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ON CLASSICAL ELECTRODYNAMICS WITH DISTRIBUTION-VALUED SOURCES

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Классическая электродинамика
с обобщенными источниками

Предлагается новый подход к классической электродинамике. Показано, что в этом подходе нет трудностей стандартной теории.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

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On classical Electrodynamics
with Distribution-Valued Sources

A new formulation of classical electrodynamics is proposed. It is shown that the new formulation is free from the troubles of the conventional formulation.

The investigation has been performed at the Laboratory of the Theoretical Physics, JINR.

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I. Introduction

Maxwell electrodynamics provides a general scheme for all classical and quantum electromagnetic phenomena. In spite of its tremendous success it still suffers from some unresolved defects. Among them the most important one is the long-standing problem of a point charge for which the standard theory predicts an infinite amount of energy. At first glance it might seem that this is only a particular and an elementary problem but we agree with the statements: "that elementary problems should be solved before attacking sophisticated problems" and that "we cannot expect that new geometries and topologies will reveal deeper insights into the physical world if we persist in ignoring the more elementary and relevant down-to-earth problems" /1/ .

Our paper is based on a very careful analysis of the foundation of Maxwell electrodynamics with the special attention to the role of distribution-valued sources and constitutive relations used to close the basic field equations. As a result, we arrive at a new formulation of classical electrodynamics which has many advantages over all previous ones. In particular, our reformulation of electrodynamics is free from all troubles of the Maxwell theory. It is also more general than the Maxwell theory and includes the latter as a particular case.

2. Maxwell electrodynamics with distribution-valued sources

Maxwell theory describes general laws of electromagnetism in terms of four electromagnetic field $\vec{E}(\vec{x},t)$, $\vec{D}(\vec{x},t)$, $\vec{B}(\vec{x},t)$ and $\vec{H}(\vec{x},t)$ connected by a particular system of differential relations called the Maxwell equations. In the rationalized system of units these equations have the form /2/ :

$$\operatorname{rot} \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{2.1}$$

$$\operatorname{div} \vec{B} = 0 \tag{2.2}$$

$$\operatorname{rot} \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{j} \tag{2.3}$$

$$\operatorname{div} \vec{D} = \varrho \qquad (2.4)$$

where $\varrho(\vec{x},t)$ and $\vec{j}(\vec{x},t)$ are the scalar density of charge and the vector density of current, respeotively. To apply the Maxwell equations to the description of any particular electromagnetic situation, we must close the system of differential relations (2.1)-(2.4) using some additional information about the electromagnetic fields. Customarily, this information is supplied by the so-called constitutive relations which are valid only for a particular medium. The constitutive relations describe the response of the medium to the electromagnetic field and they contain all the relevant eleotromagnetic characterization of the medium. In the simplest case of a vacuum the constitutive relations have the form

$$\vec{D} = \varepsilon_0 \vec{E}$$
$$\vec{B} = \mu_0 \vec{H} \qquad (2.5)$$

where ε_0 and μ_0 are the electromagnetic constants of the vacuum, and substituting these relations into (2.3) and (2.4) we get the equations

$$\operatorname{rot} \vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} + \mu_0 \vec{j}$$
$$\operatorname{div} \vec{E} = \frac{1}{\varepsilon_0}\varrho \qquad (2.6)$$

where the relation

$$c^2 \varepsilon_0 \mu_0 = 1 \qquad (2.7)$$

has been used. (c - velocity of light in vaouum). Equations (2.1), (2.2) and (2.6) form a closed Maxwell—Lorentz system of equations for the fields $\vec{E}$ and $\vec{B}$ and are the basis of the Lorentz microscopic electrodynamics [3].

