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SUPERFIELD ACTIONS FOR N = 4 WZNW - LIOUVILLE SYSTEMS

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1 Introduction

Some time ago, a new type of supersymmetric extensions of Wess-Zumino- Novikov-Witten (WZNW) sigma-models has been constructed [1, 2]. These systems possess $SO(4) \times U(1)$ $N = 4$ $2D$ superconformal symmetry [3, 4] and stem from nonlinear realizations of the latter. Their bosonic fields parametrize the manifolds $SO(4) \times U(1) \times U(1)$ or $SU(2) \times U(1)$. The bosonic fields valued in the abelian factors (as well as the fermionic fields) are originally described by free actions. However, as was shown in [1, 2], the action of one of such fields (of the $2D$ dilaton) can be modified by a potential Liouville term accompanied by appropriate Yukawa couplings to fermions. with maintaining the underlying superconformal symmetry in the $SO(4) \times U(1) \times U(1)$ case and with reducing it to $SU(2)$ $N = 4$ superconformal symmetry in the $SU(2) \times U(1)$ case. Thus, the models proposed in [1, 2] can be regarded as N=4 superextensions of at once two conformal theories: Liouville theory and WZNW model [1]. These models are expected to have interesting applications. for example in connection with stringy instantons and solitons [6] and in the theory of noncritical strings and superstrings (see, e.g., recent paper [7]).

Keeping in mind these possible uses of the models in question, and also for coupling them in an unambiguous way to appropriate superfield 2D supergravities, it is desirable to have their manifestly supersymmetric description via N=4 2D superfield. In [1, 2], the basic constrained superfields accommodating the irreducible field contents of these models have been defined and a superfield form of the relevant equations of motion has been given (as well as the on-shell actions of these systems). Later on, these constrained superfields have been used to work out the corresponding on-shell superconformal calculus, both in classical and quantum cases [8, 9]. However, it remained an open problem how to construct complete off-shell actions in terms of unconstrainted N=4 superfields (prepotentials). Recall that the superfield actions for $N = 1$ and $N = 2$ super-Liouville models were known for a long time [10, 11].

This problem is solved in the present paper [2]. We start in sect.2 with recapitulating (in some improved form) the basics of the constrained superfield description of $SU(2) \times U(1)$ and $SO(4) \times U(1) \times U(1)$ N=4 WZNW models and their super-Liouville deformations. In sect. 3 we pass to unconstrained prepotentials and study the transformation properties of the latter under $N = 4$ superconformal group. We find two possible transformation laws of the prepotential under $SU(2)$ $N = 4$ superconformal group (SCG), corresponding to the two types of N=4 twisted scalar supermultiplet [1, 13, 14]. While in the $SU(2) \times U(1)$ $N = 4$ case it suffices to have one prepotential, in the $SO(4) \times U(1) \times U(1)$ $N = 4$ case we are led to introduce two prepotentials possessing different transformation properties with respect to a fixed $SU(2)$ $N = 4$ subalgebra of $SO(4) \times U(1)$ $N = 4$ superconformal algebra. In sect. 4 we construct the off-shell superfield actions of the $SU(2) \times U(1)$ and $SO(4) \times U(1) \times U(1)$ $N = 4$ WZNW - Liouville models and show that in components and after elimination of auxiliary fields they are reduced just to the actions given earlier [2]. The superfield Liouville term in the case of $SU(2) \times U(1)$ model is given by the Fayet-Iliopoulos term of the prepotential, while in the case of $SO(4) \times U(1) \times U(1)$ model it has a bit more complicated structure, including two independent prepotentials.

[1] In the case when no Liouville terms are present, a wider class of such N=4 superconformal WZNW models has been discovered in [5].

[2] In a recent preprint [12] an off-shell action for the $SU(2) \times U(1)$ WZNW model has been constructed in $N = 4$ extended ($N = 4$ harmonic) superspace.

2 $N = 4$ WZNW - Liouville systems: description via constrained superfields

2.1 The $SO(4) \times U(1) \times U(1)$ WZNW - Liouville superfield equations

The superfied equations describing the $SO(4) \times U(1) \times U(1)$ N=4 superconformal WZNW - Liouville sigma model of [2] read

$$D_+^i q^{\underline{j}k} + D_+^k q^{\underline{j}i} = \frac{1}{2}\delta^{ik} D_+^l q^{\underline{j}l} \tag{2.1}$$

$$D_-^{\underline{i}} q^{\underline{j}k} + D_-^{\underline{j}} q^{\underline{i}k} = \frac{1}{2}\delta^{\underline{ij}} D_-^{\underline{l}} q^{\underline{l}k} \tag{2.2}$$

$$D_-^{\underline{i}} (q^{-1} D_+^k q)^{jl} = im\,(\delta^{kl} q^{\underline{ij}} + \delta^{jl} q^{\underline{i}k} - \delta^{jk} q^{\underline{i}l} + 2\alpha\, \epsilon^{kjlr} q^{\underline{i}r}) \,. \tag{2.3}$$

Here m and α are some coupling constants, $ijkl \cdots$ are vector $SO(4)$ indices, ϵ^{ijkl} is a totally antisymmetric tensor, D_+^i, $D_-^{\underline{j}}$ are covariant spinor derivatives satisfying the anticommutation relations

$$\{D_+^i, D_+^j\} = 2i\delta^{ij}\,\partial_+, \quad \{D_-^{\underline{i}}, D_-^{\underline{j}}\} = 2i\delta^{\underline{ij}}\,\partial_-$$
$$\{D_+^i, D_-^{\underline{j}}\} = 0 \tag{2.4}$$

and ordinary and underlined indices are related to the $(+)$ and $(-)$ light-cone branches of $2D$ $N = 4$ superconformal group (these are rotated by two independent Kac-Moody groups $SO_+(4)$ and $SO_-(4)$ gauged by the coordinates x^+ and x^-, respectively). The matrix $N = 4$ superfield $q^{\underline{i}j}$ is defined as follows

$$q^{\underline{i}j} \equiv \tilde{q}^{\underline{i}j} \exp\{-u\}, \tag{2.5}$$

where u is a superdilaton and $\tilde{q}^{\underline{i}j}$ is the orthogonal matrix superfield parametrizing the coset $SO_-(4) \times SO_+(4)/SO(4)$. The equations (2.1) and (2.2) are the superfield irreducibility constraints, (2.3) is the equation of motion. As a consequence of eqs. (2.1), (2.2), $q^{\underline{i}j}$ describes off shell $16+16$ degrees of freedom [2]. After exploiting eq. (2.3), this number is reduced to $8+8$. Note that one more physical bosonic field, apart from the dilaton $u(x) = u(z)\,|$ (hereafter, slash means restriction to the θ independent parts) and six $SO_-(4) \times SO_+(4)/SO(4)$ coset parameters, appears as a solution of the differential constraint

$$\partial_- A_+ - \partial_+ A_- = 0 \Rightarrow A_\pm(x) = \partial_\pm \phi(x), \tag{2.6}$$

$$A_+(x) = \frac{1}{12}\epsilon^{ijkl} D_+^i (q^{-1} D_+^j q)^{kl}\,|, \quad A_-(x) = \frac{1}{12}\epsilon^{\underline{ijkl}} D_-^{\underline{i}} (q^{-1} D_-^{\underline{j}} q)^{\underline{kl}}\,|$$

which is a consequence of (2.1), (2.2). The additional $U(1)$ symmetry characteristic of $SO(4)\times U(1)$ $N=4$ SCG [3] comes out as a freedom in solving eq. (2.6) and it is realized as constant shifts of the field $\phi(x)$ (it is the rigid part of the whole $U(1)$ Kac-Moody symmetry).

