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NONLINEAR REALIZATIONS OF W_3 SYMMETRY

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1 Introduction

Nonlinear W_N algebras, since their discovery in quantum conformal field theory by A.Zamolodchikov [1], received much attention. Recently, it has been found that these algebras have interesting implications at the classical level as the symmetry algebras of some field-theory models: the systems of free $d = 2$ bosonic fields [2, 3], Toda lattices [4], the supersymmetric extensions of the latter, etc. The gauging of these classical W_N symmetries and their various linear limiting cases, such as W_∞ and w_∞, has been performed[2, 5] and some steps towards understanding their geometric origin have been done [6]. For a deeper insight into the structure of W-gravities, W-strings and related theories (both already known and yet to be constructed) it seems very urgent to fully reveal, from different points of view, the geometry behind W symmetries.

The most natural way to understand the geometry of some symmetry group G is to consider it as a group of transformations acting in the coset space G/H with a properly chosen stability subgroup H. This is a starting point of the famous nonlinear realization method [7]. In this method, the coset space coordinates (or, at least, a part of them) are identified with the goldstone fields, thus giving a clear geometric meaning to these fields. The group transformations of the latter acquire a transparent geometric interpretation as isometries of the coset manifold (linear or nonlinear, depending on whether the corresponding generators belong to the stability subgroup or to the coset). Defining the relevant left-invariant Cartan forms, one may construct out of them all the tensor object characterizing the intrinsic geometry of given group: curvatures, torsions, complex structures, etc. The invariant actions are also built from these basic covariant quantities.

Nonlinear realizations defined in accordance with the prescriptions of [7] work nicely in the case of finite-dimensional groups G and H where they have been successfully used, e.g., for constructing nonlinear sigma models. Their applications to infinite-dimensional symmetries started with the articles of V.Ogievetsky and one of the present authors (E.I.) [8] in which ordinary gauge theories were interpreted as nonlinear realizations of gauge groups viewed as some abstract groups with an infinite number of generators. In the papers of two of us (E.I. & S.K) [9] a nonlinear realization of Virasoro symmetry (to be more precise, of its centerless contact subsymmetry) has been constructed for the first time. Later we treated along the same line various super-Virasoro symmetries [10]. Recently, the nonlinear realization techniques were applied for the geometric understanding of field realizations of W_∞ and w_∞ symmetries [3, 11].

In the present paper we suggest a new general geometric set-up for classical W_N symmetries, based on their nonlinear realizations. We address here the simplest case $N = 3$, however, no principal difficulties are seen in extending our approach to other algebras and superalgebras of this kind. The basic trick allowing us to apply the standard nonlinear realization scheme to the nonlinear algebras of the type W_N consists in replacing them by some abstract infinite-dimensional linear W_∞ type algebras (W_N^∞ in what follows) which arise if one treats as independent all the composite higher spin generators appearing in the commutators of the basic W_N generators. We construct a nonlinear realization of W_3^∞ symmetry (to be more precise, of its two commuting $d = 2$ light-cone + and − copies) in its certain infinite-dimensional coset space and demonstrate that after imposing the inverse Higgs effect [8] covariant constraints we are left with two essential goldstone fields on which the original symmetry is realized precisely as W_3 symmetry on the sl_3 Toda lattice fields. Thus the Toda realization of W_3 is recovered in a pure geometric way as a particular coset realization of some linear symmetry with an infinite number of higher spin generators. The sl_3 Toda lattice equations get an interesting geometric interpretation in this approach as the conditions which single out (together with the inverse Higgs effect constraints) a two-dimensional fully geodesic subspace in the original infinite-dimensional coset space. This subspace is intimately related to the group $SL(3,R)$, being a special coset manifold of the latter. The corresponding algebra $sl(3,R)$ is hidden in W_3 as a factor algebra of one of its infinite-dimensional subalgebras. A zero-curvature representation for the sl_3 Toda lattice equations on this $sl(3,R)$ algebra naturally emerges in the approach proposed.

2 Preliminaries: nonlinear realizations of W_2 symmetry

Having in mind that the present paper actually promotes the results of [9] to the case of W_3, it is useful to review here in brief main features of nonlinear realizations of Virasoro (W_2) symmetry.

We considered in [9] two mutually commuting copies of an infinite-dimensional group of transformations generated by the truncated centerless set of $d=2$ Virasoro generators[1]

$$L_{-1},\ L_0,\ L_1,\ L_2,\ ...L_n,\ ... \tag{2.1}$$

We have identified the $d=2$ Minkiowski light-cone coordinates x^+, x^- with the parameters of the coset of this group over its infinite-dimensional subgroup H generated by two copies of the generator sets $\{L_0, L_1, \ldots L_n, \ldots\}$ and found that the left action of the original group on this coset yields just the standard $d=2$ conformal coordinate transformations (with the parameters nonsingular at $x^\pm = 0$)

$$\delta x^{\pm} = \lambda^{\pm}(x^{\pm}) = \sum_{n=-1}^{+\infty} \lambda_n^{\pm}(x^{\pm})^{n+1}. \tag{2.2}$$

Thus these transformations appear in a pure geometric way as left shifts on a two-parametric coset of $d=2$ conformal group. Further, one may extend the set of the coset generators by including into it the sum of generators L_0 from two commuting copies (the difference of these generators should be placed into the stability subgroup as it generates linear $d=2$ Lorentz $SO(1,1)$ transformations). If we treat the corresponding coset parameter as a $d=2$ goldstone field, $u(x^+,x^-)$, we immediately obtain that the left action of the conformal group on this new coset manifold induces for $u(x^+,x^-)$ the following transformations

$$\delta u(x) = -\lambda^+\partial_+ u(x) - \lambda^-\partial_- u(x) + \frac{1}{2}\left(\partial_+\lambda^+ + \partial_-\lambda^-\right), \tag{2.3}$$

where the first two pieces are due to the coordinate transformations (2.2). This transformation law is recognized as that of $d=2$ dilaton (or, in other words, of $d=2$ Liouville field). So, the conformal transformation of this field also has a clear geometric interpretation within the nonlinear realization in question as a coordinate transformation in an extended coset manifold.