The above-presented scheme works perfectly for oharges and currents for whioh the oorresponding densities ϱ and $\vec{j}$ are represented by smooth functions of space-time coordinates. Unfortunately, it crashes when we try to apply it to problems for which both ϱ and $\vec{j}$ are represented by generalized functions called distributions. In fact, from the Maxwell - Lorentz equations (2.6) it follows that for distribution-valued sources ϱ and $\vec{j}$ the electromagnetio fields $\vec{E}$ and $\vec{B}$ also must be represented by generalized vector--valued functions and the whole Maxwell - Lorentz electrodynamios

must be treated in the mathematical language of distributions. A sufficiently comprehensive treatment of the Maxwell -Lorentz field equations in the framework of the theory of generalized functions has been given, for example, in Ref. 4 . But the discussion of Ref. 4 is physically incomplete because field equations deal only with the linear part of electrodynamics. As it is well-known, in addition to this linear part, electrodynamics contains a lot of important nonlinear expressions, such as, for example, the Lorentz force

$$\vec{F}(t) = \int d^3x \left[\varrho(\vec{x},t)\vec{E}(\vec{x},t) + \vec{j}(\vec{x},t)\times\vec{B}(\vec{x},t)\right], \qquad (2.8)$$

the Poyinting vector

$$\vec{S}(t) = \int d^3x \; \vec{E}(\vec{x},t)\times\vec{H}(\vec{x},t), \qquad (2.8)$$

the Joule heat

$$Q(t) = \int d^3x \; \vec{j}(\vec{x},t)\cdot\vec{E}(\vec{x},t) \qquad (2.10)$$

or the energy balance equation

$$\frac{dW(t)}{dt} = \int d^3x \left(\vec{E}(\vec{x},t)\cdot\frac{\partial \vec{D}(\vec{x},t)}{\partial t} + \vec{H}(\vec{x},t)\cdot\frac{\partial \vec{B}(\vec{x},t)}{\partial t}\right). \qquad (2.11)$$

From the mathematical point of view all these quantities are meaningless for distribution-valued fields and sources because generalized functions cannot be multiplied [5].

Tha lack of a multiplication law for generalized functions is the primary source of all troubles of the conventional treatment of the problems with point charges in the framework of Maxwell electrodynamics. For instance, the point charge q moving along a trajectory $\vec{X}(t)$ is described by the following densities

$$\varrho(\vec{x},t) = q\,\delta^{(3)}(\vec{x}-\vec{X}(t)), \qquad (2.12)$$
$$\vec{j}(\vec{x},t) = q\,\dot{\vec{X}}(t)\,\delta^{(3)}(\vec{x}-\vec{X}(t)).$$

For such sources the Lorentz foroe (2.8) and the Joule heat (2.10) are meaningful quantites only provided the electromagnetic fields $\vec{E}$ and $\vec{B}$, as funotions of the variable $\vec{x}$, have the properties of the test functions used in the theory of generalized functions [5]. However, from the field equations (2.6) it follows that along a trajectory of the point charge these fields are singular functions of the variable $\vec{x}$ and an apparent contradiotion arises. Moreover, the field quantities at infinity behave quite differently from the test functions. The fact that the fields for point charges turn out to be ordinary singular functions should not confuse us because mathematics of the field equations with distribution-valued sources uniquely requires that all these singular functions must be treated in the framework of generalized functions. Therefore, we cannot calculate the energy of the field with the usual formula

$$W = \frac{1}{2}\int d^3x \left(\varepsilon_0 \vec{E}^2(\vec{x},t) + \mu_0 \vec{B}^2(\vec{x},t)\right) \tag{2.13}$$

because it is meaningless for distribution-valued fields. The whole discussion of the problem of infinite energy of the electromagnetic field of a point charge, contained in all text-books on electrodynamics, is based on an inadequate mathematics and has therefore no physical meaning because physics cannot be based on a wrong mathematics.

To see what is really going on, let us observe first that the conclusion about the distribution character of the fields $\vec{E}$ and $\vec{B}$ follows not from the original system of Maxwell equations (2.1)-(2.4) but from the Maxwell - Lorentz equations (2.6). The original Maxwell equations require that for distribution-valued cources only the fields $\vec{D}$ and $\vec{H}$ must be distributions while they leave open the question of the mathematical properties of the fields $\vec{E}$ and $\vec{B}$. We may therefore use this freedom to assign meaning to all non-linear electromagnetic quantities listed above. For this purpose it is sufficient to assume that the fields $\vec{E}$ and $\vec{B}$ always serve as vector-valued test functions for the vector-valued generalized functions $\vec{D}$, $\vec{H}$, $\vec{j}$ and the scalar generalized function ϱ because all physically interesting non-linear electromagnetic quantities by definition are always linear functionals of the fields $\vec{E}$ and $\vec{B}$. The widely spread non-linear dependences on the fields $\vec{E}$ and $\vec{B}$ are always introduced by constitutive relations the meaning of which will be discussed at a moment. Using the notation of the theory of distributions we may rewrite formulas (2.8)-(2.11) in the form:

$$F_j(t) = \langle \varrho_t, E_{jt} \rangle + \sum_{k,l=1}^{3} \varepsilon_{jkl} \langle j_{kt}, B_{lt} \rangle, \tag{2.14}$$

$$S_j(t) = \sum_{k,l=1}^{3} \varepsilon_{jkl} \langle H_{kt}, E_{lt} \rangle, \tag{2.15}$$

$$Q(t) = \sum_{k=1}^{3} \langle j_{kt}, E_{kt} \rangle, \tag{2.16}$$

$$\frac{dW(t)}{dt} = \sum_{k=1}^{3} \left(\langle \frac{\partial D_{kt}}{\partial t}, E_{kt} \rangle + \langle H_{kt}, \frac{\partial B_{kt}}{\partial t} \rangle \right) \equiv \equiv \sum_{k=1}^{3} \left(\langle H_{kt}, \frac{\partial B_{kt}}{\partial t} \rangle - \langle D_{kt}, \frac{\partial E_{kt}}{\partial t} \rangle \right) \tag{2.17}$$

where ε_{jkl} is the three-dimensional Levi - Civita symbol and the bracket $\langle f_t, \varphi_t \rangle$ denotes the value of the generalized function $f_t(\vec{x}) \equiv f(\vec{x},t)$ on the test function $\varphi_t(\vec{x}) \equiv \varphi(\vec{x},t)$ where the time variable is treated as a parameter of both the generalized and test functions. It is now clear that all these formulas are perfectly well-defined for all distribution-valued sources provided the fields $\vec{E}$ and $\vec{B}$ will be treated as test functions of the theory. Obviously, our assumption on the role of the fields $\vec{E}$ and $\vec{B}$ is just opposite to that usually made on these fields in electrodynamics with singular sources. The advantage of our assumption over the usual ones consists in the fact that it assigns meaning to all non-linear quantities which are notoriously ill-defined in other approaches.

The difference in the mathematical properties of the electromagnetic fields, necessary for further development of the theory, is possible only in the framework of the original Maxwell equations and not in the widely used Maxwell- Lorentz vacuum electrodynamics. Since the latter arises from the former after using constitutive relations (2.5) we come to the conclusion that the troubles with distribution-valued sources are not inherent in electrodynamics itself but their source lies in the constitutive relations. In fact, having decided, that the fields $\vec{E}$ and $\vec{B}$ belong to the class of test functions for the distributions $\vec{D}$ and $\vec{H}$ (and for $\vec{j}$ and ϱ, as well) we have lost all possibilities to write relations of the type (2.5) because all relations like that are now meaningless. The efforts to express non-trivial generalized functions in terms of text functions either fail completely or lead to such complicated constructions/5/ that they are deprived of any physical interest. Therefore, in electrodynamics with distribution-valued sources we must reject all constitutive relations which will relate the fields $\vec{D}$ and $\vec{H}$ to the fields $\vec{E}$ and $\vec{B}$. This conclusion excludes however the usual way of closing Maxwell equations. We shall show in the next section how to reformulate the original Maxwell theory so that all constitutive relations between fields may be omitted.

3. Reformulation of Maxwell electrodynamics

In the previous section we have shown that the mathematics of Maxwell electrodynamics with distribution-valued sources is well-defined only if the fields $\vec{E}$ and $\vec{B}$ are mathematically quite different from the fields $\vec{D}$ and $\vec{H}$. Combining this with the fact that also physically the fields $\vec{E}$ and $\vec{B}$ are quite different from the fields $\vec{D}$ and $\vec{H}$, we arrive at the fundamental question: is it really necessary to have that asymmetry at the fundamental level of the theory?