Now, let us rewrite (2.1) - (2.3) in a more convenient form. To this end, we represent $SO_{\pm}(4)$ as direct products of two $SU(2)$'s and substitute the $SO(4)$ vector indices by the doublet $SU(2)$ ones according to the rule

$$A^i \rightarrow A^{\alpha a} = \frac{1}{\sqrt{2}} A^i (\sigma^i)^{\alpha a}$$
$$A^{\underline{i}} \rightarrow A^{\underline{\alpha a}} = \frac{1}{\sqrt{2}} A^{\underline{i}} (\sigma^{\underline{i}})^{\underline{\alpha a}}.$$

Here, the greek and latin indices refer to two different groups $SU(2)$ forming $SO_+(4)$ $(SO_-(4))$ $(SU_I(2)$ and $SU_{II}(2)$ in what follows). As was shown in [9], the matrix superfield $q^{\underline{\alpha a},\beta b}$ can be divided, without loss of generality, into the product

$$q^{\underline{\alpha a},\beta b} \equiv q_1^{\underline{\alpha}\beta}\, q_2^{\underline{a}b}, \qquad (2.7)$$

where

$$\begin{aligned} q_1^{\underline{\alpha}\beta} &= \exp\{-u_1\}\, \tilde{q}_1^{\underline{\alpha}\beta}, \quad q_2^{\underline{a}b} = \exp\{-u_2\}\, \tilde{q}_2^{\underline{a}b} \\ u_1 &= \frac{u}{2} + \frac{\phi}{4}, \quad u_2 = \frac{u}{2} - \frac{\phi}{4}, \quad \tilde{q}^{\,\underline{\alpha a},\beta b} \equiv \tilde{q}_1^{\underline{\alpha}\beta}\, \tilde{q}_2^{\underline{a}b}, \end{aligned} \qquad (2.8)$$

$\tilde{q}_1$, $\tilde{q}_2$ are 2×2 unitary (and unimodular) matrices and $\phi(x) = \phi(z)\,|$. The original constraints (2.1), (2.2) and their consequence (2.6) prove to be satisfied ($A_{\pm}(x)$ are expressed through $\phi(x)$ just according to eq. (2.6)), if the superfields q_1, q_2 obey the following constraints

$$D_+^{\alpha a} q_1^{\underline{\alpha}\beta} + D_+^{\beta a} q_1^{\underline{\alpha}\alpha} = 0, \quad D_-^{\underline{\alpha a}} q_1^{\underline{\beta}\alpha} + D_-^{\underline{\beta a}} q_1^{\underline{\alpha}\alpha} = 0 \qquad (2.9a)$$
$$D_+^{\alpha a} q_2^{\underline{a}b} + D_+^{\alpha b} q_2^{\underline{a}a} = 0, \quad D_-^{\underline{\alpha a}} q_2^{\underline{b}a} + D_-^{\underline{\alpha b}} q_2^{\underline{a}a} = 0. \qquad (2.9b)$$

Such superfields, first defined in [1] and re-discovered in [13], comprise two $8+8$ $N=4$ scalar off-shell supermultiplets called in [13] "$N=4$ twisted ". Thus the considered $N=4$ WZNW - Liouville sigma model is represented in terms of two different $N=4$ twisted scalar multiplets. Using the relation

$$D_-^{\underline{i}} D_+^{i}\, \phi = 4im\alpha\, q^{\underline{j}i}, \qquad (2.10)$$

which follows from eqs. (2.3) and (2.6), the dynamical equation (2.3) can also be rewritten via q_1, q_2

$$D_-^{\underline{\alpha a}} \left(q_1^{-1\beta}{}_{\underline{\beta}}\, D_+^{\alpha a} q_1^{\underline{\beta}\gamma} \right) = im(1-2\alpha)\, \epsilon^{\alpha\gamma} q_1^{\underline{\alpha}\beta}\, q_2^{\underline{a}a} \qquad (2.11a)$$
$$D_-^{\underline{\alpha a}} \left(q_2^{-1b}{}_{\underline{b}}\, D_+^{\alpha a} q_2^{\underline{b}c} \right) = im(1+2\alpha)\, \epsilon^{ac} q_1^{\underline{\alpha}\alpha}\, q_2^{\underline{a}b}. \qquad (2.11b)$$

Thus we have equivalently rewritten eqs. (2.1) - (2.3) as the set of coupled equations (2.9), (2.11) for two $N=4$ twisted superfields $q_1^{\underline{\alpha}\alpha}$, $q_2^{\underline{a}a}$. This representation is most convenient for our further purposes.

2.2 Component content

The off-shell component structure of the superfields q_1, q_2 can be revealed by the standard method of projections, acting on these superfields by spinor derivatives and taking account of the constraints (2.9). Proceeding in this way, we find that $q_1^{\underline{\alpha}\alpha}$ includes the following $8+8$ independent components

$$\begin{aligned} q_1^{\underline{\alpha}\alpha}(x) &= \exp\{-u_1\}\, \tilde{q}_1^{\underline{\alpha}\alpha}\,|, \quad B_1^{\underline{\alpha}a}(x) = \frac{1}{16} q_{1\underline{\alpha}}^{\beta}\, D_-^{\underline{\alpha a}} \left(q_1^{-1}{}_{\beta\underline{\beta}}\, D_+^{\gamma a} q_{1\gamma}^{\underline{\beta}} \right)| \\ \xi_{1+}^{\alpha a}(x) &= \frac{1}{4} q_1^{-1\alpha}{}_{\underline{\beta}}\, D_+^{\lambda a} q_1^{\underline{\beta}}{}_{\lambda}\,|, \quad \xi_{1-}^{\underline{\alpha a}}(x) = \frac{1}{4} D_-^{\underline{\lambda a}} q_{1\underline{\lambda}}^{\beta}\, q_{1\beta}^{-1\underline{\alpha}}\,|. \end{aligned} \qquad (2.12)$$

(this definition is chosen for further convenience). Other components either vanish or are expressed through (2.12) with the aid of the relations

$$\begin{aligned} D_+^{\delta d} D_+^{\alpha a}\, q_1^{\underline{\lambda}\gamma} &= -2i\epsilon^{da}\epsilon^{\alpha\gamma}\, \partial_+ q_1^{\underline{\lambda}\delta} \\ D_-^{\underline{\delta d}} D_-^{\underline{\alpha a}}\, q_1^{\underline{\lambda}\gamma} &= -2i\epsilon^{\underline{da}}\epsilon^{\underline{\alpha\lambda}}\, \partial_- q_1^{\underline{\delta}\gamma} \end{aligned} \qquad (2.13)$$

following from the constraints (2.9) and the algebra of $N=4$ spinor derivatives (2.4).

Analogously, for $q_2^{\underline{a}a}$ we have

$$\begin{aligned} q_2^{\underline{a}a}(x) &= \exp\{-u_2\}\, \tilde{q}_2^{\underline{a}a}\,|, \quad B_2^{\underline{\alpha}a}(x) = \frac{1}{16} q_{2\underline{a}}^{b}\, D_-^{\underline{\alpha a}} \left(q_2^{-1}{}_{b\underline{b}}\, D_+^{\alpha c} q_{2c}^{\underline{b}} \right)| \\ \xi_{2+}^{\alpha a}(x) &= \frac{1}{4} q_2^{-1a}{}_{\underline{b}}\, D_+^{\alpha c} q_{2c}^{\underline{b}}\,|, \quad \xi_{2-}^{\underline{\alpha a}}(x) = \frac{1}{4} D_-^{\underline{\alpha c}} q_{2\underline{c}}^{b}\, q_{2b}^{-1\underline{a}}\,|. \end{aligned} \qquad (2.14)$$

As will be shown later, in both sets the fields u, $\tilde{q}$, ξ_+, ξ_- are physical and B_1, B_2 are auxiliary (actually, this is clear already by the dimensionality reasons).

Acting on the superfield equations (2.11) by spinor derivatives and using the identities listed in Appendix, we find the component equations of our $SO(4)\times U(1)\times U(1)$ $N=4$ WZNW - Liouville sigma model

$$\begin{aligned} B_1^{\underline{\alpha}a} &= -\frac{1}{4}\, im(1-2\alpha)\, q_2^{\underline{a}a} \exp\{-2u_1\} \\ \partial_+\partial_- u_1 &= 1/2\,(1-2\alpha)\, [m^2 \exp\{-2(u_1+u_2)\} \\ &\quad - im\, (\xi_{2-\,\underline{\alpha a}}\, q_2^{\underline{a}a}\, q_1^{\underline{\alpha}\alpha}\, \xi_{2+\,\alpha a} + \xi_{2-\,\underline{\alpha a}}\, q_2^{\underline{a}a}\, q_1^{\underline{\alpha}\alpha}\, \xi_{1+\,\alpha a} \\ &\quad + \xi_{1-\,\underline{\alpha a}}\, q_2^{\underline{a}a}\, q_1^{\underline{\alpha}\alpha}\, \xi_{2+\,\alpha a} + \xi_{1-\,\underline{\alpha a}}\, q_2^{\underline{a}a}\, q_1^{\underline{\alpha}\alpha}\, \xi_{1+\,\alpha a})] \\ \partial_- \left(\tilde{q}_1^{-1}{}_{\beta\underline{\beta}}\, \partial_+ \tilde{q}_1^{\underline{\beta}\alpha} \right) &= -im(1-2\alpha)\, (\xi_{2-\,\underline{\alpha a}}\, q_2^{\underline{a}a}\, q_1^{\underline{\alpha}\alpha}\, \xi_{2+\,\beta a} + \xi_{2-\,\underline{\alpha a}}\, q_2^{\underline{a}a}\, q_1^{\underline{\alpha}\alpha}\, \xi_{1+\,\beta a} \\ &\quad + \xi_{1-\,\underline{\alpha a}}\, q_2^{\underline{a}a}\, q_1^{\underline{\alpha}\alpha}\, \xi_{2+\,\beta a} + \xi_{1-\,\underline{\alpha a}}\, q_2^{\underline{a}a}\, q_1^{\underline{\alpha}\alpha}\, \xi_{1+\,\beta a}) \\ \partial_- \xi_{1+}^{\alpha a} &= \frac{m}{2}(1-2\alpha)\, (\xi_{2-\,\underline{\alpha a}} + \xi_{1-\,\underline{\alpha a}})\, q_2^{\underline{a}a}\, q_1^{\underline{\alpha}\alpha} \\ \partial_+ \xi_{1-}^{\underline{\alpha a}} &= -\frac{m}{2}(1-2\alpha)(\xi_{2+\,\alpha a} + \xi_{1+\,\alpha a})\, q_2^{\underline{a}a}\, q_1^{\underline{\alpha}\alpha}. \end{aligned} \qquad (2.15)$$