Some subtleties come out when one tries to construct the Cartan forms on these cosets. Because of the specific structure of the commutation relations of Virasoro algebra the standard

[1]Later on, the relevance of this subgroup for the classical Virasoro $d=2$ gravity has been pointed out by K.Schoutens and A.Sevrin and J.van Holten [12].

procedure of [7] in this case fails to give reasonable covariant objects[2] and one is led to seek for a way out. Paradoxically, the simplest and most suggestive decision is to add to the above three coset generators an infinite set of the remaining ones, leaving in H only the $d=2$ Lorentz generator [9]. The transformation properties of $x^\pm$, $u(x)$ do not change and at the same time we become able to define an infinite set of well-behaved covariant Cartan forms on the $d=2$ conformal group, entirely following the procedure of [7]. As a cost, we gain an infinite number of new goldstone fields. These, however, are redundant, in the sense that all can be eliminated in terms of $u(x)$ and its derivatives by putting certain projections of the relevant Cartan forms equal to zero. This manifestly covariant procedure of getting rid of the superfluous goldstone fields is called inverse Higgs effect [8] and it is widely applied now in nonlinear realizations of space-time symmetries. In [9] we have found that in the Virasoro case this effect can be extended so as to yield a dynamics for the essential coset parameter $u(x)$: by imposing some additional covariant constraints on the Cartan forms we were able to obtain for $u(x)$ either free or Liouville equations

$$\partial_+\partial_- u(x) = 0 \quad \text{or} \quad \partial_+\partial_- u(x) = m^2 e^{-2u(x)} \quad . \tag{2.4}$$

which so turned out to be intimately related to the intrinsic geometry of classical Virasoro symmetry. These equations, together with the kinematic Higgs effect constraints, have been shown to play a role of the conditions singling out certain fully geodesic finite-dimensional subspaces ($E_2/SO(1,1)$ and $SO(1,2)/SO(1,1)$) in the infinite-dimensional coset manifold we started with.

Classical W_3 symmetry is a nonlinear extension of Virasoro symmetry by a new infinite set of the spin 3 generators and it has a natural realization on two $d=2$ fields described either by the free action or by that of sl_3 Toda lattice [2, 4]. With this in mind, we may guess that, having somehow generalized the above scheme to the W_3 case, we might rederive these theories and the corresponding realizations of W_3 in a pure geometric way. This is indeed so and our further incentive here will be to prove this. It is useful first to recall some basic facts about sl_3 Toda lattice and the corresponding classical realization of W_3.

3 sl_3 Toda lattice and its W_3 invariance

Let us begin by writing down the action of sl_3 Toda lattice

$$S = \gamma^2 \int dx^+ dx^- \left(\frac{1}{2}\partial_- u\partial_+ u + \frac{1}{2}\partial_-\phi\partial_+\phi + \frac{m_1^2}{8}e^{-2(u+\sqrt{3}\phi)} + \frac{m_2^2}{8}e^{-2(u-\sqrt{3}\phi)}\right) \tag{3.1}$$

and the corresponding equations of motion for two scalar fields $u(x^+,x^-)$, $\phi(x^+,x^-)$

$$\begin{aligned} \partial_+\partial_- u &= -\frac{m_1^2}{4}e^{-2(u+\sqrt{3}\phi)} - \frac{m_2^2}{4}e^{-2(u-\sqrt{3}\phi)} \\ \partial_+\partial_-\phi &= -\frac{\sqrt{3}m_1^2}{4}e^{-2(u+\sqrt{3}\phi)} + \frac{\sqrt{3}m_2^2}{4}e^{-2(u-\sqrt{3}\phi)} \end{aligned} \tag{3.2}$$

[2]As was recently noticed [11], analogous difficulties appear when considering finite-dimensional coset spaces of the W_∞ type symmetries. Moreover, these are present already at the elementary level of nonlinear realization of the $4D$ conformal group $SO(4,2)$ in Minkowski space regarded as a coset space of $SO(4,2)$.

As is shown in [4] , the system (3.2) is invariant under conformal transformations generated by the sress-tensor $T^{(-2)}$ [3]

$$\gamma^{-2}T^{(-2)} = -\frac{1}{2}(\partial u)^2 - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}\partial^2 u \tag{3.3}$$

and under the transformations, generated by the spin 3 current $J^{(-3)}$

$$\gamma^{-2}J^{(-3)} = \frac{1}{4}\partial^3\phi + \frac{3}{2}\partial^2\phi\partial u + \frac{1}{2}(\partial u)^2\partial\phi + 2(\partial u)^2\partial\phi - \frac{2}{3}(\partial\phi)^3 \quad . \tag{3.4}$$

The conformal transformation of $u(x)$ is given by eq.(2.3) while $\phi(x)$ is transformed only due to conformal shifts of its arguments

$$\delta\phi(x) = -\lambda^+\partial_+\phi(x) - \lambda^-\partial_-\phi(x) \quad . \tag{3.5}$$

The spin 3 transformations are a bit more complicated

$$\begin{aligned} \delta u &= -\frac{1}{2}a'\partial\phi + a\left(\partial^2\phi + 4\partial u\partial\phi\right) \\ \delta\phi &= \frac{1}{4}a'' - \frac{3}{2}a'\partial u + a\left(2(\partial u)^2 - 2(\partial\phi)^2 - \partial^2 u\right) \quad . \end{aligned} \tag{3.6}$$

The Lie bracket of two such transformations yields a conformal transformation and a new spin 4 transformation which looks as the conformal one with field-dependent parameters

$$[\delta_{a_1}, \delta_{a_2}] = \delta_\lambda$$

$$\lambda = a_1'''a_2 - \frac{3}{2}a_1''a_2' + 8a_1a_2'\left((\partial u)^2 + (\partial\phi)^2 + \partial^2 u\right) - (1 \leftrightarrow 2) \quad . \tag{3.7}$$

This group structure is typical for W_3 symmetry [1]. Note an important distinction of (3.7) from the Lie bracket structure of what is called classical W_3 symmetry in [5]. It consists in the presence of ordinary $d = 2$ conformal transformations on the right-hand side of (3.7). As we shall see, the nonlinear realization of just this type of W_3 leads to a nontrivial output.