Trying to give an answer to this question we start with the observation that the usual Maxwell macroscopic electrodynamics is not a theory of a single medium but it is a theory of two quite different media. In fact, the fields $\vec{E}$ and $\vec{B}$ are operationally defined only in a vacuum while the fields $\vec{D}$ and $\vec{H}$ describe electromagnetism in a given medium which may be quite different from the classical vacuum. In the usual formulation of classical electrodynamics the classical vacuum, with some hypothetical properties, always serves as a reference medium for all other media. The comparison of fields in a medium with the fields in a vacuum is implemented by the constitutive relations. But the experimental verification of these relations involves many assumptions which often cannot be really verified. Obviously, all that introduces into the theory unnecessary uncertainties and undeterminable restrictions which cannot be even explicitly stated. This circumstance is always neglected in the formulation of Maxwell electrodynamics and it is assumed that this theory is applicable to all electromagnetic phenomena with an absolute accuracy!

The experience from solid state physics, however, unambiguously shows that in many, if not in all, cases it is not reasonable to compare a given medium with the vacuum. On the contrary, it is much more convenient to describe each medium in its own language which reflects the properties of the medium in the most economical way. We should also take into account the fact that our present-day understanding of the notion of a vacuum is quite different from the old-fashioned point of view according to which the vacuum is merely an empty space. The properties of the vacuum depend on the required accuracy of the theory and are quite different at the classical and quantum levels. It is one of the tasks of the theory to predict these properties and therefore at any early stage of the theory we have no right to make any assumption concerning the possible complicated structure of the vacuum. The vacuum by no means may be simpler than any other medium and therefore there is no gain to use it as a reference medium.

The above-presented observation leads us to the following brave idea: the best way of removing the aforementioned asymmetry in the foundation of electrodynamics is to resign from the fields $\vec{E}$ and $\vec{B}$ at all. But, we know that the fields $\vec{D}$ and $\vec{H}$ themselves are not sufficient to describe all electromagnetic phenomena because working only with these fields we cannot extract from the theory all the information on the behaviour of the medium itself. To resolve this problem, let us recall that in Maxwell theory the most general constitutive relations are of the form

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P} \tag{3.1}$$

$$\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M} \tag{3.2}$$

where the two vector fields $\vec{P}(\vec{x},t)$ and $\vec{M}(\vec{x},t)$ describe the polarization and magnetization properties of the medium. In the standard approach /6/ to macroscopic electrodynamics these vectors fields are considered as given quantities and the relations (3.1) and (3.2) are used to eliminate the fields $\vec{D}$ and $\vec{H}$ from the theory. For the remaining fields $\vec{E}$ and $\vec{B}$ we then get the following complete system of field equations:

$$\mathrm{rot}\,\vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \tag{3.3}$$

$$\mathrm{div}\,\vec{B} = 0 \tag{3.4}$$

$$\mathrm{rot}\,\vec{B} - \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} = \mu_0\left(\vec{j} + \frac{\partial \vec{P}}{\partial t} + \mathrm{rot}\,\vec{M}\right) \tag{3.5}$$

$$\mathrm{div}\,\vec{E} = \frac{1}{\varepsilon_0}\left(\varrho - \mathrm{div}\,\vec{P}\right). \tag{3.6}$$

It is however not difficult to see that in this way we arrive at a theory which uses electromagnetic fields in a vacuum but describes electromagnetism in a given medium which has nothing to do with the vacuum. The presence of the matter is taken into account solely as some corrections to the source terms. For sitribution-valued sources such an approach suffers however from the difficulty mentioned in the previous section and therefore we cannot follow it. To find the way of resolving this trouble, let us observe that it is possible to hold the opposite point of view. Since we already know that the fields $\vec{E}$ and $\vec{B}$ should be eliminated from the theory, let us treat the fields $\vec{P}$ and $\vec{M}$ not as given quantities but as fields to be determined from their own set of field equations. To obtain these field equations, we may just use the general constitutive relations (3.1) and (3.2) not as a tool for elimination from the theory of the fields $\vec{D}$ and $\vec{H}$ but as a tool for elimination from the theory of the now unwanted fields $\vec{E}$ and $\vec{B}$. Proceeding in this way we arrive at a reformulation of the Maxwell electrodynamics in which the whole electromagnetism in a given medium will be described in terms of two pairs of vector fields $(\vec{D},\vec{H})$ and $(\vec{P},\vec{M})$ both of which refer solely to the same oonsidered medium without any reference to the vacuum. We shall see below that for distribution-valued sources also the fields $\vec{P}$ and $\vec{M}$ are generalized functions and therefore the above-discussed asymmetry in the foundation of electrodynamics will disappear.