$$B_2^{\underline{\alpha}\alpha} = -\frac{1}{4}\, im(1+2\alpha)\, q_1^{\underline{\alpha}\alpha} \exp\{-2u_2\}$$

$$
\begin{aligned}
\partial_+\partial_- u_2 &= 1/2\,(1+2\alpha)\big[m^2 \exp\{-2(u_1+2u_2)\} \\
&\quad -im\,\big(\xi_{2-\,\underline{\alpha a}}\, q_2^{\underline{a}a}\, q_1^{\underline{\alpha}+\alpha}\, \xi_{2+\,\alpha a} + \xi_{2-\,\underline{\alpha a}}\, q_2^{\underline{a}a}\, q_1^{\underline{\alpha}\alpha}\, \xi_{1+\,\alpha a} \\
&\quad + \xi_{1-\,\underline{\alpha a}}\, q_2^{\underline{a}a}\, q_1^{\underline{\alpha}\alpha}\, \xi_{2+\,\alpha a} + \xi_{1-\,\underline{\alpha a}}\, q_2^{\underline{a}a}\, q_1^{\underline{\alpha}\alpha} \xi_{1+\,\alpha a}\big)\big] \\
\partial_-\left(\tilde{q}^{-1}_{2\,b\underline{b}}\, \partial_+ \tilde{q}_2^{\underline{b}a}\right) &= -im(1+2\alpha)\big(\xi_{2-\,\underline{\alpha a}}\, q_2^{\underline{a}a}\, q_1^{\underline{\alpha}\alpha}\, \xi_{2+\,\alpha b} + \xi_{2-\,\underline{\alpha a}}\, q_2^{\underline{a}a}\, q_1^{\underline{\alpha}\alpha}\, \xi_{1+\,\alpha b} \\
&\quad + \xi_{1-\,\underline{\alpha a}}\, q_2^{\underline{a}a}\, q_1^{\underline{\alpha}\alpha}\, \xi_{2+\,\alpha b} + \xi_{1-\,\underline{\alpha a}}\, q_2^{\underline{a}a}\, q_1^{\underline{\alpha}\alpha}\, \xi_{1+\,\alpha b}\big) \qquad (2.16) \\
\partial_- \xi_{2+}^{\alpha a} &= \frac{m}{2}(1+2\alpha)\big(\xi_{2-\,\underline{\alpha a}} + \xi_{1-\,\underline{\alpha a}}\big)\, q_2^{\underline{a}a}\, q_1^{\underline{\alpha}\alpha} \\
\partial_+ \xi_{2-}^{\underline{\alpha a}} &= -\frac{m}{2}(1+2\alpha)\,(\xi_{2+\,\alpha a} + \xi_{1+\,\alpha a})\, q_2^{\underline{a}a}\, q_1^{\underline{\alpha}\alpha}
\end{aligned}
$$

2.3 Transformation properties

We turn now to discussing the invariances of the superfield system (2.9), (2.11). As was shown in [2, 3, 4], the pure $N=4$ WZNW sigma model following from (2.9), (2.11) in the limit $m=0$ respects $SO(4)\times U(1) \sim SU_I(2)\times SU_{II}(2)\times U(1)$ $N=4$ superconformal symmetry. The same turns out to be true in the general case of $m \neq 0$. As it follows from the results of [9], the corresponding $N=4$ superconformal algebra can be obtained as a closure of two its different $SU_{I,II}(2)$ $N=4$ subalgebras, so henceforth we limit our consideration to the transformations which belong to these defining subalgebras. The transformation laws of the superfields q_1, q_2 under the full $SO(4)\times U(1)$ $N=4$ superconformal symmetry are given in [9].

The $N=4$ superspace coordinates transform under these two $SU_I(2)$ and $SU_{II}(2)$ $N=4$ SCG's as follows (as usual, we limit our consideration to the (+) light-cone subspace and the related branch of $N=4$ SCG)

$$
\begin{aligned}
\delta\theta^{+\,\alpha a} &= \frac{1}{2i} D_+^{\alpha a} E_I + \frac{1}{2i} D_+^{\alpha a} E_{II} \\
\delta x^+ &= E_I + \frac{1}{2}\, D_+^{\alpha a} E_I\, \theta^+_{\alpha a} + E_{II} + \frac{1}{2}\, D_+^{\alpha a} E_{II}\, \theta^+_{\alpha a}\,, \qquad (2.17)
\end{aligned}
$$

where the superparameters E_I and E_{II} obey the constraints

$$
D_+^{\lambda(a} D_{+\lambda}^{b)}\, E_I = 0\,, \quad D_+^{(\alpha a} D_{+a}^{\beta)}\, E_{II} = 0 \qquad (2.18)
$$

and collect the relevant sets of the $SU_{I,II}(2)$ $N=4$ superconformal transformation parameters (those of two independent $2D$ conformal transformations, four local supersymmetries and $SU_I(2)$ and $SU_{II}(2)$ Kac-Moody symmetries forming the $SO(4)$ one). The associate superfield transformations leaving invariant (separately) eqs. (2.9) and (2.11) are given by

$$
\begin{aligned}
\delta q_1^{\underline{\alpha}\alpha} &= -\frac{1}{4i} q_1^{\underline{\alpha}\beta}\, D_+^{\alpha a} D_{+\beta a} E_I \\
\delta q_2^{\underline{a}a} &= -\frac{1}{4i} q_2^{\underline{a}b}\, D_+^{\alpha a} D_{+\alpha b} E_{II}\,. \qquad (2.19)
\end{aligned}
$$

It is worth remarking that $q_1^{\underline{\alpha}\alpha}$ ($q_2^{\underline{a}a}$) is scalar under $SU_{II}(2)$ ($SU_I(2)$) SCG and is rotated in its $SU(2)$ indices by $SU_I(2)$ ($SU_{II}(2)$) SCG. To our knowledge, the existence of these

two $N=4$ twisted multiplets having different transformation properties with respect to a fixed $SU(2)$ $N=4$ SCG has been first noticed in [14].

2.4 Reductions to the $SU(2)\times U(1)$ WZNW and WZNW - Liouville models

As has been already mentioned, the $SU_I(2)$ and $SU_{II}(2)$ $N=4$ superconformal invariances of eqs. (2.9), (2.11) (and, hence, their commutant, $SO(4)\times U(1)$ $N=4$ superconformal invariance) are also maintained in the limit $m=0$, where this system splits into the two decoupled sets of equations for q_1 and q_2

$$
D_+^{\alpha a} q_1^{\underline{\alpha}\beta} + D_+^{\beta a} q_1^{\underline{\alpha}\alpha} = 0, \quad D_-^{\underline{\alpha a}} q_1^{\underline{\beta}\alpha} + D_-^{\underline{\beta a}} q_1^{\underline{\alpha}\alpha} = 0 \qquad (2.20a)
$$

$$
D_-^{\underline{\alpha a}}\left(q^{-1\beta}_{1\;\underline{\beta}}\, D_+^{\alpha a} q_1^{\underline{\beta}\gamma}\right) = 0, \qquad (2.20b)
$$

and analogously for q_2. Eqs. (2.20) describe the $SU(2)\times U(1)$ $N=4$ WZNW sigma model [1, 2]. It is a simple exercise to check that eqs. (2.20a) (the off-shell constraint) and (2.20b) (the equation of motion) are separately covariant under the tranfsormations (2.17), (2.19).

A less trivial reduction is effected by putting in (2.9), (2.11)

$$
\alpha = -\frac{1}{2},\; q_2^{\underline{a}b} = \epsilon^{\underline{a}b} \quad \text{or} \quad \alpha = \frac{1}{2},\; q_1^{\underline{\alpha}\beta} = \epsilon^{\underline{\alpha}\beta}\,. \qquad (2.21)
$$

These two options are equivalent up to the interchange $SU_I(2) \leftrightarrow SU_{II}(2)$, so it suffices to consider, say, the first one. It reduces (2.9), (2.11) to the following set

$$
D_+^{\alpha a} q_1^{\underline{\alpha}\beta} + D_+^{\beta a} q_1^{\underline{\alpha}\alpha} = 0, \quad D_-^{\underline{\alpha a}} q_1^{\underline{\beta}\alpha} + D_-^{\underline{\beta a}} q_1^{\underline{\alpha}\alpha} = 0 \qquad (2.22a)
$$

$$
D_-^{\underline{\alpha a}}\left(q^{-1\beta}_{1\;\underline{\beta}}\, D_+^{\alpha a} q_1^{\underline{\beta}\gamma}\right) = 2im\, \epsilon^{\alpha\gamma}\, \epsilon^{\underline{a}a}\, q_1^{\underline{\alpha}\beta}\,. \qquad (2.22b)
$$

These are the equations of $SU(2)\times U(1)$ $N=4$ WZNW - Liouville sigma model constructed for the first time in [1] [3]. The $SO(4)\times U(1)$ $N=4$ superconformal symmetry of the initial model is now reduced to the semi-direct product of $SU_I(2)$ $N=4$ superconformal symmetry and global $SU(2)$ symmetry which rotates in a uniform way the latin doublet indices $\underline{a}$, a (so as to leave invariant the antisymmetric tensor $\epsilon^{\underline{a}a}$). This global $SU(2)$ is the only trace of the second $SU(2)$ $N=4$ superconformal symmetry to survive the above reduction. Of course, in the limit $m=0$ this second $SU(2)$ $N=4$ superconformal symmetry and, hence, the whole $SO(4)\times U(1)$ $N=4$ one are restored.