In contrast to (2.3), (3.5), the W_3 transformations (3.6) contain higher derivatives on the fields, so they cannot be immediately interpreted as resulting from some coset manifold realization of W_3. Moreover, in view of nonlinear character of W_3 algebra, one may wonder how the very notion of such realizations, well defined in the case of symmetries with ordinary, linear algebras, could be extended to the present case. In order to answer all these questions, let us inspect in more detail the structure of W_3.

4 From W_3 to W_3^∞

The most general classical nonlinear W_3 algebra is the classical version of Zamolodchikov's W_3 [13]

$$\begin{aligned} T(z)T(x) &= \frac{\frac{c}{2}}{(z-x)^4} + \frac{2T(x)}{(z-x)^2} + \frac{T'(x)}{z-x} \\ T(z)J(x) &= \frac{3J(x)}{(z-x)^2} + \frac{J'(x)}{z-x} \\ J(z)J(x) &= -\frac{\frac{5c}{2}}{(z-x)^6} - \frac{15T(x)}{(z-x)^4} - \frac{\frac{15}{2}T'(x)}{(z-x)^3} - \frac{\frac{9}{4}T''(x) + \frac{48}{c}\Lambda(x)}{(z-x)^2} - \\ &\quad -\frac{\frac{1}{2}T'''(x) + \frac{24}{c}\Lambda'(x)}{(z-x)} \quad , \end{aligned} \tag{4.1}$$

[3]From now on we omit the Lorentz indices $\{\pm\}$ in the currents, keeping in mind a full symmetry between the $\{+\}$ and $\{-\}$ branches.

where the operator $\Lambda(x)$ is defined as a square of $T(x)$:

$$\Lambda(x) = T(x)T(x) \quad . \tag{4.2}$$

The operator product expansions (4.1) amount to the following commutation relations

$$\begin{aligned} [L_n, L_m] &= (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0} \\ [L_n, J_m] &= (2n-m)J_{n+m} \\ [J_n, J_m] &= -\frac{24}{c}(n-m)\Lambda_{n+m} - \frac{n-m}{2}\left[(n+m)^2 - \frac{5}{2}nm - 4\right]L_{n+m} - \\ &\quad -\frac{c}{48}(n^2-4)(n^2-1)n\delta_{n+m,0} \quad , \end{aligned} \tag{4.3}$$

where

$$L_n = \int dx x^{n+1}T(x) \quad , \quad J_n = \int dx x^{n+2}J(x) \quad , \quad \Lambda_n = \int dx x^{n+3}\Lambda(x) = \sum_m L_{m-n}L_n . \tag{4.4}$$

The algebra generated by the sl_3 Toda lattice currents (3.3),(3.4) with respect to the Poisson brackets (between the canonical fields $\tilde{u}(x) \equiv \gamma u(x)$ and $\tilde{\phi}(x) \equiv \gamma\phi(x)$) is just (4.3) with

$$c = 3\gamma^2 \quad . \tag{4.5}$$

The fields $u(x)$, $\phi(x)$ are inert under the action of central charge, hence, so far as the group variations (2.3),(3.5),(3.6) are concerned, the algebra (4.3) is reduced to

$$\begin{aligned} [L_n, L_m] &= (n-m)L_{n+m} \quad , \quad [L_n, J_m] = (2n-m)J_{n+m} \\ [J_n, J_m] &= -\frac{24}{c}(n-m)\Lambda_{n+m} - \frac{1}{2}(n-m)\left[(n+m)^2 - \frac{5}{2}nm - 4\right]L_{n+m} \quad . \end{aligned} \tag{4.6}$$

Nonetheless, we cannot completely put the central charge equal to zero when we consider the realization of W_3 on the fields. This situation becomes more clear if we consider for example the following commutator

$$[L_n, \Lambda_m] = (3n-m)\Lambda_{n+m} + \frac{c}{6}(n^3 - n)L_{n+m} \quad . \tag{4.7}$$

It is easy to check that the last term in the right hand side of (4.7) does not drop from the algebra and so the presence of the central charge in the W_3 algebra (4.3) is crucial for our consideration. In other words, we may put central charge equal to zero only when it acts directly on the fields (because it does not produce any symmetry), however should keep it in the places where it enters as a structure constant, i.e. the commutators involving the spin 4 generator Λ_n (4.7) and the higher spin composite generators.

Let us forget for a moment about any relation of the algebra (4.3) to Toda lattice and regard it as some abstract nonlinear algebra completely defined by the commutation relations (4.3) and the bilinear relation (4.4).

As was already mentioned, the nonlinear realization techniques we intend to apply to the W_3 symmetry have been worked out in [7] for symmetries based on Lie algebras, i.e. linear algebras. How to generalize these techniques to symmetries with nonlinear algebras? Our proposal is to treat all the composite higher-spin generators appearing in the enveloping algebra of (4.3) as independent ones. In other words, one replaces (4.3) by some linear infinite-dimensional higher-spin algebra W_3^∞

$$W_3^\infty = \{L_n,\ J_n,\ \Lambda_n,\ ...J_n^S,\ ...\}\ , \quad S = 5,6,... \tag{4.8}$$

in which the commutation relations between generators of the lowest spins (2 and 3) are given by (4.3) and all the remaining relations involving the higher-spin generators $\Lambda_n, ...J_n^S, ...$ are computed proceeding from these basic relations and the quadratic relation (4.4). In principle, any commutator can be computed in this way and a generic form of higher-order commutators can be indicated. For our purpose it is of no need to know the detailed structure of these commutators. We only note that the central charge c appears in the r.h.s. of these commutators as a structure constant multiplying the lower spin generators.

Surprisingly, in spite of seemingly complicated structure of such a huge algebra, it is rather easy to single out its some important subalgebras. Of major relevance for our purpose is the following infinite-dimensional subalgebra which is the genuine generalization of the truncated Virasoro algebra (2.1)

$$\begin{array}{cccccccc} & & L_{-1} & L_0 & L_1 & L_2 & \dots \\ & J_{-2} & J_{-1} & J_0 & J_1 & J_2 & \dots \\ \Lambda_{-3} & \Lambda_{-2} & \Lambda_{-1} & \Lambda_0 & \Lambda_1 & \Lambda_2 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{array} \tag{4.9}$$

We call it $\tilde W_3^\infty$. Like in the case of (2.1), the central charge drops from those commutation relations of W_3^∞ in which it is present on its own. However, it retains in the higher-spin commutation relations as the structure constant. To avoid a misunderstanding, let us point out that the higher-spin generators in (4.9), when treated as composite, still belong to the enveloping algebra of the whole W_3 which involves the generators with all negative and positive conformal dimensions. Nevertheless, it is a simple exercise to verify that the generators (4.9) indeed form a closed set.