To arrive at the wanted reformulation of Maxwell electrodynamics, let us remind that in an arbitrary medium the charges and currents described by the densities $\varrho(\vec{x},t)$ and $\vec{j}(\vec{x},t)$, respectively, may, in general, induce two other kinds of charges and currents: the polarization charges and ourrents described by the densities $\varrho_P(\vec{x},t)$ and $\vec{j}_P(\vec{x},t)$, respectively, and the magnetization charges and ourrents described by the densities $\varrho_m(\vec{x},t)$ and $\vec{j}_m(\vec{x},t)$, respectively. Note, however, that ϱ_m does not describe magnetic monopoles because the induced magnetic charge density is present in a medium also in the absence of magnetic monopole. The presence of magnetic monopoles will require one more charge and current density but we shall not oonsider this case here. All the charge and current densities obey their own conservation laws in the usual form:

$$\frac{\partial \varrho}{\partial t} + \operatorname{div}\vec{j} = 0\,, \tag{3.7}$$

$$\frac{\partial \varrho_P}{\partial t} + \operatorname{div}\vec{j}_P = 0\,, \tag{3.8}$$

$$\frac{\partial \varrho_m}{\partial t} + \operatorname{div}\vec{j}_m = 0 \tag{3.9}$$

but only the densities ϱ and $\vec{j}$ may be regulated by external sources. The induced densities $(\varrho_P, \vec{j}_P)$ and $(\varrho_m, \vec{j}_m)$ depend on the properties of the medium and may depend on the external sources ϱ and $\vec{j}$. The functional dependenoe of the induced densities on the external densities may be desoribed by a new type of constitutive relations, we shall discuss at a moment.

The complete sets of field equations for the fields $\vec{D}$ and $\vec{H}$ as well as for the fields $\vec{P}$ and $\vec{M}$ will be established from the usual Maxwell equations (2.1)-(2.4), from the constitutive relations (3.1) and (3.2), and from the generalized Helmholtz theorem [7] which says that the most economical way of characterizing any two vector fields $\vec{V}_1(\vec{x},t)$ and $\vec{V}_2(\vec{x},t)$ is given by the following set of field equations:

$$\operatorname{div}\vec{V}_1 = s_1 \tag{3.10}$$

$$\operatorname{div}\vec{V}_2 = s_2 \tag{3.11}$$

$$\operatorname{rot}\vec{V}_1 + \eta_2\frac{\partial \vec{V}_2}{\partial t} = -\eta_2\vec{c}_2 \tag{3.12}$$

$$\operatorname{rot}\vec{V}_2 + \eta_1\frac{\partial \vec{V}_1}{\partial t} = -\eta_1\vec{c}_1 \tag{3.13}$$

where $s_i(\vec{x},t)$ and $c_i(\vec{x},t)$ for i=1,2 are the sources of the fields and η_1 and η_2 are, in general dimensional, constants which determine the character of propagation of the fields $\vec{V}_1(\vec{x},t)$ and $\vec{V}_2(\vec{x},t)$ in space-time. In particular, the hyperbolio propagation law requires that

$$\eta_1\eta_2 = -\frac{1}{c^2} \tag{3.14}$$

where c is the velocity of propagation. It is easy to see that equations (3.10)-(3.13) imply that each pair of souroes $(s_i, \vec{c}_i)$ satisfies its own oonservation law

$$\frac{\partial s_i}{\partial t} + \operatorname{div}\vec{c}_i = 0\,. \tag{3.15}$$

Now, let us take $\vec{V}_1 = \vec{D}$ and $\vec{V}_2 = \vec{H}$. From the Maxwell equations (2.3) and (2.4) we get then

$$s_1 = \varrho \tag{3.16}$$

$$\vec{c}_1 = \vec{j} \tag{3.17}$$

$$\eta_1 = -1\,. \tag{3.18}$$

From the relation (3.14) we get immediately that

$$\eta_2 = \frac{1}{c^2} \tag{3.19}$$

Moreover, by definition we have

$$\operatorname{div} \vec{H} = \varrho_m \tag{3.20}$$

and therefore

$$s_2 = \varrho_m . \tag{3.21}$$

The continuity equations (3.9) and (3.15) lead then to the identification

$$\vec{c}_2 = \vec{j}_m \tag{3.22}$$

and we arrive at the following complete set of Maxwell equations for the fields $\vec{D}$ and $\vec{H}$:

$$\operatorname{div} \vec{D} = \varrho , \tag{3.23}$$

$$\operatorname{div} \vec{H} = \varrho_m , \tag{3.24}$$

$$\operatorname{rot} \vec{D} + \frac{1}{c^2} \frac{\partial \vec{H}}{\partial t} = -\frac{1}{c^2} \vec{j}_m , \tag{3.25}$$

$$\operatorname{rot} \vec{H} - \frac{\partial \vec{D}}{\partial t} = \vec{j} . \tag{3.26}$$