In terms of independent components of q_1, eq. (2.22b) amounts to the following set of equations (we suppress the suffix 1)

$$
B^{\underline{a}a} = -\frac{1}{2}\, im\, \epsilon^{\underline{a}a} \exp\{-2u\}
$$

[3] This model has been recently re-discovered in [7] in a $N=1$ superfield formulation.

$$
\begin{aligned}
\partial_+\partial_- u &= m^2 \exp\{-2u\} - im\, \epsilon^{\underline{a}a}\, \xi_{-\underline{\alpha a}}\xi_{+\alpha a}\, q^{\underline{\alpha}\alpha} \\
\partial_-\left(\tilde{q}^{-1}_{\alpha\underline{\beta}}\, \partial_+ \tilde{q}^{\underline{\beta}\lambda}\right) &= -2im\, \epsilon^{\underline{a}a}\, \xi_{-\underline{\lambda a}}\xi_{+\alpha a}\, q^{\underline{\lambda}\lambda} \\
\partial_+\xi_-^{\underline{\alpha a}} &= -m\, \epsilon^{\underline{a}a}\, q^{\underline{\alpha}\alpha}\, \xi_{+\alpha a} \\
\partial_-\xi_+^{\alpha a} &= m\, \epsilon^{\underline{a}a}\, q^{\underline{\alpha}\alpha}\, \xi_{-\underline{\alpha a}}
\end{aligned}
\qquad (2.23)
$$

Correspondingly, in this model we have 8 bosons and 8 fermions off shell and 4+4 on shell.

It is easy to write the off-shell component actions which lead to eqs. (2.15), (2.16), (2.23) and, after eliminating auxiliary fields, are reduced to those given in [2]. However, we postpone giving these actions to constructing an unconstrained superfield description of the systems under consideration. They will be obtained from the corresponding superfield actions.

We conclude this Section by several comments concerning the terminology.

The full symmetry of eqs. (2.9), (2.11) in the limit $m = 0$, where they describe pure $N = 4$ super-WZNW model, is actually wider than $SO(4) \times U(1) \times U(1)$ $N = 4$ superconformal one: it is given by two independent $SO(4) \times U(1) \times U(1)$ SCG's acting independently on q_1 and q_2 (the corresponding $SU(2)$ $N = 4$ SCG's are represented by the transformations (2.17), (2.19) with two independent sets of the superparameters E_I, E_{II}). The fields $u_1(x) = u_1 \mid$, $u_2(x) = u_2 \mid$ can be regarded as the parameters of two independent $U(1)$ Kac-Moody groups, hence the bosonic manifold of this model can be identified with $SU_I(2) \times SU_{II}(2) \times U(1) \times U(1) \sim SO(4) \times U(1) \times U(1)$ [4], which explains the denomination we use for this WZNW model.

Analogously, the abbreviation "$SU(2) \times U(1)$ $N = 4$ WZNW" for the reduced model (2.20) originates from the fact that this model possesses one $SU_I(2) \times SU_{II}(2) \times U(1)$ $N = 4$ superconformal symmetry, with $u(x) = u \mid$ and $\tilde{q}_1^{\underline{\alpha}\alpha}(x) = \tilde{q}_1^{\underline{\alpha}\alpha} \mid$ parametrizing the groups $U(1)$ and $SU_I(2)$ and, hence, with $SU(2) \times U(1)$ as the bosonic target manifold.

The Liouville $m \neq 0$ deformation (2.11), (2.9) of the $SO(4) \times U(1) \times U(1)$ $N = 4$ WZNW sigma model enjoys the invariance only with respect to a diagonal SCG in the product of the two aforementioned $SO(4) \times U(1)$ SCG's, so the full internal symmetry group of this model is $SO(4) \times U(1)$ Kac-Moody group. Accordingly, one of the involved scalar fields (the combination $u_1(x) + u_2(x)$) ceases to be the $U(1)$ group parameter, being rather a goldstone field describing the spontaneous breakdown of $2D$ conformal symmetry (the $2D$ dilaton). The Liouville deformation (2.22) of the $SU(2) \times U(1)$ $N = 4$ WZNW sigma model (2.20) displays a restricted superconformal invariance with respect to $SU(2)$ $N = 4$ SCG multiplied by some extra rigid $SU(2)$ realized on spinor fields. Thus the full internal symmetry acting on the bosonic fields is reduced to $SU(2)$ Kac-Moody symmetry and the field $u(x)$, like in the previous case, loses the status of the $U(1)$ group parameter. In spite of these reductions of the internal symmetry groups, we name the $N = 4$ WZNW - Liouvile systems in question the $SO(4) \times U(1) \times U(1)$ and $SU(2) \times U(1)$ ones, in order to point out their genetic relationship with the corresponding $N = 4$ WZNW sigma models.

[4] Depending on the boundary conditions imposed on the fields u_1, u_2, one may regard them taking values either in $U(1)$ or in its non-compact version $R(1)$.

3 Solving the constraints. Prepotentials

The problem of construction of superfield actions for the above systems is difficult to solve dealing with the superfields $q_1^{\underline{\alpha}\alpha}$ and $q_2^{\underline{a}a}$ because they are constrained. One way (perhaps, most perspective) to overcome this difficulty is to re-interpret the constraints (2.9) as a kind of the harmonic analiticity conditions [15], to represent q_1 and q_2 as unconstrained superfields given on some analytic subspace of a harmonic extension of $N = 4$ $2D$ superspace and to write the relevant actions as integrals over this analytic subspace. Basically keeping to this strategy, Roček, Schoutens and Sevrin have recently constructed an off-shell superfield action for the $SU(2) \times U(1)$ $N = 4$ WZNW sigma model [12]. We propose here an alternative approach to constructing such actions, staying in the framework of the conventional $N = 4$ $2D$ superspace. It is based upon solving the constraints (2.9) via unconstrained prepotentials.

In fact, the general solution to these constraints has been given by Siegel several years ago [16]. It is easy to show that (2.9a), (2.9b) become identities after representing

$$
q_1^{\underline{\alpha}\alpha} = 16 \hat{D}_{1-}^{3\,\underline{\alpha a}} \hat{D}_{1+}^{3\,\alpha a}\, V_{1\,\underline{a}a} \qquad (3.1)
$$

$$
q_2^{\underline{a}a} = 16 \hat{D}_{2-}^{3\,\underline{\alpha a}} \hat{D}_{2+}^{3\,\alpha a}\, V_{2\,\underline{\alpha}\alpha} \qquad (3.2)
$$

$$
\hat{D}_1^{3\,\alpha a} = D_\tau^{\ \alpha} D^{\lambda(a} D_\lambda^{\ \tau)}, \quad \hat{D}_2^{3\,\alpha a} = D_\gamma^{\ a} D^{(\gamma\tau} D_\tau^{\ \alpha)}, \qquad (3.3)
$$

where $V_{1\,\underline{a}a}$, $V_{2\,\underline{\alpha}\alpha}$ are unconstrained $N = 4$ superfields, the prepotentials. The representation (3.1), (3.2) exhibits a freedom under the following gauge transformations of the prepotentials

$$
\begin{aligned}
\delta V_1^{\underline{a}a} &= D_{+\alpha b} K_1^{\underline{a}(ab)\alpha} + D_{-\underline{\alpha b}} K_1^{(\underline{ab})a\alpha} \\
\delta V_2^{\underline{\alpha}\alpha} &= D_{+\beta a} K_2^{\underline{\alpha}(\beta\alpha)a} + D_{-\underline{\beta a}} K_2^{(\underline{\alpha\beta})\alpha a},
\end{aligned}
\qquad (3.4)
$$

where K_1, K_2 are unconstrained supergauge parameters.