As opposed to the algebra (2.1) which contains the nontrivial finite-dimensional subalgebra $sl(2,R) = \{L_{-1},\ L_0,\ L_1\}$, no finite-dimensional subalgebras with more than one J generator exist in (4.9). A curious fact which, to our knowledge, was never mentioned in literature, is as follows. Let us consider the infinite-dimensional "triangular"subalgebra $W_\wedge$ in $\tilde W_3^\infty$

$$W_\wedge = \left\{ \begin{array}{ccccccc} & & L_{-1} & L_0 & L_1 & & \\ & J_{-2} & J_{-1} & J_0 & J_1 & J_2 & \\ \Lambda_{-3} & \Lambda_{-2} & \Lambda_{-1} & \Lambda_0 & \Lambda_1 & \Lambda_2 & \Lambda_3 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{array} \right\} \tag{4.10}$$

(dots mean higher-spin generators with proper indices) which is a sort of the so called wedge algebra [14]. Then it is straightforward to check that the factor algebra of (4.10) over an infinite-dimensional ideal including all the generators in (4.10) except for those present in the first two lines is the algebra $sl(3,R)$

$$W_\wedge/\{\Lambda_{-3},\dots,\Lambda_3,\dots J_{-4}^5\dots\} \sim sl(3,R)\ . \tag{4.11}$$

Finally, we would like to note that the version of classical W_3 symmetry treated in [5] can be obtained from (4.3) by a kind of contraction with c as the contraction parameter. Indeed, let us rescale the generators J_n as $J_n = c^{-1/2}\tilde J_n$ and then let c go to zero. It is easy to see that no singularities in c appear at any step and (4.3) go over in this limit to

$$\begin{aligned} [L_n, L_m] &= (n-m)L_{n+m}\ , \quad [L_n, J_m] = (2n-m)J_{n+m}\ , \\ [J_n, J_m] &= -\frac{1}{2}(n-m)\Lambda_{n+m}\ . \end{aligned} \tag{4.12}$$

This algebra seems to be not too interesting for constructing a nonlinear realization because all its generators, beginning with those of spin 3, and the higher spin ones (treated in the above spirit, as independent) form an infinite-dimensional ideal in it. The corresponding factor-algebra is just the centerless Virasoro algebra (2.1). Respectively, the aforementioned finite-dimensional factor-algebra $sl(3,R)$ degenerates in the limit $c = 0$ into its contraction of the type discussed in [5].

5 Nonlinear realizations of W_3^∞

Having replaced W_3 by a linear algebra W_3^∞, we are ready to construct a nonlinear realization of the latter along the lines of [9], following the generic prescriptions of [7] and employing the inverse Higgs effect [8].

By reasonings of minimality and for the correspondence with the nonlinear realization of Virasoro group [9] we limit ourselves to the truncated algebra $\tilde W_3^\infty$ (4.9). Like in the Virasoro case, in order to have manifest $d = 2$ Lorentz symmetry, we start with the product of two commuting copies of the $\tilde W_3^\infty$ symmetry groups

$$G = \tilde W_{3+}^\infty \times \tilde W_{3-}^\infty\ . \tag{5.1}$$

As a next step we need to choose an appropriate coset of G, which is actually reduced to choosing the stability subgroup H. We have checked that, in contrast to the Virasoro case, no finite-dimensional cosets analogous to those discussed in Sect.2 can be found in G. So, even before constructing Cartan forms, in the present case one is forced to deal with infinite-dimensional coset manifolds. Fortunately, it is not so difficult as it could seem. For the correspondence with the Virasoro case [9] the coset should include the generators $L_{-1}^\pm$, $L_0^+ + L_0^-$, $L_1^\pm$, $...L_n^\pm$, ..., with the $d = 2$ coordinates $x^\pm$ and the Liouville field $u(x)$ being parameters corresponding to the first three generators. Also, having as a goal to eventually come to sl_3 Toda lattice, we need to reserve a place for the second Toda field $\phi(x)$ as the coset parameter. The only appropriate generators having zero conformal dimension are $J_0^\pm$, so we are led to include their linear combination into the set of the coset generators. Finally, it would be desirable to place all higher-spins generators into the stability subgroup and in what follows not to care about them.

All these wishes are met with the choice of the following two-parameter stability subgroup[4]

$$H \propto \{J^{\pm}_{-2} - \frac{m_1^2 m_2^2}{4} J_2^{\mp}, J^{+}_{-1} + \frac{\sqrt{3}}{2} L^{+}_{-1} - \frac{m_1^2}{2}\left(J_1^{-} + \frac{\sqrt{3}}{2} L_1^{-}\right), J_0^{+} - J_0^{-}, \qquad (5.2)$$

$$L_0^{+} - L_0^{-}, J^{-}_{-1} - \frac{\sqrt{3}}{2} L^{-}_{-1} - \frac{m_2^2}{2}\left(J_1^{+} - \frac{\sqrt{3}}{2} L_1^{+}\right); \Lambda^{\pm}_{-3}, \Lambda^{\pm}_{-2}, \ldots ; \text{higher spin generators}\}.$$

From the point of view of the $sl(3,R)$ factor algebra (4.11), these combinations of generators form the maximal parabolic subgroup in the diagonal $SL(3,R)$. Note also that just the above combination of $J^{\pm}_{-1}$, $L^{\pm}_{-1}$, but not each of these generators separately, forms a closed algebra with the remaining H-generators.

