сообщения
объединенного
института
ядерных
исследований
дубна

E2-91-316

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BEYOND TWO DIMENSIONS

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1991

Conformal invariance has become an essential ingredient in contemporary quantum field theory [1,2]. Conformal field theories can be constructed in any dimension but only for d=2 they exhibit a radically different behaviour. Main between them is the fact that the symmetry group becomes infinite-dimensional. After conveniently parametrise its generators in terms of Fourier components one obtains the familiar Virasoro algebra [3]. To be more precise the group is Vir⊗Vir, with one Virasoro algebra for each space-time direction. The fact that the symmetry group is infinite-dimensional makes the theory an exactly solvable conformal theory. Furthermore, two-dimensional field theories hold several other properties which are missing when formulated into higher-dimensional space-times. Main between them are those properties which make quantum field theory a physically sensible theory.

The ideal situation would be to have a theory formulated from the very beginning in four dimensions and exhibiting the previously mentioned properties: conformal invariance and the appearance of an infinite-dimensional symmetry group. An attempt in this direction was recently done [4] by relying on infinite-dimensional extensions of conformal algebras similar to the Virasoro extension of the little conformal algebra for d=1,2. However, the existence of this infinite-dimensional symmetry group is more a postulate than a consequence of the theory. Here we undertake a radically different approach to obtain the desired behaviour of the theory.

In a conformal field theory the symmetry generators are the conformal Killing vectors. They are solutions of the conformal Killing equation. Only for two-dimensional spaces the solutions are infinitely many giving rise to an infinite-dimensional symmetry group. A closer analysis of the conformal Killing equation shows that this critical dimension is closely related to the rank of the metric. In fact, since the metric is a second-rank tensor, in the conformal Killing equation will appear two terms containing derivatives of the Killing vectors. After contraction with the metric a 2 is contributed which leads to the critical dimension d=2. Therefore, the critical dimension for which the theory exhibits the critical behaviour is determined by the rank of the metric. In order to get a conformal behaviour for higher-dimensions we need a higher-rank metric and a corresponding higher-rank conformal Killing equation.

In order to prove our claim we consider space-times described by higher-rank metrics. Throughout all the work we use algebraic and differential properties of higher-rank forms which are currently under study [5]. We make the analysis for the physically interesting case of rank four but the generalisation to any rank is straightforward. Associated there would be a fourth-rank completely symmetric metric tensor $G_{\mu\nu\lambda\rho}$. This metric is used to raise and lower indices and to make contractions. The second-rank case is exceptional in the sense that the metric takes vectors into vectors, i.e., it is an isomorphism between the tangent and cotangent bundles. In the fourth-rank case however a contravariant vector is mapped into a third-rank covariant tensor. Now the conformal Killing equation involves a fourth-rank metric and therefore 4 terms containg derivatives of the third-rank Killing tensor will appear. After contraction with the metric a 4 is contributed which leads to the critical dimension d=4. One can easily check that in this case the symmetry group is infinite-dimensional.

To our regret, due to the nature of this approach, we must bore the reader by exhibiting some standard and well known results in order to illustrate where the new approach departs from the standard one. Let us start by making some elementary considerations on field theory. We will restricts our considerations to generic fields ϕ^A described by a Lagrangian

$$\mathcal{L} = \mathcal{L}(\phi^A, \phi^A{}_\mu) , \tag{1}$$

where $\phi^A{}_\mu = \partial_\mu \phi^A$. The field equations are

$$\frac{\delta\mathcal{L}}{\delta\phi^A} = \frac{\partial\mathcal{L}}{\partial\phi^A} - d_\mu \Pi_A{}^\mu = 0 , \tag{2}$$

where we have introduced the generalised canonical momentum

$$\Pi_A{}^\mu = \frac{\partial\mathcal{L}}{\partial\phi^A{}_\mu} . \tag{3}$$

The energy-momentum tensor is given by

$$\mathcal{H}^\mu{}_\nu = \phi^A{}_\nu \Pi_A{}^\mu - \delta^\mu_\nu \mathcal{L} , \tag{4}$$

and satisfies the continuity equation

$$d_\mu \mathcal{H}^\mu{}_\nu = - \phi^A{}_\nu \frac{\delta\mathcal{L}}{\delta\phi^A} = 0 . \tag{5}$$