To obtain the corresponding complete set of Maxwell equations for the fields $\vec{P}$ and $\vec{M}$, let us now take $\vec{V}_1 = \vec{P}$ and $\vec{V}_2 = \vec{M}$. From the constitutive relations (3.1) and (3.2) and Maxwell equations (2.2) and (2.3) we get the equalities

$$\operatorname{div} \vec{M} = -\operatorname{div} \vec{H} \tag{3.27}$$

and

$$\operatorname{rot} \vec{D} + \frac{1}{c^2} \frac{\partial \vec{H}}{\partial t} = \operatorname{rot} \vec{P} - \frac{1}{c^2} \frac{\partial \vec{M}}{\partial t} . \tag{3.28}$$

Comparing these equalities with equations (3.24) and (3.25) we come now to the identification

$$s_2 = -\varrho_m , \tag{3.29}$$

$$\vec{c}_2 = -\vec{j}_m , \tag{3.30}$$

$$\eta_2 = -\frac{1}{c^2} . \tag{3.31}$$

From the relation (3.14) it follows now that

$$\eta_1 = 1 . \tag{3.32}$$

This time, by definition we have the relation

$$\operatorname{div} \vec{P} = -\varrho_P \tag{3.33}$$

and therefore

$$s_1 = -\varrho_P . \tag{3.34}$$

The continuity equations (3.8) and (3.15) give then

$$\vec{c}_1 = -\vec{j}_P \tag{3.35}$$

and we arrive at the following set of Maxwell equations for the fields $\vec{P}$ and $\vec{M}$:

$$\operatorname{div} \vec{P} = -\varrho_P , \tag{3.36}$$

$$\operatorname{div} \vec{M} = -\varrho_m , \tag{3.37}$$

$$\operatorname{rot} \vec{P} - \frac{1}{c^2} \frac{\partial \vec{M}}{\partial t} = -\frac{1}{c^2} \vec{j}_m , \tag{3.38}$$

$$\operatorname{rot} \vec{M} + \frac{\partial \vec{P}}{\partial t} = -\vec{j}_P . \tag{3.39}$$

We have already argued that for distribution-valued sources ϱ and $\vec{j}$ the fields $\vec{D}$ and $\vec{H}$ must be distributions. From the field equations (3.24) and (3.25) it follows now that also the induced densities ϱ_m and $\vec{j}_m$ must be distributions. But for distribution--valued sources ϱ_m and $\vec{j}_m$ the field equations (3.37) and (3.38) predict that the fields $\vec{P}$ and $\vec{M}$ are distributions as well. Then, finally, from the field equations (3.36) and (3.39) it follows that the induced sources ϱ_P and $\vec{j}_P$ also must be distributions. Therefore, we see that for distribution-valued sources ϱ and $\vec{j}$ all electromagnetic quantities must be represented by distributions. In our reformulation of the Maxwell theory by eliminating from it the fields $\vec{E}$ and $\vec{B}$ and introducing into it as primary fields the fields $\vec{P}$ and $\vec{M}$ we have not only removed the physical asymmetry between basic fields but also the asymmetry in their mathematical character. All basic electromagnetic fields refer now only to a single given medium and all they are distributions for distribution-valued sources. For ordinary sources represented by ordinary smooth functions we may immediately go back to the standard Maxwell theory by defining the fields $\vec{E}$ and $\vec{B}$ as secondary fields given by the relations

$$\vec{E} = \varepsilon_0^{-1}(\vec{D}-\vec{P}), \tag{3.40}$$

$$\vec{B} = \mu_0(\vec{H}+\vec{M}). \tag{3.41}$$

In our approach, all basic fields are Maxwellian, i.e., the unique fields determined by the Maxwell equations supplemented by the corresponding boundary or initial conditions. We may treat the sources either as given quantities or relate the induced densities to the external ones by some kind of constitutive relations. As all sources are distributions, for non-linear media, these constitutive relations may be written only as convolutions of distributions and not as simple products at the same space-time point because such products are meaningless for distributions. All that belongs to the range of applicability of our formalism, we do not consider in the present paper.