The transformation rules of the prepotentials under two $SU(2)$ $N = 4$ superconformal symmetries can be deduced from the requirement that for q_1, q_2 the transformations (2.19) are reproduced. After some labour (using the identities (A.11) - (A.13) listed in Appendix), these are found to be

$$
\begin{aligned}
\delta V_{1\,\underline{a}a} &= \partial_+ E_I\, V_{1\,\underline{a}a} + \frac{3}{2}\, \partial_+ E_{II}\, V_{1\,\underline{a}a} - \frac{1}{4i}\, D^{\lambda}_{+(a} D_{+\lambda c)} E_{II}\, V_{1\,\underline{a}}^{\ c} \\
\delta V_{2\,\underline{\alpha}\alpha} &= \partial_+ E_{II}\, V_{2\,\underline{\alpha}\alpha} + \frac{3}{2}\, \partial_+ E_I\, V_{2\,\underline{\alpha}\alpha} - \frac{1}{4i}\, D^{\tau}_{+(\alpha} D_{+\gamma)\tau} E_I\, V_{2\,\underline{\alpha}}^{\ \gamma}.
\end{aligned}
\qquad (3.5)
$$

It is amusing that these transformations are in a sense complementary to those of q_1, q_2: with respect to $SU_I(2)$ ($SU_{II}(2)$) $N = 4$ SCG the prepotential V_1 (V_2) behaves as a scalar density (with the conformal weight 1, in contrast to $q_1(q_2)$ which has zero weight) while with respect to $SU_{II}(2)$ ($SU_I(2)$) it undergoes also a local $SU(2)$ rotation in external indices (and possesses the conformal weight 3/2).

After fixing Wess-Zumino gauge with respect to (3.4), the component content of V_1, V_2 is reduced to that of q_1, q_2. The precise relation of the WZ gauge components of V_1 to (2.12) is as follows (we omit the suffix 1)

$$
\begin{aligned}
\hat{D}_-^{3\,\underline{\alpha a}}\hat{D}_+^{\alpha a}V_{\underline{a}a}\,| &= \frac{1}{16}\,q^{\underline{\alpha}\alpha} \\
D_{+\alpha b}\hat{D}_-^{3\,\underline{\alpha a}}\hat{D}_+^{3\,\alpha a}V_{\underline{a}a}\,| &= q^{\underline{\alpha}\alpha}\,\xi_{+\,\alpha b} \\
D_{-\underline{\alpha b}}\hat{D}_-^{3\,\underline{\alpha a}}\hat{D}_+^{3\,\alpha a}V_{\underline{a}a}\,| &= \xi_{-\,\underline{\alpha b}}\,q^{\underline{\alpha}\alpha} \\
D_{-\underline{\alpha b}}D_{+\alpha b}\hat{D}_-^{3\,\underline{\alpha a}}\hat{D}_+^{3\,\alpha a}V_{\underline{a}a}\,| &= 16\left(B_{\underline{b}b}+\xi_{-\underline{\alpha b}}\,\xi_{+\alpha b}\,q^{\underline{\alpha}\alpha}\right).
\end{aligned}
\tag{3.6}
$$

The relation between V_2 in WZ gauge and q_2 is basically the same, up to the interchange of greek and latin doublet indices.

4 $N=4$ WZNW - Liouville superfield actions

4.1 $SU(2)\times U(1)$ model

Now we are sufficently armed to construct off-shell superfield actions for the models we are interested in. We begin with the simpler case of $SU(2)\times U(1)$ $N=4$ WZNW - Liouville system.

Our aim is to find an unconstrained $N=4$ superfield action which would respect $SU_I(2)$ $N=4$ superconformal and gauge (3.4) symmetries and result in eq. (2.22b). To this end, it is advantageous to make use of the constraints (2.22a) to rewrite (2.22b) in the following equivalent form

$$
q_{\underline{\alpha}}^{-1\,\beta}\,D_-^{\underline{\alpha a}}\left(q^{-1}_{\underline{\beta}\beta}\,D_+^{\alpha a}q^{\underline{\beta}}_{\;\alpha}\right)-8im\epsilon^{\underline{a}a}=0. \tag{4.1}
$$

The two terms in eq. (4.1) are separately covariant under $SU(2)$ $N=4$ SCG, so the action must consist of the two independent parts,

$$
S^{SU(2)}_{WZL}=S^{SU(2)}_{WZ}+S_m\,, \tag{4.2}
$$

producing, respectively, the first and second items in (4.1).

It is easy to guess the form of S_m: it is none other than the Fayet-Iliopoulos term for the prepotential $V_{\underline{a}a}$

$$
S_m\sim-8im\int d^2xd^8\theta\,\epsilon^{\underline{a}a}V_{\underline{a}a} \tag{4.3}
$$

(up to an overall coupling constant in front of the action). It is invariant under gauge transformations (3.4) and, with taking account of the $SU_I(2)$ $N=4$ SCG transformation property of the superspace integration measure

$$
\delta\left(d^2xd^8\theta\right)=-\partial_+E_I\left(d^2xd^8\theta\right)\,, \tag{4.4}
$$

also under $SU_I(2)$ $N=4$ superconformal symmetry.

The form of the action responsible for the first piece in (4.1) is not immediately obvious. To understand how it could be constructed, let us apply to the $N=0$ (purely bosonic) case and recall the procedure of restoring the action of bosonic $WZNW$ sigma model by the corresponding equations of motion.

To be more specific, we consider the equation of $SU(2)$ WZNW model

$$
\partial_-\left(\tilde{q}^{-1}_{\alpha\underline{\beta}}\,\partial_+\tilde{q}^{\underline{\beta}\lambda}\right)=0. \tag{4.5}
$$

The most direct way to reproduce this equation from the action principle is to allow $\tilde{q}$ to depend on some extra parameter t and to consider the action (discarding the overall coupling constant)

$$
\begin{aligned}
S_{WZ} &= \int d^2x\int_0^1 dt\,\partial_-\left(\tilde{q}^{-1}_{\alpha\underline{\beta}}\,\partial_+\tilde{q}^{\underline{\beta}\lambda}\right)\left(\tilde{q}^{-1}_{\lambda\underline{\lambda}}\,\partial_t\,\tilde{q}^{\underline{\lambda}\alpha}\right) \\
&\tilde{q}^{\underline{\alpha}\beta}(t=1)\equiv\tilde{q}^{\underline{\alpha}\beta},\quad \tilde{q}^{\underline{\alpha}\beta}(t=0)=0\,.
\end{aligned}
\tag{4.6}
$$

After simple manipulations including integration by part, this action can be recasted into the familiar form of the WZNW sigma model action

$$
\begin{aligned}
S_{WZ} &= \frac{1}{2}\int d^2x\,\partial_+\tilde{q}^{\underline{\nu}\lambda}\,\partial_-\tilde{q}^{-1}_{\lambda\underline{\nu}} \\
&+ \frac{1}{2}\int d^2x\int_0^1 dt\,(\tilde{q}^{\underline{\gamma}\phi}\,\partial_t\,\tilde{q}^{-1}_{\phi\underline{\lambda}})\left[(\tilde{q}^{\underline{\lambda}\gamma}\,\partial_+\tilde{q}^{-1}_{\gamma\underline{\omega}})(\tilde{q}^{\underline{\omega}\nu}\,\partial_-\tilde{q}^{-1}_{\nu\underline{\gamma}})-(\partial_+\leftrightarrow\partial_-)\right],
\end{aligned}
$$

where the second term is recognized as the WZNW one. It is important for us that the action can be taken in the form (4.6). The integral over t entirely drops out only in the variation of (4.6). To show this, it is convenient to introduce the quantities

$$
A_\pm=\tilde{q}^{-1}\,\partial_\pm\tilde{q},\quad A_\delta=\tilde{q}^{-1}\,\delta\tilde{q},\quad A_t=\tilde{q}^{-1}\,\partial_t\,\tilde{q} \tag{4.7}
$$

which are related by the identities of the sort

$$
\delta A_+=-A_\delta A_++\partial_+A_\delta+A_+A_\delta. \tag{4.8}
$$

With making use of these identities, the process of varying (4.6) is substantially simplified

$$
\delta\int_0^1 dt\,\partial_-A_+A_t=\int_0^1 dt\,\frac{d}{dt}(\partial_-A_+A_\delta)=\partial_-A_+A_\delta\,. \tag{4.9}
$$

Equating this variation for arbitrary $\delta\tilde{q}$ to zero, we come to (4.5).

The superfield action $S^{SU(2)}_{WZ}$ producing the first term in eq. (4.1) can be constructed analogously to the action (4.6)

$$
\begin{aligned}
S^{SU(2)}_{WZ} &= \frac{1}{64\,f^2}\int d^2xd^8\theta\int_0^1 dt\,\left[q_{\underline{\alpha}}^{-1\,\beta}\,D_-^{\underline{\alpha a}}\left(q^{-1}_{\beta\underline{\beta}}\,D_+^{\alpha a}q^{\underline{\beta}}_{\;\alpha}\right)\right]\partial_tV_{\underline{a}a}, \\
&q^{\underline{\alpha}\beta}(t=1)\equiv q^{\underline{\alpha}\beta},\quad q^{\underline{\alpha}\beta}(t=0)=0\,,
\end{aligned}
\tag{4.10}
$$

where the numerical coefficient 1/64 was included to have a proper normalization of the component action.