Now, an element of the coset space G/H can be parametrized as follows

$$g \equiv G/H = e^{x^{\pm} L^{\pm}_{-1}} e^{\psi_1^{\pm} J_1^{\pm}} e^{\xi_1^{\pm} L_1^{\pm}} e^{\psi_2^{\pm} J_2^{\pm}} \ldots e^{u(L_0^{+} + L_0^{-})} e^{\phi(J_0^{+} + J_0^{-})} . \qquad (5.3)$$

Here, in accordance with the previous reasonings, $x^{\pm}$ are the $d = 2$ Minkowski space coordinates, and the parameters-fields $u(x), \phi(x), \psi_1^{\pm}(x), \xi_1^{\pm}(x), \ldots$ constitute an infinite tower of the goldstone fields. The group G acts on the coset (5.3) from the left

$$g_0(\lambda) g(x, u, \phi, \ldots) = g(x', u', \phi', \ldots) \cdot h , \qquad (5.4)$$

where $g_0(\lambda)$ is an arbitrary element of G and h belongs to the subgroup H. The arrangement of the group factors as in (5.3) is convenient in that the transformation laws of coordinates under conformal transformations ($g_0 = \exp \sum_{n=-1}^{+\infty} \lambda_n L_n$) coincide with the ordinary ones (2.2), while the variations of $u(x)$ and $\phi(x)$ depend only on the space coordinates $x^{\pm}$, but not on the coordinates-fields:

$$\begin{aligned} \delta_\lambda u(x) &= u'(x') - u(x) = \frac{1}{2}(\partial_+ \lambda^+ + \partial_- \lambda^-) \\ \delta_\lambda \phi(x) &= \phi'(x') - \phi(x) = 0 . \end{aligned} \qquad (5.5)$$

We see that conformal transformations of these fields coincide with the Toda lattice ones (2.3), (3.5).

The J_n transformations of the Minkowski space coordinates and parameters-fields can be also deduced from the general formula (5.4). For $x^{\pm}$ and $u(x)$, $\phi(x)$ we get the following transformations (we write down here only the transformations generated by the + branch of the group G)

$$\begin{aligned} \delta_a x &= -\frac{\sqrt{3}}{2} a'(x) + 2\sqrt{3}\left(\frac{\sqrt{3}}{2}\psi_1 + \xi_1\right) a(x) \\ \delta_a u(x) &= -\frac{\sqrt{3}}{2}\left(\frac{\sqrt{3}}{2}\psi_1 + \xi_1\right) a'(x) + \left(6\psi_2 + 2\sqrt{3}\xi_1^2\right) a(x) \\ \delta_a \phi(x) &= \frac{1}{4} a''(x) - \frac{3}{2}\left(\frac{\sqrt{3}}{2}\psi_1 + \xi_1\right) a'(x) + 3\left[\left(\frac{\sqrt{3}}{2}\psi_1 + \xi_1\right)^2 - \xi_2\right] a(x) , \end{aligned} \qquad (5.6)$$

[4]The parameters m_1, m_2 reflect a freedom in extracting the diagonal $SL(3,R)$ group in $SL^{+}(3,R) \times SL^{-}(3,R)$.

where the function $a(x)$ collects constant parameters of the group element g_0:

$$g_0 = \exp \sum_{n=-2}^{+\infty} a_n J_n \quad , \quad a(x) = \sum_{n=-2}^{+\infty} a_n x^{n+2} .$$

The main peculiarity of transformations (5.6) and their crucial difference from the conformal ones is that J_n have no realizations on the coordinates $x^{\pm}$ alone – these generators necessarily mix the space coordinates with the goldstone fields $\psi_1, \psi_2, \xi_1, \xi_2$. Moreover, in fact we deal here with an infinite-dimensional nonlinear representation of $\tilde{W}_3^{\infty}$, because the fields ψ_1, ξ_1 are transformed through higher-spin fields $\psi_2, \xi_2, \psi_3, \xi_3$ and so on. It is impossible to single out in our coset space any finite-dimensional coordinate subset closed under $\tilde{W}_3^{\infty}$. We stress that at this step the transformations of goldstone fields taken at a fixed point contain in each term no more than one field derivative which is due to the field-dependent shift of x in (5.6). The same is true, of course, for higher-spin transformations which appear in the Lie brackets of (5.6). How to obtain the standard higher-derivative realization of W_3 proceeding from these pure geometric transformations? A key to this problem is provided by the inverse Higgs effect. In order to utilize it, one needs to construct covariant Cartan forms on the coset space G/H.

These are introduced by the standard relation [7]

$$g^{-1} dg = \sum_{n=-1}^{+\infty} \omega^n_{\pm} L^{\pm}_n + \sum_{n=-2}^{+\infty} \Omega^n_{\pm} J^{\pm}_n + \ldots , \qquad (5.7)$$

where dots stand for the forms entering with the higher spin generators. Let us stress that the transformation law of Cartan forms (5.7) is more complicated than in standard nonlinear realizations (including those of Virasoro group) [7], because the coset (5.3) is not ortonormal in Cartan's sense[5]. Nevertheless, for our purpose it suffices to be sure that all the forms associated with the coset generators still transform homogeneously. Let us quote several first forms in (5.7)

$$\begin{aligned} \omega^{\pm}_{-1} &= e^{-u} ch(\sqrt{3}\phi) dx^{\pm} \quad , \quad \omega_0^{\pm} = du - 2\xi_1^{\pm} dx^{\pm} \\ \omega_1^{\pm} &= e^{u}\left[\left(d\xi_1^{\pm} + \xi_1^{\pm 2} dx^{\pm} + \frac{9}{4}\psi_1^{\pm 2} dx^{\pm} - 3\xi_2^{\pm} dx^{\pm}\right) ch(\sqrt{3}\phi) + \right. \\ &\left. + \frac{\sqrt{3}}{2}\left(d\psi_1^{\pm} + 6\psi_1^{\pm}\xi_1^{\pm} dx^{\pm} - 4\psi_2^{\pm} dx^{\pm}\right) sh(\sqrt{3}\phi)\right] \\ \omega_2^{\pm} &= e^{2u}\left(d\xi_2^{\pm} + 4\xi_1^{\pm}\xi_2^{\pm} dx^{\pm} - 4\xi_3^{\pm} dx^{\pm}\right) \end{aligned} \qquad (5.8)$$

$$\begin{aligned} \Omega^{\pm}_{-2} &= 0 \quad , \quad \Omega^{\pm}_{-1} = -\frac{2}{\sqrt{3}} e^{-u} sh(\sqrt{3}\phi) dx^{\pm} \quad , \quad \Omega_0^{\pm} = d\phi - 3\psi_1^{\pm} dx^{\pm} \\ \Omega_1^{\pm} &= e^{u}\left[\frac{2}{\sqrt{3}}\left(d\xi_1^{\pm} + \xi_1^{\pm 2} dx^{\pm} + \frac{9}{4}\psi_1^{\pm 2} dx^{\pm} - 3\xi_2^{\pm} dx^{\pm}\right) sh(\sqrt{3}\phi) + \right. \\ &\left. + \left(d\psi_1^{\pm} + 6\psi_1^{\pm}\xi_1^{\pm} dx^{\pm} - 4\psi_2^{\pm} dx^{\pm}\right) ch(\sqrt{3}\phi)\right] \end{aligned} \qquad (5.9)$$