4. Test functions for distribution-valued electromagnetic fields

We have arrived at the reformulation of Maxwell electrodynamics in which all basic electromagnetic quantities are distributions for disctribution-valued sources. To make this theory complete, we must now find the space of test functions for all these distributions and establish the physical meaning of the test functions.

It is well-known that classical electrodynamics has, in general, two interrelated aspects: the first one connects the fields with their sources and the second one describes the action of the electromagnetic field on matter. In our approach the first aspects is contained in the sets of Maxwell equations (3.23)-(3.26) and (3.36)--(3.38) and we pass now to the discussion of the second aspect of electrodynamics.

In classical electrodynamics it is customary to consider the action of the electromagnetic field solely in the framework of classical physics in which a crucial role is played by the Lorentz force (2.8) and the Joule heat (2.10). This approach necessitates the introduction of the fields $\vec{E}$ and $\vec{B}$ which in our scheme are defined by the relations (3.40) and (3.41). However, we have seen that the

fields $\vec{E}$ and $\vec{B}$ should possess different mathematical properties than the fields $\vec{D}$, $\vec{H}$, $\vec{P}$ and $\vec{M}$ have and we arrive at an important restriction on these fields which consists in the following: The difference of the distribution-valued fields $\vec{D}$ and $\vec{P}$ and the sum of the distribution-valued fields $\vec{H}$ and $\vec{M}$ must be smooth test function. This condition obviously introduce a strong correlation between the singularities of the fields which up to now were quite independent. For example, for non-polarizable media we may put

$$\varrho_P = \varrho_m = 0 , \tag{4.1}$$

$$\vec{j}_P = \vec{j}_m = 0$$

and we obtain homogeneous field equations for the fields $\vec{P}$ and $\vec{M}$. It might seem that we may take

$$\vec{P} = \vec{M} = 0 \tag{4.2}$$

as solutions to these equations but in view of the above-formulated restriction on the distribution-valued fields this is impossible for distribution-valued sources because the resulting fields $\vec{D}$ and $\vec{H}$ will be distributions and under the assumption (4.2) there is nothing to compensate the singularities of these fields. The relations (3.40) and (3.41) will not therefore lead to smooth fields $\vec{E}$ and $\vec{B}$ and an apparent contradiction will arise. This shows that the Maxwell electrodynamics possesses the following very important mechanism: the theory does not allow us to make unphysical assumptions. In fact, the physically meaningful electrodynamics with distribution-valued sources must admit that we always may achieve such an accuracy of measuring the properties of the sources that allows us to establish their distributional character. In particular, this means that for δ-type sources of the kind (2.12) we may really locate the charge at a single point with absolute accuracy and we really may neglect all effects connected with a possible spatial spread of the charge. Consequently, we must accept that we may also measure all polarization effects with absolute accuracy. The impossibility of the solution (4.2) means that in the presence of distribution-valued sources all media exhibit polarization effects. This is in sharp contradiction with the usual assumption on the classical vacuum as an empty space. Our discussion shows that the classical vacuum may be approximated by on empty space only for smooth external sources. In the presence of distribution-valued sources it must exhibit some polarization phenomena which one controlled by the singularities of the electromagnetic fields. The classical vacuum gains therefore the properties of a complicated medium what is usually appreciated only at the quantum level. This slightly unexpected result was one of our arguments for rejecting the standard classical vacuum as a reference medium for all other media. The detailed discussion of this problem is however out of the scope of the present paper.