To prove the correctness of (4.10), one should, like in the bosonic case, compute the variation of (4.10) with respect to the prepotential $V_{\underline{a}a}$ and check that the t integral in this variation can be removed. Calculations are simplified to a great extent with using the irreducible projection superfields defined in eq. (2.12). The superfield Lagrangian in (4.10) can be rewritten in their terms as

$$L_{WZ}^{SU(2)} = -16 \int_0^1 dt \exp\{2u\}\, B^{\underline{a}a}\, \partial_t V_{\underline{a}a} \,. \tag{4.11}$$

Discarding full spinor derivatives, the variation of (4.11) can be put into the form

$$\begin{aligned} \delta L_{WZ}^{SU(2)} \sim\; & D_+^{\sigma a} D_-^{\underline{\sigma a}} \exp\{2u\}\, \partial_t V_{\underline{a}a}\, \delta q_{\underline{\sigma}\sigma} + D_-^{\underline{\sigma a}} \exp\{2u\}\, D_+^{\sigma a}\, \partial_t V_{\underline{a}a}\, \delta q_{\underline{\sigma}\sigma} \\ - \; & D_+^{\sigma a} \exp\{2u\}\, D_-^{\underline{\sigma a}}\, \partial_t V_{\underline{a}a}\, \delta q_{\underline{\sigma}\sigma} + \exp\{2u\}\, D_+^{\sigma a} D_-^{\underline{\sigma a}}\, \partial_t V_{\underline{a}a}\, \delta q_{\underline{\sigma}\sigma} \\ - \; & 16 \exp\{2u\}\, B^{\underline{a}a}\, \delta\partial_t V_{\underline{a}a} \,. \end{aligned} \tag{4.12}$$

Going from δq to δV by eqs. (3.1), (3.2), discarding full derivatives and exploiting identities (A.4) - (A.11), we finally put $\delta L_{WZ}^{SU(2)}$ into the form with δ and ∂_t interchanged, that is,

$$\delta L_{WZ}^{SU(2)} \sim -\int_0^1 dt\, \frac{d}{dt}\left(\exp\{2u\}\, B^{\underline{a}a}\, \delta V_{\underline{a}a}\right) = -\exp\{2u\}\, B^{\underline{a}a}\, \delta V_{\underline{a}a} \,. \tag{4.13}$$

This is the desirable result.

The action (4.10) exhibits invariance under both $SU(2)$ $N = 4$ SCG's defined in sec.2.3 and, hence, under $SO(4) \times U(1)$ $N = 4$ SCG. This may be checked either by using the above formula for the variation and substituting the active form of the infinitesimal superconformal transformations of $V_{\underline{a}a}$ (at a fixed point of $N = 4$ superspace), or by considering a passive form of $SU(2)$ $N = 4$ superconformal transformations, i.e. making use of the transformation laws (2.19), (3.5), (4.4) and those of covariant spinor derivatives

$$\delta D_+^{\alpha a} = -\frac{1}{2}\, D_+^{\alpha a} D_{+\,\gamma c}(E_I + E_{II})\, D_+^{\gamma c} \,. \tag{4.14}$$

With the help of the identities (A.3) and (A.8) one also proves the invariance of (4.10) under gauge transformations (3.4).

Thus we have found that the off-shell superfield action of $SU(2) \times U(1)$ $N = 4$ WZNW sigma model is given by the expression (4.10) while its Liouville extension by

$$S = \frac{1}{64\, f^2} \int d^2x d^8\theta \left\{ \int_0^1 dt\, \left[q^{-1\,\beta}_{\underline{\alpha}}\, D_-^{\underline{\alpha a}} \left(q^{-1}_{\beta\underline{\beta}}\, D_+^{\alpha a} q_{\alpha}^{\underline{\beta}} \right) \right] \partial_t V_{\underline{a}a} - 8im\, \epsilon^{\underline{a}a}\, V_{\underline{a}a} \right\} . \tag{4.15}$$

We stress once more that the action (4.15), in contrast to (4.10), respects a restricted superconformal invariance, that is under $SU_I(2)$ $N = 4$ SCG and an additional global $SU(2)$ group which uniformly rotates the indices $\underline{a}$, a. The maximal $2D$ $N = 4$ superconformal

symmetry is restored only in the limit $m = 0$, when (4.15) becomes the action of pure $SU(2) \times U(1)$ $N = 4$ WZNW sigma model.

Now, choosing WZ gauge for $V_{\underline{a}a}$, it is straightforward to integrate in (4.15) over θ's (using again identities (A.3) - (A.7)) and thus to derive the off-shell component action of $SU(2) \times U(1)$ $N = 4$ WZNW - Liouville model

$$\begin{aligned} S_{WZL}^{SU(2)} = \; & \frac{1}{f^2} \int d^2x \left\{ \partial_+ u \partial_- u - i\partial_- \xi_+^{\alpha a} \xi_{+\alpha a} - i\partial_+ \xi_-^{\underline{\alpha a}} \xi_{-\underline{\alpha a}} + 2\exp\{2u\}\, B^{\underline{a}a} B_{\underline{a}a} \right. \\ + \; & \left. 2im\, \epsilon^{\underline{a}a} \left(B_{\underline{a}a} + \xi_{-\underline{\alpha a}}\, q^{\underline{\alpha}\alpha} \xi_{+\alpha a} \right) \right\} + \frac{1}{f^2} S_{WZ} \,, \end{aligned} \tag{4.16}$$

where S_{WZ} is defined by eq. (4.6). Varying (4.18) yields just eqs. (2.23). For completeness, we also quote the form of the physical fields action

$$\begin{aligned} S_{WZL}^{SU(2)} = \; & \frac{1}{f^2} \int d^2x \left\{ \partial_+ u \partial_- u - i\partial_- \xi_+^{\alpha a} \xi_{+\alpha a} - i\partial_+ \xi_-^{\underline{\alpha a}} \xi_{-\underline{\alpha a}} - m^2 \exp\{-2u\} \right. \\ + \; & \left. 2im\, \epsilon^{\underline{a}a} \xi_{-\underline{\alpha a}}\, q^{\underline{\alpha}\alpha} \xi_{+\alpha a} \right\} + \frac{1}{f^2} S_{WZ} \,. \end{aligned} \tag{4.17}$$

It coincides with the action given in [2]. Recall that the coupling constant f is quantized on topological grounds [17, 2]

$$\frac{f^2}{4\pi} = \frac{1}{K} \,. \tag{4.18}$$

with K integer.

4.2 $SO(4) \times U(1) \times U(1)$ model

This $N = 4$ WZNW - Liouville system is distinguished in that it is invariant under the maximally extended $SO(4) \times U(1)$ $N = 4$ superconformal symmetry.

In the limit $m = 0$ the $SO(4) \times U(1) \times U(1)$ WZNW - Liouville equations (2.11) become two independent $SU(2) \times U(1)$ WZNW ones, so the kinetic term of the relevant off-shell superfield action is given by a sum of two actions of the type (4.10) depending, respectively, on the prepotentials $V_{1\,\underline{a}b}$ and $V_{2\,\underline{\alpha}\beta}$. After rewriting (2.11), with the help of constraints (2.9), in a form similar to eq. (4.1), the superfield potential term is also easy to construct. The total off-shell action of this model is eventually written in terms of superfields as

$$\begin{aligned} S_{WZL}^{SO(4)} = \; & \frac{1}{32\, f^2} \int d^2x d^8\theta \left\{ \int_0^1 dt\, \left[\frac{1}{1-2\alpha}\, q^{-1\beta}_{1\underline{\alpha}}\, D_-^{\underline{\alpha a}} \left(q^{-1}_{1\,\beta\underline{\beta}} D_+^{\alpha a} q_{1\;\alpha}^{\underline{\beta}} \right) \partial_t V_{1\,\underline{a}a} \right. \right. \\ & \left. \left. + \frac{1}{1+2\alpha}\, q^{-1\,b}_{2\,\underline{a}}\, D_-^{\underline{\alpha a}} \left(q^{-1}_{2\,b\underline{b}}\, D_+^{\alpha a} q_{2\;a}^{\underline{b}} \right) \partial_t V_{2\,\underline{\alpha}\alpha} \right] - 2im \left(q_1^{\underline{\alpha}\alpha} V_{2\,\underline{\alpha}\alpha} + q_2^{\underline{a}a} V_{1\,\underline{a}a} \right) \right\} \end{aligned} \tag{4.19}$$

and, in terms of components,

$$S_{WZL}^{SO(4)} = \frac{2}{f^2} \int d^2x \left\{ \frac{1}{1-2\alpha} \left(\partial_- u_1 \partial_+ u_1 - i\partial_- \xi_{1+}^{\alpha a} \xi_{1+\,\alpha a} - i\partial_+ \xi_{1-}^{\underline{\alpha a}} \xi_{1-\,\underline{\alpha a}} \right. \right. \tag{4.20}$$