The first comment relevant to our work is in order here. The definition (4), of the energy-momentum tensor, guarantees, through (5), its conservation on-shell. This definition is independent of the existence of a metric. This is what we need in the next stages where we are going to disentangle from the usual second-rank metric.

Let us now make some considerations on conformal field theory. The main properties that a conformal theory must hold are:

C1. Translational invariance, which implies that the energy-momentum tensor $\mathcal{H}^\mu{}_\nu$ is conserved, i.e., eq.(5).

C2. Invariance under scale transformations which implies the existence of the dilaton current. This current can be constructed to be [6]

$$D^\mu = \mathcal{H}^\mu{}_\nu x^\nu . \tag{6}$$

The conservation of D^μ implies that $\mathcal{H}^\mu{}_\nu$ is traceless

$$d_\mu D^\mu = \mathcal{H}^\mu{}_\mu = 0 , \tag{7}$$

where the conservation of $\mathcal{H}^\mu{}_\nu$, eq.(5), has been used.

Now we look for the possibility of constructing further conserved quantities. We concentrate on currents of the form

$$J^\mu = \mathcal{H}^\mu{}_\nu \xi^\nu , \tag{8}$$

then

$$d_\mu J^\mu = \mathcal{H}^\mu{}_\nu d_\mu \xi^\nu = 0 . \tag{9}$$

In order to obtain more information from this equation we need to introduce a further geometrical object allowing us to raise and low indices. To start with we consider the usual second-rank metric but we will see that other, higher-rank, objects can also do the game. A further restriction will be to flat, constant, metrics. The generalisation to the curved case is straightforward and involves only minor technical details.

Let us then consider the usual Minkowski metric $\eta_{\mu\nu}$. Then we define

$$\mathcal{H}^{\mu\nu} = \mathcal{H}^\mu{}_\lambda \eta^{\lambda\nu} , \tag{10}$$

$$\xi^\mu = \eta^{\mu\nu} \xi_\nu . \tag{11}$$

If (10) happens to be symmetric eq.(9) can be written as

$$d_\mu J^\mu = \frac{1}{2} \mathcal{H}^{\mu\nu} (\partial_\mu \xi_\nu + \partial_\nu \xi_\mu) = 0 . \tag{12}$$

Furthermore, if the energy-momentum tensor is traceless the most general solution to (12) is

$$\partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \frac{2}{d} \eta_{\mu\nu} (\eta^{\alpha\beta} \partial_\alpha \xi_\beta) = 0 , \tag{13}$$

i.e., the ξ's are conformal Killing vectors for the metric $\eta_{\mu\nu}$

Now we make some considerations on the solution to eq.(13) for d=2. In two dimensions one can always find null coordinates ζ^+, ζ^-, such that the line element can be written as

$$ds^2 = 2\, d\zeta^+ d\zeta^- . \tag{14}$$

The only non-null components of the metric are

$$\eta_{+-} = \eta^{+-} = 1 . \tag{15}$$

The solutions to eq.(13) are then

$$\xi^+ = f(\zeta^+) \ , \tag{16a}$$

$$\xi^- = g(\zeta^-) \ , \tag{16b}$$

where f and g are arbitrary functions. Only for d=2 the set of solutions (16) is infinite-dimensional. Now we define