[5]Recall that the ortonormality of some coset G/H means that the generators belonging to the coset can be chosen so that, being commuted with those from the stability subgroup, they always yield themselves but not the H-generators. This is obviously not the case for our coset (5.3).

$$
\begin{aligned}
\Omega_2^{\pm} = {} & e^{2u}\Big[4\phi\left(d\xi_2^{\pm} + 4\xi_1^{\pm}\xi_2^{\pm}dx^{\pm} - 4\xi_3^{\pm}dx^{\pm}\right) + d\psi_2^{\pm} - \xi_1^{\pm}d\psi_1^{\pm} - 3\psi_1^{\pm}{\xi_1^{\pm}}^2dx^{\pm} + \\
& + \frac{3}{4}{\psi_1^{\pm}}^3dx^{\pm} + 4\xi_1^{\pm}\psi_2^{\pm}dx^{\pm} + 12\psi_1^{\pm}\xi_2^{\pm}dx^{\pm} - 5\psi_3^{\pm}dx^{\pm}\Big] \quad .
\end{aligned}
$$

We see that the fields $\psi_{1,2}^{\pm}$, $\xi_{1,2}^{\pm}$ enter into some coset space forms linearly and homogeneously. Hence, according to a general theorem of [8], they can be covariantly expressed in terms of u and ϕ by equating some projections of these forms to zero. The same is true for the higher-spin goldstone fields. Without entering into details, the complete infinite set of the inverse Higgs covariant constraints expressing all the goldstone fields through u, ϕ and derivatives of the latter is as follows

$$
\begin{aligned}
\omega_0^+ + \omega_0^- = 0 \quad , & \quad \Omega_0^+ + \Omega_0^- = 0 \quad , \\
\omega_n^+|_+ = 0 \quad , \quad \omega_n^-|_- = 0 \quad , & \quad \Omega_n^+|_+ = 0 \quad , \quad \Omega_n^-|_- = 0 \quad \text{for all } n \geq 1 \quad ,
\end{aligned}
\tag{5.10}
$$

where $|_{\pm}$ means that a given Cartan form is projected onto $dx^{\pm}$, respectively.

From the explicit expressions for the lowest Cartan forms (5.8)-(5.9) we obtain the following expressions for some lower spin parameters-fields

$$
\begin{aligned}
\xi_1^{\pm} &= \partial_{\pm}u(x) \quad , \quad \psi_1^{\pm} = \frac{2}{3}\partial_{\pm}\phi(x) \\
\xi_2^{\pm} &= \frac{1}{3}\left[\partial_{\pm}^2u(x) + (\partial_{\pm}u(x))^2 + (\partial_{\pm}\phi(x))^2\right] \\
\psi_2^{\pm} &= \frac{1}{6}\partial_{\pm}^2\phi(x) + \partial_{\pm}u(x)\partial_{\pm}\phi(x), \quad \text{etc.}
\end{aligned}
\tag{5.11}
$$

Let us stress that the constraints (5.10) are purely kinematical and do not imply equations of motion. Their role is to covariantly express all the parameters of our coset (5.3) in terms of $u(x), \phi(x)$ and derivatives of the latter.

Now we may substitute the expressions (5.11) into the transformation laws (5.6) and check that the resulting spin 3 transformations of $\tilde{W}_3^{\infty}$ precisely coincide with those of the Toda lattice fields (3.6):

$$
\begin{aligned}
\tilde{\delta}u(x) \equiv u'(x) - u(x) &= -\frac{1}{2}a'(x)\partial\phi + a(x)\left(\partial^2\phi + 4\partial u\partial\phi\right) \\
\tilde{\delta}\phi(x) \equiv \phi'(x) - \phi(x) &= \frac{1}{4}a''(x) - \frac{3}{2}a'(x)\partial u + a(x)\left(2(\partial u)^2 - 2(\partial\phi)^2 - \partial^2u\right) \quad .
\end{aligned}
\tag{5.12}
$$

Thus we have succeeded in deducing the Toda realization of nonlinear W_3 symmetry starting from a pure geometric coset realization of some linear higher-spin symmetry $\tilde{W}_3^{\infty}$. In other words, W_3 arises as a particular field realization of this huge algebra. The crucial role in this phenomenon is played by the inverse Higgs constraints reducing an infinite number of the initial goldstone fields to the two essential ones $u(x)$ and $\phi(x)$. Moreover, it turns out that the sl_3 Toda lattice equations of motion acquire a new geometric meaning within this approach. This is discussed in the next Section.

6 sl_3 Toda lattice as a nonlinear realization

So far, our fields $u(x)$ and $\phi(x)$ were not subject to any dynamical equation. To get a dynamics for these fields, we have to carry out the covariant reduction of the coset G/H.