$$
\begin{aligned}
&+ 2\exp\{2u_1\}\, B_1^{\underline{a}a} B_{1\,\underline{a}a}) \\
&+ \frac{1}{1+2\alpha}\left(\partial_- u_2 \partial_+ u_2 - i\partial_- \xi_{2+}^{\alpha a}\xi_{2+\,\alpha a} - i\partial_+ \xi_{2-}^{\underline{\alpha a}}\xi_{2-\,\underline{\alpha a}}\right. \\
&+ 2\exp\{2u_2\}\, B_2^{\underline{a}\alpha} B_{2\,\underline{a}\alpha}) \\
&+ im \left(q_1^{\underline{\lambda}\lambda} B_{2\,\underline{\lambda}\lambda} + q_2^{\underline{a}a} B_{1\underline{a}a} + \xi_{2-\,\underline{\lambda a}}\, q_2^{\underline{a}a}\, q_1^{\underline{\lambda}\lambda}\, \xi_{2+\,\lambda a} \right. \\
&\left. + \xi_{2-\,\underline{\lambda a}}\, q_1^{\underline{\lambda}\lambda}\, q_2^{\underline{a}a}\, \xi_{1+\,\lambda a} + \xi_{1-\,\underline{\lambda a}}\, q_2^{\underline{a}a}\, q_1^{\underline{\lambda}\lambda}\, \xi_{2+\,\lambda a} + \xi_{1-\,\underline{\lambda a}}\, q_2^{\underline{a}a}\, q_1^{\underline{\lambda}\lambda}\, \xi_{1+\,\lambda a} \right) \Big\} \\
&+ \frac{2}{f^2}\frac{1}{1-2\alpha} S_{1\,WZ} + \frac{2}{f^2}\frac{1}{1+2\alpha} S_{2\,WZ}\,.
\end{aligned}
$$

Here $S_{2\,WZ}$ is given by the expression analogous to (4.6)

$$
S_{2\,WZ} = \int d^2x \int_0^1 dt\, \partial_- \left(\tilde{q}^{-1}_{a\underline{b}}\, \partial_+ \tilde{q}^{\,\underline{b}c} \right) \tilde{q}^{-1}_{c\underline{c}}\, \partial_t\, \tilde{q}^{\,\underline{c}a}. \tag{4.21}
$$

Eliminating the auxiliary fields B_1, B_2 in (4.20) yields

$$
\begin{aligned}
S^{SO(4)}_{WZL} \;=\; & \frac{2}{f^2}\int d^2x \left\{ \frac{1}{1-2\alpha}\left(\partial_- u_1 \partial_+ u_1 - i\partial_- \xi_{1+}^{\alpha a}\xi_{1+\,\alpha a} - i\partial_+ \xi_{1-}^{\underline{\alpha a}}\xi_{1-\,\underline{\alpha a}}\right)\right. \\
&+ \frac{1}{1+2\alpha}\left(\partial_- u_2 \partial_+ u_2 - i\partial_- \xi_{2+}^{\alpha a}\xi_{2+\,\alpha a} - i\partial_+ \xi_{2-}^{\underline{\alpha a}}\xi_{2-\,\underline{\alpha a}}\right) \\
&+ im \left(im \exp\{-2(u_1+u_2)\} + \xi_{2-\,\underline{\lambda a}}\, q_2^{\underline{a}a}\, q_1^{\underline{\lambda}\lambda}\, \xi_{2+\,\lambda a} \right. \\
&\left.\left. + \xi_{2-\,\underline{\lambda a}}\, q_1^{\underline{\lambda}\lambda}\, q_2^{\underline{a}a}\, \xi_{1+\,\lambda a} + \xi_{1-\,\underline{\lambda a}}\, q_2^{\underline{a}a}\, q_1^{\underline{\lambda}\lambda}\, \xi_{2+\,\lambda a} + \xi_{1-\,\underline{\lambda a}}\, q_2^{\underline{a}a}\, q_1^{\underline{\lambda}\lambda}\, \xi_{1+\,\lambda a} \right) \right\} \\
&+ \frac{2}{f^2}\frac{1}{1-2\alpha} S_{1\,WZ} + \frac{2}{f^2}\frac{1}{1+2\alpha} S_{2\,WZ}
\end{aligned} \tag{4.22}
$$

that coincides with the physical fields action of ref. [2]. The quantization conditions for the coupling constants f and α are as follows [2]

$$
\frac{f^2}{4\pi} = \frac{1}{K_1} + \frac{1}{K_2}, \quad \alpha = \frac{1}{2}\frac{K_1 - K_2}{K_1 + K_2}\,, \tag{4.23}
$$

where the two independent integers K_1, K_2 come from two $SU(2)$ WZNW actions in (4.20), (4.22).

Let us inspect symmetries of the superfield action (4.19). As was already explained in the previous Subsection, each of two kinetic terms in (4.19) is invariant under the whole $SU_I(2) \times SU_{II}(2) \times U(1)$ $N = 4$ SCG. It remains to prove the invariance of the superfield Liouville terms. We first note that these two terms actually coincide modulo a total spinor derivative, as this can be easily seen from eqs. (3.1), (3.2). Then, it suffices to study the transformation properties of one of them, say, of $V_{1\underline{a}a} q_2^{\underline{a}a}$. The corresponding piece of the action is invariant under $SU_I(2)$ SCG by the same reasonings as (4.3), since $q_2^{\underline{a}a}$ is a scalar with respect to this SCG and V_1 transforms so as to compensate the transformation of the $N = 4$ superspace integration measure. As concerns $SU_{II}(2)$ $N = 4$ superconformal

transformations, both q_2 and V_1 transform with nontrivial conformal weights (see eqs. (2.19), (3.5)), the total weight being just opposite in sign to the weight of the measure (which transforms under both SCG's according to the same rule (4.4)). The $SU_{II}(2)$ local rotation part of these transformatons does not contribute because of manifest $SU_{II}(2)$ invariance of $V_{1\underline{a}a} q_2^{\underline{a}a}$. Thus the $N = 4$ super-Liouville term in (4.19) is invariant under both $SU(2)$ $N = 4$ superconformal symmetries discussed in sect. 2.3 and, hence, under the maximally extended $SU_I(2) \times SU_{II}(2) \times U(1)$ $N = 4$ superconforaml symmetry. The invariance of this term under the gauge transformations (3.4) can also be easily checked: gauge variation of the integrand is proportional, up to a total spinor derivative, to the constraint on q_2 (or on q_1, if one prefers to deal with the term $\sim q_1^{\underline{a}\alpha} V_{2\,\underline{a}\alpha}$). Finally, we recall (see the end of sect. 2) that the symmetry of (4.19) in the $N = 4$ WZNW limit $m = 0$ becomes the product of two independent $SO(4) \times U(1)$ $N = 4$ superconformal symmetries realized on $q_1(V_1)$ and $q_2(V_2)$, respectively. The super-Liouville term couples q_1 to q_2 and so breaks this product symmetry down to the diagonal $SO(4) \times U(1)$ $N = 4$ SCG.

5 Conclusion

In this paper we re-formulated $N = 4$ WZNW - Liouville systems of ref. [1, 2] in terms of unconstrained superfield prepotentials, constructed the relevant off-shell superfield and component actions and examined their symmetry properties. The next steps could be to promote these actions to local $N = 4$ superconformal symmetry by coupling them to superfield $N = 4$ supergravity, to interpret them as results of fixing superconformal gauge in the appropriate coupled $N = 4$ supergravity - matter systems and to establish their relationship with non-critical $N = 4$ fermionic strings and superstrings on the quantum level. For $SU(2) \times U(1)$ WZNW - Liouville model this range of problems was explored in the component approach in [7]. It remains to reveal implications of the maximally extended $SO(4) \times U(1) \times U(1)$ WZNW - Liouville model (4.19). We believe that the manifestly invariant unconstrained superfield approach developed here is most suitable for analyzing these and, hopefully, more general systems (e.g., those involving Liouville or Toda extensions of the models listed in ref. [5]) from a common point of view.

It is also tempting to relate the superfield actions (4.15), (4.19), via a kind of Hamiltonian reduction, to the superfield actions of some more general super-WZNW sigma models, like this has been done for bosonic Louville and Toda actions in [18]. This would allow to establish a link between $N = 4$ superconformal symmetries of the $N = 4$ WZNW - Liouville systems and generalized super Kac-Moody symmetries of the prototype super-WZNW models.

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Appendix

Here we quote a number of useful identities which can help an inquisitive reader in checking some proofs and statements.

1. Identities following from constraints (2.9)

We limit our consideration to the superfield $q_1^{\underline{\alpha}\alpha}$; for $q_2^{\underline{a}a}$ the same relations are valid, up to the replacement of greek indices by latin ones and vice versa. In subsequent formulas we omit index 1 altogether.