$$U_+(f) = \xi^+ \partial_+ = f(\zeta^+) \partial_+ \ , \tag{17a}$$

$$U_-(g) = \xi^- \partial_- = g(\zeta^-) \partial_- \ . \tag{17b}$$

These quantities satisfy the commutation relations

$$\{U_+(f_1), U_+(f_2)\} = U_+(f_1 f_2' - f_2 f_1') \ , \tag{18a}$$

$$\{U_+(f), U_-(g)\} = 0 \ , \tag{18b}$$

$$\{U_-(g_1), U_-(g_2)\} = U_-(g_1 g_2' - g_2 g_1') \ . \tag{18c}$$

Relations (18) are essentially the algebra of two-dimensional diffeomorphisms. After conveniently parametrise them in terms of Fourier components one gets the familiar Virasoro algebra. To be more precise there is one Virasoro algebra for each null direction.

Some observations are in order here. The parametrisation (14) is independent of the signature of the ground manifold. Euclidean and Minkowskian signatures are obtained with

$$\zeta^\pm = u \pm i\ v \ , \tag{19a}$$

$$\zeta^\pm = t \pm x \ , \tag{19b}$$

respectively. Furthermore, only in a flat space one can write the line element as in (14). It will be this the notion of flatness, which we call null flatness, that will be used for the generalisation to higher-rank case.

Let us now generalise the previous results to fourth-rank metrics. Let us start by considering a space-time described by a fourth-rank line element

$$ds^4 = G_{\mu\nu\lambda\rho}\ dx^\mu\ dx^\nu\ dx^\lambda\ dx^\rho \ , \tag{20}$$

where $G_{\mu\nu\lambda\rho}$ is the completely symmetric fourth-rank metric. The determinant is defined by

$$G = \det(G_{\mu\nu\lambda\rho}) = \frac{1}{d!}\ \epsilon^{\mu_1\cdots\mu_d} \cdots \epsilon^{\rho_1\cdots\rho_d}\ G_{\mu_1\cdots\rho_1} \cdots G_{\mu_d\cdots\rho_d} \ . \tag{21}$$

The inverse metric $G^{\mu\nu\lambda\rho}$ is defined by

$$G^{\mu\nu\lambda\rho} = \frac{1}{(d-1)!G}\ \epsilon^{\mu\mu_2\cdots\mu_d} \cdots \epsilon^{\rho\rho_2\cdots\rho_d}\ G_{\mu_2\cdots\rho_2} \cdots G_{\mu_d\cdots\rho_d} \ , \tag{22}$$

and satisfies the relation

$$G^{\mu\alpha\beta\gamma}\ G_{\nu\alpha\beta\gamma} = \delta^\mu_\nu \ . \tag{23}$$

We start by defining currents as in (8). As before they will give rise to conserved quantities if their divergences happen to vanish. In analogy with (10) and (11) we define

$$\mathcal{H}^{\mu\nu\lambda\rho} = \mathcal{H}^\mu{}_\alpha\ G^{\alpha\nu\lambda\rho} \ , \tag{24}$$

$$\xi^\mu = G^{\mu\alpha\beta\gamma}\ \xi_{\alpha\beta\gamma} \ . \tag{25}$$

If the energy-momentum tensor happens to be symmetric eq.(9) can be written as

$$d_\mu J^\mu = \frac{1}{4}\ \mathcal{H}^{\mu\nu\lambda\rho}\ (\partial_\mu \xi_{\nu\lambda\rho} + \partial_\nu \xi_{\lambda\rho\mu} + \partial_\lambda \xi_{\rho\mu\nu} + \partial_\rho \xi_{\lambda\mu\nu}) = 0 \ . \tag{26}$$

Furthermore, if the energy-momentum tensor happens to be traceless we obtain

$$\partial_\mu \xi_{\nu\lambda\rho} + \partial_\nu \xi_{\lambda\rho\mu} + \partial_\lambda \xi_{\rho\mu\nu} + \partial_\rho \xi_{\mu\nu\lambda} - \frac{4}{d}\ G_{\mu\nu\lambda\rho}\ (G^{\alpha\beta\gamma\delta}\ \partial_\alpha \xi_{\beta\gamma\delta}) = 0 \ . \tag{27}$$

This is the obvious generalisation of the equation for the second-rank case, (13).