This reduction goes as follows. Given Cartan forms (5.7) defined from the beginning over the infinite-dimensional algebra $\tilde{W}_3^{\infty}$ (4.9), one imposes on them the covariant constraint

$$
g^{-1}dg = \sum_{n=-1}^{+\infty}\omega_{\pm}^nL_n^{\pm} + \sum_{n=-2}^{+\infty}\Omega_{\pm}^nJ_n^{\pm} + \ldots = g_{red}^{-1}dg_{red} \in \tilde{\mathcal{G}} \quad ,
\tag{6.1}
$$

where $\tilde{\mathcal{G}}$ is some subalgebra in $\tilde{W}_3^{\infty}$ (4.9). Constraint (6.1) means that all Cartan's forms associated with the generators which do not belong to the subalgebra $\tilde{\mathcal{G}}$ must be put equal to zero. There is only one limitation on $\tilde{\mathcal{G}}$: to ensure covariance of the constraint (6.1) under group G, it must include the algebra of H (5.2)[6]. In the case at hand, the most general subalgebra $\tilde{\mathcal{G}}$ consists of the following generators

$$
\tilde{\mathcal{G}} \propto \begin{cases}
R^{\pm} = J_{-1}^{\pm} + \frac{\sqrt{3}}{2}L_{-1}^{\pm} - \frac{m_1^2}{2}\left(J_1^{\mp} + \frac{\sqrt{3}}{2}L_1^{\mp}\right) \\
S^{\pm} = J_{-1}^{\pm} - \frac{\sqrt{3}}{2}L_{-1}^{\pm} - \frac{m_2^2}{2}\left(J_1^{\mp} - \frac{\sqrt{3}}{2}L_1^{\mp}\right) \\
B^{\pm} = J_{-2}^{\pm} - \frac{m_1^2m_2^2}{4}J_2^{\mp} \\
U = L_0^+ - L_0^- \quad , \quad T = J_0^+ - J_0^- \\
\text{Higher spin generators.}
\end{cases}
\tag{6.2}
$$

This algebra includes the stability subgroup algebra (5.2) as a subalgebra. Let us remind that all the higher spin generators in the algebra $\tilde{\mathcal{G}}$ constitute its ideal. It is a simple task to check that the factor-algebra $\tilde{\mathcal{G}}$/(higher spin generators) is the diagonal $sl(3, R)$ with the following commutation relations:

$$
\begin{aligned}
& [B^+, B^-] = -3m_1^2m_2^2U \quad , \quad [U, B^{\pm}] = \pm 2B^{\pm} \quad , \quad [U, R^{\pm}] = \pm R^{\pm} \quad , \quad [U, S^{\pm}] = \pm S^{\pm}, \\
& [T, R^{\pm}] = \pm\sqrt{3}R^{\pm} \quad , \quad [T, S^{\pm}] = \mp\sqrt{3}S^{\pm} \quad , \quad [B^{\pm}, R^{\mp}] = -\sqrt{3}m_1^2S^{\pm}, \\
& [B^{\pm}, S^{\mp}] = -\sqrt{3}m_2^2R^{\pm} \quad , \quad [R^+, R^-] = \frac{3m_1^2}{2}U + \frac{3\sqrt{3}m_1^2}{2}T, \\
& [S^+, S^-] = \frac{3m_2^2}{2}U - \frac{3\sqrt{3}m_2^2}{2}T \quad ; \quad [R^{\pm}, S^{\pm}] = -\sqrt{3}B^{\pm}
\end{aligned}
\tag{6.3}
$$

and the parameters m_1 and m_2 have the meaning of some inverse constant curvatures.[7]

The covariant reduction constraints (6.1) read in terms of the forms $\omega_n^{\pm}$ and $\Omega_n^{\pm}$ as

$$
\begin{aligned}
\omega_0^+ + \omega_0^- &= 0 \quad , \quad \Omega_0^+ + \Omega_0^- = 0 \\
\omega_1^{\pm} &= -\frac{m_1^2 + m_2^2}{4}\omega_{-1}^{\mp} + \frac{\sqrt{3}(m_2^2 - m_1^2)}{8}\Omega_{-1}^{\mp} \\
\Omega_1^{\pm} &= \frac{m_2^2 - m_1^2}{2\sqrt{3}}\omega_{-1}^{\mp} - \frac{m_1^2 + m_2^2}{4}\Omega_{-1}^{\mp} \\
\Omega_2^{\pm} &= \frac{m_1^2m_2^2}{4}\Omega_{-2}^{\mp} \\
\omega_n^{\pm} &= 0 \quad , \quad \Omega_{n+1}^{\pm} = 0 \quad \text{for all } n \geq 2 \quad .
\end{aligned}
\tag{6.4}
$$

[6] Let us remind that the Cartan's forms belonging to the algebra of the stability subgroup H are transformed inhomogeneously under G. So, putting them equal to zero would immediately break the G invariance.

[7] We are at liberty to put m_1^2 as well m_2^2 equal to zero. This gives rise to alternative reductions, with $u(x)$ and/or $\phi(x)$ described by free actions.

Note that the set (6.4) includes our previously established inverse Higgs phenomenon constraints (5.10) as a subset, so all the parameters of our coset are expressed in terms of $u(x), \phi(x)$ by the same formulas (5.11).

It is straightforward to check that the constraints (6.4) result in an additional infinite set of equations for the coset parameters, which actually prove to be equivalent to two equations for the fields $u(x), \phi(x)$. These equations are just the equations of motion of the sl_3 Toda lattice (3.2).

It is worth noting that a zero-curvature representation for the system (3.1) on the $sl(3,R)$ algebra [15] automatically arises in this picture. Indeed, after imposing the constraints (6.4), we are left with the forms on $\tilde{\mathcal{G}}$ (6.2) involving the two fields $u(x), \phi(x)$:

$$\begin{aligned} \Omega_{red} &= \frac{1}{\sqrt{3}} e^{-u-\sqrt{3}\phi} dx^{\pm} R^{\pm} - \frac{1}{\sqrt{3}} e^{-u+\sqrt{3}\phi} dx^{\pm} S^{\pm} + (\partial_- u dx^- - \partial_+ u dx^+) U + \\ &\quad + (\partial_- \phi dx^- - \partial_+ \phi dx^+) T + \text{ higher spin generators.} \end{aligned} \tag{6.5}$$

However, as we have already mentioned in Section 3, the higher spin generators constitute an ideal in the algebra $\tilde{\mathcal{G}}$ (6.2), so the Maurer-Cartan equations for the forms

$$\begin{aligned} \Omega_{sl(3,R)} &= \frac{1}{\sqrt{3}} e^{-u-\sqrt{3}\phi} dx^{\pm} R^{\pm} - \frac{1}{\sqrt{3}} e^{-u+\sqrt{3}\phi} dx^{\pm} S^{\pm} - \\ &\quad + (\partial_- u dx^- - \partial_+ u dx^+) U + (\partial_- \phi dx^- - \partial_+ \phi dx^+) T \end{aligned} \tag{6.6}$$

are closed without any contribution from the higher-spin forms. Thus, the Maurer-Cartan equation for the group $SL(3,R)$ immediately yields the zero-curvature condition for $\Omega_{sl(3,R)}$:

$$d^{\text{ext}} \Omega_{sl(3,R)} = \Omega_{sl(3,R)} \wedge \Omega_{sl(3,R)} \tag{6.7}$$

It is a simple exercise to verify that eq.(6.7) is equivalent to the Toda lattice equations (3.1).

