V_c(\varphi_o)+V_{sc}(\varphi_o)\,, \\ V_{sc}(\varphi_o) &= -\lim_{\Omega\to\infty}\frac{1}{\Omega}\ln J_\Omega(\varphi_o)\,, \end{aligned} \tag{2.13}$$

where,

$$J_\Omega(\varphi_o)\equiv e^{-\Omega V_c(\varphi_o)}=\int d\sigma_\mu\cdot\exp\left\{\int_\Omega d^n\mathrm{x}\right.$$

$$N_\mu\left\{-\frac{g}{4}\left[\phi^4(\mathrm{x})+4\phi^3(\mathrm{x})(\varphi_o+b(\mathrm{x}))+12\varphi_o b(\mathrm{x})\phi^2(\mathrm{x})\right]-\right.$$

$$\left.\left.\left[\frac{1}{2}A_\mu\phi^2(\mathrm{x})+A_\mu b(\mathrm{x})\,\phi(\mathrm{x})+\delta E_\mu+\frac{1}{2}(b^2+\varphi_o^2)\,A_\mu\right]\right\}\right\}, \tag{2.14}$$

Eqs.(2.10) and (2.12)-(2.14) define completely the effective potential at arbitrary coupling g.

3 Strong Coupling Regime

Our idea is to calculate the coefficient $\alpha(G)$ in (1.1) at strong self-interaction constant. It will be convenient to work in units of m dealing with numerical results. We define

$$\xi\equiv(\mu/m)^{4-n}\,,\quad \Phi_o{}^2\equiv 4\pi m^{2-n}\varphi_o^2\quad\text{and}\quad B^2\equiv 4\pi m^{2-n}b^2\,. \tag{3.1}$$

A. Two-dimensions

In two-dimensions the counter-terms A_m, A_μ, δE_m and δE_μ are zero and

$$\Delta=\frac{1}{4\pi}\ln\frac{\mu^2}{m^2}+\frac{1}{\mu^2\Omega}\;\xrightarrow{\Omega\to\infty}\;\frac{1}{4\pi}\ln\xi\,. \tag{3.2}$$

Let us consider the nontrivial solutions b^2 and μ in (2.12), obeying:

$$\begin{aligned} &\xi+2-3G\left\{\,ln\xi-\Phi_o{}^2\right\}=0\,, \\ &\xi-1+\tfrac{3}{2}G\left\{\,ln\xi-\Phi_o{}^2-B^2\right\}=0\,, \end{aligned} \tag{3.3}$$

The first of Eqs.(3.3) has a solution for $G>G_o=1.4397$ if $\Phi_o{}^2=0$. In the strong coupling regime the solutions of (3.3) are

$$\begin{aligned} \xi &= 3\,G\,lnG+O(G\,lnlnG)\,, \\ B^2 &= 3\,lnG+O(lnlnG)\,. \end{aligned} \tag{3.4}$$

It means that the effective coupling constant

$$G_{eff}\equiv\frac{g}{2\pi\mu^2}=\frac{G}{\xi}=\frac{1}{3\,lnG}\left\{1+O\!\left(\frac{lnlnG}{lnG}\right)\right\} \tag{3.5}$$

becomes small as $G\to\infty$ and one can successfully develop perturbation expansion in G_{eff} series for the functional integral (2.14):

$$V_{sc}(\varphi_o)=\sum_{n=1}^{\infty}G_{eff}^n\cdot V_{sc}^{(n)}(\varphi_o)\,. \tag{3.6}$$

Taking into account the "cactus"-type potential

$$V_c(\Phi_o) = \frac{m^3}{8\pi}\left\{E_c(G) + \frac{3G}{2}(G\,ln\xi - \xi)\,\Phi_o{}^2 + O(\Phi_o{}^4)\right\}, \quad (3.18)$$

we finally obtain the effective potential

$$V(\Phi_o) \equiv V_c(\Phi_o) + V_{sc}(\Phi_o) = \frac{m^3}{8\pi}\left\{E(G) + \alpha(G)\cdot\Phi_o{}^2 + O(\Phi_o{}^4)\right\}, \quad (3.19)$$

where the desired coefficient

$$\alpha(G) = \frac{3G^2}{2}\,lnG\cdot\left\{1 + \frac{\sqrt{96}C_1}{(lnG)^{3/2}} + O\left(\frac{1}{(lnG)^{5/2}}\right)\right\}, \quad (3.20)$$

is positive.

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