Comparison of this equation with eq.(13) illustrates the comments at the introduction concerning the appearance of a number of terms equal to the rank of the metric.

Now we make some considerations on the solutions to eq.(27) for d=4. For fourth-rank metrics there is no intuitive notion of flatness as for the second-rank case. In analogy with (14) we assume that flat spaces are described by fourth-rank line elements of the form

$$ds^4 = 24\ d\zeta^1\ d\zeta^2\ d\zeta^3\ d\zeta^4 \ . \tag{28}$$

In this case the only non-null components of the metric are

$$G_{1234} = 1 \ ,$$

$$G^{1234} = \frac{1}{6} \ . \tag{29}$$

The solution to eqs.(27) involves a finite-dimensional piece and an infinite-dimensional one. The infinite-dimensional one is

$$\xi^\mu = f^\mu(\zeta^\mu) \ . \tag{30}$$

where the f's are arbitrary functions of the single coordinate μ. As before we define

$$U_\mu(f) = \xi^\mu\ \partial_\mu = f^\mu(\zeta^\mu)\ \partial_\mu \ . \tag{31}$$

In the previous formulae there is no summation over repeated indices. The previous quantities satisfy the commutation relations

$$\{U_\mu(f_1),\ U_\nu(f_2)\} = \delta_{\mu\nu}\ U_\nu(f_1\ f_2' - f_2\ f_1')\ . \tag{32}$$

As in the two-dimensional case, relations (32) are the algebra of four-dimensional diffeomorphisms. After conveniently parametrise them in terms of Fourier components one gets one Virasoro algebra for each null direction. The full symmetry group is Vir⊗Vir⊗Vir⊗Vir. Furthermore, only in four dimensions the solution to eqs.(27) gives rise to an infinite-dimensional symmetry group.

Therefore, we have succeded into implementing conformal invariance for d=4. We have seen furthermore that the rank of the metric is essential to implement conformal invariance in higher dimensions. It must be furthermore observed that the appearance of the conformal behaviour for some critical dimension is a geometrical property of the base space and therefore model independent. Therefore any attempt at the implementation of conformal invariance in four dimensions by relying only on the second-rank metric is condemned to fail. The next step is of course to construct a realistic model on lines, for example, similar to the Polyakov string. This will be reported separately [7].

Some final speculative remarks. In the eventuality of constructing a sensible conformal field theory in four dimensions, the metric $G_{\mu\nu\lambda\rho}$, in analogy with the second rank case, the string, would be an induced metric, or an effective field. One would be therefore automatically describing the effects of gravitation. The natural question is how this theory would connect with the usual theories of gravitation based on the second-rank metric. In trying to answer this question we have discovered the existence of spaces which are separable in the sense that the metric can be written as $G_{\mu\nu\lambda\rho}=g_{(\mu\nu}h_{\lambda\rho)}$. This does not imply any singular behaviour for the four-metric, as can be verified from (21). The line element is written as

$$ds^4 = (ds^2)_g \otimes (ds^2)_h\ . \tag{33}$$

More surprising is the fact that in the flat (null), low energy, regime the signatures of both pieces are almost fixed. There are only two possibilities: (4,0)⊗(1,3) and (4,0)⊗(2,2). Of course, the first one is to be chosen as that describing the low energy behaviour of such a theory. In this case only the Minkowskian piece would be observable. This would explain why if the universe is described by a fourth-rank metric, at large, low energy, scales it looks Riemannian. The results on this direction will be reported somewhere else [8].

Acknowledegements

This work was done during a visit of the author to the Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, Dubna, Soviet Union. The author thanks the directorate of JINR and LTP for the kind hospitality.

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Received by Publishing Department
on July 8, 1991.