A few comments are needed concerning the geometric meaning of the above procedure. As was explained in [9, 16] on simple examples, the essence of the covariant reduction consists in reducing a given group space to its some lower-dimensional fully geodesic subspace, in a way covariant under the original nonlinear realization. The dynamical equations (Liouville equation [9], the equations of conformal mechanics [16], etc) come out as the basic constraints accomplishing this reduction. In the present case the relevant fully geodesic subspace, defined like in other cases as the quotient of the covariant reduction subgroup over the stability subgroup, coincides with the two-dimensional coset of the group $SL(3,R)$ over its six-parameter parabolic subgroup. In the theory of sl_3 Toda lattice this manifold is expected to play the role analogous to $SL(2,R)/SO(1,1)$ in the Liouville (sl_2 Toda lattice) theory. Recall that the sl_n Toda lattice equations can alternatively be obtained by a kind of the Hamiltonian reduction [17] from those of the $SL(N,R)$ WZNW sigma model [18]. This suggests obvious parallels between the Hamiltonian and covariant reductions. An essential difference between these two procedures lies in the fact that the covariant reduction produces dynamical systems starting from the manifolds which were not originally subject to any dynamical restrictions, while in the Hamiltonian reduction one begins with a dynamical system and then constrains it in a proper way[8]. At present, the precise correspondence between these two types of reduction is not quite clear to us.

[8]We thank A.Isaev for suggesting this argument to us.

Finally, we briefly discuss one more interesting application of the covariant reduction techniques. If we start from the group $W_3^- \times W_\wedge^+$, the relevant Cartan forms are given by the same expression (5.8),(5.9) in which one has to put $\xi_n = 0, \psi_{n+1} = 0, n = 2, 3, \ldots$. After imposing the same constraints (6.4), we again obtain the Toda lattice equations (3.1) for $u(z), \phi(x)$ but they are now accompanied by the additional constraints

$$T^{(-2)} = 0 \ , \quad J^{(-3)} = 0 \ . \tag{6.8}$$

Constraints (6.8) are compatible with the equations (3.1) but severely fix the coordinate dependence of $u(x)$ and $\phi(x)$, thus selecting a particular class of solutions of the equations (3.1). Let us remark that the Toda lattice system with the constraints (6.8) is closely related to the chiral boson theory [19].

7 Summary and comments

In this paper we have constructed, for the first time, a geometric realization of classical W_3 symmetry in some infinite-dimensional coset space and have shown an intimate relation of this realization to the sl_3 Toda lattice equations. The central point of our construction is the substitution of the nonlinear W_3 algebra by the linear algebra W_3^∞ which includes all the higher-spin composites of the spin 2 and spin 3 generators as independent generators. This allowed us to apply to the present case the powerful techniques of nonlinear realizations [7] supplemented with the inverse Higgs effect [8] and the covariant reduction method [9, 10, 16]. The Toda lattice realization of W_3 on two scalar fields $u(x)$ and $\phi(x)$ amounts to that of W_3^∞ in a coset space originally involving an infinite number of the coordinates-goldstone fields which are covariantly expressed afterwards via $u(x)$ and $\phi(x)$ by the inverse Higgs effect. The sl_3 Toda lattice equations also arise geometrically as the result of utilizing a dynamical version of this effect, covariant reduction. Their geometric role is to single out (together with the kinematic inverse Higgs constraints) a fully geodesic subspace in the initial coset space. This subspace is homeomorphic to the two-dimensional quotient of $SL(3,R)$ over its maximal parabolic subgroup. Thus, sl_3 Toda system turns out to be associated with this special manifold, in the same way as Liouville theory is associated with the coset $SL(2,R)/SO(1,1)$ [9].

Among possible generalizations and applications of our approach, we first mention extending it to other W_N symmetries (associated with the algebras $sl(N,R)$, as well as with the algebras from other Cartan's series, e.g. $so(N)$). We expect the one-to-one correspondence between nonlinear realizations of these symmetries (to be more precise, of the appropriate W_N^∞ symmetries) and the related two-dimensional Toda lattices. The intrinsic relation between Toda systems and the geometry of W_N symmetries established here could have many interesting consequences, e.g. in W gravities, matrix models, etc. The fact that the Toda lattice fields can be interpreted as parameters of some coset manifolds of W_N^∞, i.e. as a kind of generalized angular variables, raises an interesting problem of embedding them into linear representations of W_N (W_N^∞). Exploring this might shed more light on the interplay between Toda systems and the representation theory of W algebras, both on classical and quantum levels.

It is also desirable to understand in full how the linear higher-spin algebras W_N^∞ constructed by W_N are related to the algebras W_∞, w_∞, $w_{1+\infty}$. It seems that all W_N^∞'s can be obtained from W_∞ via appropriate contractions and truncations. It is worth noting that W_N^∞ with different N are by no means embedded into each other; rather, the lower N algebras are contractions and truncations of the higher N ones. For example, the regular part of W_4^∞ is expected to involve the same set of generators as in eq. (4.9), however, these generators satisfy different commutation relations and this is the point where the specificity of W_4 as compared to W_3 manifests itself. Also, in eq.(4.10), the generators L_n, J_m and Λ_k, though possessing the $SL(2,R)$ multiplet structure of $SL(4,R)$ generators, generate some contraction of $sl(4,R)$ (modulo an infinite-dimensional ideal). On the other hand, in W_4^∞ the same set of generators forms the genuine $sl(4,R)$ (once again, modulo an ideal).

The most perspective and ambitious extension of our nonlinear realization approach is to apply it to other nonlinear algebras and superalgebras, e.g., Knizhnik-Bershadsky superalgebras [20], and, perhaps, to quantum algebras. In this way we expect to obtain new integrable systems and to render a new geometric interpretation to the known ones, such as the KdV, MkdV and KP hierarchies.

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