As a consequence of the definition (2.8), $q^{\underline{\alpha}\alpha}$ obeys the relations

$$\bar{\tilde{q}}^{\underline{\alpha}\alpha} = \epsilon^{\underline{\alpha\beta}}\,\epsilon^{\alpha\beta}\,(q^{\underline{\beta}\beta})^{\dagger} = \epsilon^{\underline{\alpha\beta}}\,\epsilon^{\alpha\beta}\,\tilde{q}^{-1}_{\beta\underline{\beta}} = -\tilde{q}^{\underline{\alpha}\alpha} \qquad (A.1)$$

$$\epsilon^{\underline{\alpha\beta}}\,\epsilon^{\alpha\beta} q^{-1}_{\beta\underline{\beta}} = -\exp\{2u\}\, q^{\underline{\alpha}\alpha}. \qquad (A.2)$$

Then, exploiting the constraints (2.9a) and the algebra of spinor derivatives (2.4), we deduce the following identities

$$D_+^{\alpha a}\exp\{2u\} = 2\,\xi_+^{\alpha a}\exp\{2u\}\,, \qquad D_-^{\underline{\alpha a}}\exp\{2u\} = 2\,\xi_-^{\underline{\alpha a}}\exp\{2u\} \qquad (A.3)$$

$$D_-^{\underline{\alpha a}}\,D_+^{\alpha a}\,q_{\underline{\alpha}\alpha} = 16\left(B^{\underline{a}a} + \xi_{-\,\underline{\alpha a}}\,q^{\underline{\alpha}\alpha}\xi_+^{\alpha a}\right) \qquad (A.4)$$

$$D_-^{\underline{\gamma a}}\,\xi_+^{\beta a} = -2\,q^{-1\,\beta\underline{\gamma}}\,B^{\underline{a}a}\,, \qquad D_+^{\alpha a}\,\xi_-^{\underline{\mu a}} = 2\,q^{-1\,\alpha\underline{\mu}}\,B^{\underline{a}a} \qquad (A.5)$$

$$D_-^{\underline{\alpha a}}\,\xi_{-\,\underline{\mu}}^{\;\underline{b}} = i\,\epsilon^{\underline{ab}}\,\partial_- q^{\underline{\alpha}\gamma}q^{-1}_{\gamma\underline{\mu}} - 2\,\xi_-^{\underline{ab}}\,\xi_{-\,\underline{\mu}}^{\;\underline{a}}\,, \qquad D_+^{\alpha a}\,\xi_{+\,\mu}^{\;b} = i\,\epsilon^{ab}\,q_{\mu\underline{\beta}}\partial_+ q^{\underline{\beta}\alpha} - 2\,\xi_+^{ab}\,\xi_{+\,\mu}^{\;a} \qquad (A.6)$$

$$D_{-\,\underline{\lambda a}}\,B^{\underline{a}a} = 2\,\xi_{-\,\underline{\lambda a}}B^{\underline{a}a} + 2i\,q^{\beta}_{\underline{\lambda}}\partial_-\xi_{+\,\beta}^{\;a}\,, \qquad D_{+\,\lambda a}\,B^{\underline{a}a} = 2\,\xi_{+\,\lambda a}\,B^{\underline{a}a} - 2i\,\partial_+\xi_{-\,\underline{\alpha}}^{\;\underline{a}}q^{\underline{\alpha}}_{\lambda} \qquad (A.7)$$

$$D_+^{\gamma(c}\,B^{\underline{a}a)} = -2\,\xi_+^{\gamma(c}\,B^{\underline{a}a)} \qquad (A.8)$$

$$D_\pm^{\gamma(c}\,D_{\pm\,\gamma}^{a)}\exp\{2u\} = 0\,. \qquad (A.9)$$

The proof of these identities follows a common pattern. As an illustration, we derive several relations.

Proof of (A.3):

$$D_+^{\gamma a}\exp\{2u\} = D_+^{\gamma a}\left(-\frac{1}{2}\,q^{-1}_{\lambda\underline{\lambda}}\,q^{-1\,\lambda\underline{\lambda}}\right) = -D_+^{\gamma a}\,q_{\lambda\underline{\lambda}}\,q^{-1\,\lambda\underline{\lambda}}$$

$$= \left(q^{-1}_{\lambda\underline{\sigma}}\,D_+^{\gamma a}\,q^{\underline{\sigma}\sigma}\right)\left(q^{-1}_{\sigma\underline{\lambda}}\,q^{-1\,\lambda\underline{\lambda}}\right) = -\frac{1}{2}\,\xi_{+\,\lambda}^{\;a}\,q^{-1\,\gamma}_{\underline{\lambda}}\,q^{-1\,\lambda\underline{\lambda}} = \frac{1}{2}\exp\{2u\}\,\xi_+^{\gamma a}\,.$$

Proof of (A.4):

$$B^{\underline{a}a} = \frac{1}{16}\,q^{\beta}_{\underline{\alpha}}D_-^{\underline{\alpha a}}\left(q_{\beta\underline{\beta}}\,D_+^{\gamma a}\,q^{\underline{\beta}}_{\gamma}\right) = \frac{1}{16}\,q^{\beta}_{\underline{\alpha}}\,D_-^{\underline{\alpha a}}\,q^{-1}_{\beta\underline{\beta}}\,D_+^{\gamma a}\,q^{\underline{\beta}}_{\gamma}$$

$$+\frac{1}{16}\,q^{\beta}_{\underline{\alpha}}\,q^{-1}_{\beta\underline{\beta}}\,D_-^{\underline{\alpha a}}\,D_+^{\gamma a}\,q^{\underline{\beta}}_{\gamma} = -\xi_{-\,\underline{\sigma}}^{\;\underline{a}}\,q^{\underline{\sigma}\sigma}\xi_{+\,\sigma}^{\;a} + \frac{1}{16}\,D_-^{\underline{\alpha a}}\,D_+^{\gamma a}\,q_{\underline{\alpha}\gamma}\,.$$

Proof of (A.9):

$$D_-^{\underline{\gamma}(\underline{a}}D_{-\,\underline{\gamma}}^{\underline{c})}\exp\{2u\} = 2\,D_-^{\underline{\gamma}(\underline{a}}\left(\exp\{2u\}\,\xi_{-\,\underline{\gamma}}^{\underline{c})}\right)$$

$$= 4\exp\{2u\}\,\xi_-^{\underline{\gamma}(\underline{a}}\,\xi_{-\,\underline{\gamma}}^{\underline{c})} + 2\exp\{2u\}\,D_-^{\underline{\gamma}(\underline{a}}\,\xi_{-\,\underline{\gamma}}^{\underline{c})}$$

$$= 4\exp\{2u\}\,\xi_-^{\underline{\gamma}(\underline{a}}\,\xi_{-\,\underline{\gamma}}^{\underline{c})} - 2i\,\epsilon^{(\underline{ac})}\,\partial_- q^{\underline{\gamma}\gamma}\,q^{-1}_{\gamma\underline{\gamma}} - 4\exp\{2u\}\,\xi_-^{\underline{\gamma}(\underline{c}}\,\xi_{-\,\underline{\gamma}}^{\underline{a})} = 0\,.$$

2. Identities with $N=4$ spinor derivatives

$$D^{\alpha a}\,D^{\beta b} = i\,\epsilon^{\alpha\beta}\,\epsilon^{ab}\partial - \frac{1}{2}\,\epsilon^{\alpha\beta}\,D^{\gamma(a}\,D_{\gamma}^{b)} - \frac{1}{2}\,\epsilon^{ab}\,D^{(\alpha c}\,D_c^{\beta)} \qquad (A.10)$$

$$D^{\gamma(a}D_{\gamma}^{b)}\,D^{(\alpha c}D_c^{\beta)} = 0 \qquad (A.11)$$

$$D^{\alpha(a}\,D^{\beta b)} = -D^{\beta(a}\,D^{\alpha b)}\,. \qquad (A.12)$$

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Суперполевые действия для N = 4 ВЗНВ сигма модели с лиувиллевским взаимодействием

Получены явно суперсимметричные формулировки двухмерной SU(2)xU(1) и SO(4)xU(1)xU(1) N = 4 суперконформной ВЗНВ сигма модели и ее лиувиллевского расширения через неограниченный N = 4 суперполевой препотенциал. Для этих систем построены действия вне массовой оболочки как в N = 4 2D суперпространстве, так и в обычном пространстве и изучена их инвариантность относительно SO(4)xU(1) и SU(2) N = 4 суперконформных симметрий.

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Superfield Actions for N = 4 WZNW - Liouville Systems

We present a manifestly supersymmetric formulation of two-dimensional SU(2)xU(1) and SO(4)xU(1)xU(1) N = 4 superconformal WZNW sigma models as well as their Liouville extensions via unconstrained N = 4 superfield prepotentials. We construct off-shell actions for these systems, both in N = 4 2D superspace and ordinary space, and study the invariance properties of the actions under SO(4)xU(1) and SU(2) N = 4 superconformal symmetries.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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