As it is well-known, the classical electromagnetic field acts on matter also in quantum physics where its action is implemented through the principle of local gauge invariance of all material wave equations. In this approach, the electromagnetic interaction is implemented by a four-vector $A_\mu(\vec{x},t)$ (μ =0,1,2,3 and from now on we shall use relativistic notation) that undergoes the gauge transformations

$$A_\mu(\vec{x},t) \rightarrow A'_\mu(\vec{x},t) = A_\mu(\vec{x},t) + \partial_\mu \Lambda(\vec{x},t) \tag{4.3}$$

where $\Lambda(\vec{x},t)$ is an arbitrary smooth function of space-time coordinates. In the standard approach to electrodynamics, the four-vector $A_\mu(\vec{x},t)$ is treated as a potential for the fields $\vec{E}$ and $\vec{B}$ through the relations

$$\begin{aligned} \vec{E} &= -\vec{\nabla} A_0 - \dot{\vec{A}} \\ \vec{B} &= \mathrm{rot}\, \vec{A} \end{aligned} \tag{4.4}$$

but it is also possible to treat just the fields A_μ as primary physical fields [8]. To arrive at a unified point of view, we assume that the action of the classical electromagnetic field on matter is always implemented by some four-vector field $A_\mu(\vec{x},t)$ that may be related to the distribution-valued electromagnetic vectors $\vec{D}, \vec{H}, \vec{P}$ and $\vec{M}$ by some kind of constitutive relations. In the standard electrodynamics, this relations is given by (3.40), (3.41) and (4.4) but in general we may admit other relations as well. The important point is the assumption that the fields $A_\mu(\vec{x},t)$ have all the properties of the test functions for all electromagnetic distributions. This assumption follows not only from the particular relations (3.40), (3.41) and (4.4) but also from the fact that the four-vectors $A_\mu(\vec{x},t)$ always multiply the matter wave functions which are distributions themselves, and distributions may be multiplied only by smooth functions. Only then the wave equations with the covariant derivatives

$$D_\mu = \partial_\mu - i A_\mu(\vec{x},t) \tag{4.5}$$

will be well-defined for distribution-valued wave functions.

In the relativistic notation, the electromagnetic fields $\vec{D}$ and $\vec{H}$ are organized as components of an antisymmetric tensor $H^{\mu\nu}(\vec{x},t)$ and the vectors $\vec{P}$ and $\vec{M}$ enter into another antisymmetric polarization and magnetization tensor $P^{\mu\nu}(\vec{x},t)$. The Maxwell equations for these tensor fields are of the form

$$\begin{aligned} \partial_\mu H^{\mu\nu} &= j^\nu \\ \partial_\mu {}^*H^{\mu\nu} &= j^\nu_m \end{aligned} \tag{4.6}$$

and

$$\begin{aligned} \partial_\mu P^{\mu\nu} &= j^\nu_m \\ \partial_\mu {}^*P^{\mu\nu} &= j^\nu_P \end{aligned} \tag{4.7}$$

where $j^\nu = (\varrho, \vec{j})$, $j^\nu_m = (\varrho_m, \vec{j}_m)$ and $j^\nu_P = (\varrho_P, \vec{j}_P)$ are the corresponding four-currents and the star denotes the Hodge dual of the corresponding tensor. The overall picture of our approach to electrodynamics in the relativistic notation is therefore the following.

We start with some distribution-valued four-current $j^\mu(\vec{x},t)$ that satisfies the continuity equation. In terms of j^μ we express the induced four-currents j^ν_m and j^ν_P by using various constitutive relations which describe the properties of the medium. From Maxwell equations (4.6) and (4.7) supplemented by the corresponding boundary conditions we find the distribution-valued electromagnetic fields $H^{\mu\nu}$ and $P^{\mu\nu}$ defined over the space of four-vector-valued test functions $A_\mu(\vec{x},t)$. The test functions carry the gauge symmetry of electrodynamics given by (4.3) and are related to the distribution-valued fields $H^{\mu\nu}$ and $P^{\mu\nu}$ by

$$H^{\mu\nu} - P^{\mu\nu} = \varepsilon_0 g^{\mu\varrho} g^{\nu\lambda} (\partial_\varrho A_\lambda - \partial_\lambda A_\varrho) \tag{4.8}$$

where $g^{\mu\varrho}$ is the Minkowski metric tensor. Note, however, that the relation (4.8) is typical only of the Maxwell electrodynamics and may be replaced by other relations, which will lead to non-Maxwellian theories. The electromagnetic four-vector A_μ expresses the action of the electromagnetic field on matter either through the Lorentz force or through wave equations. The observables of the theory are obtained as values of the corresponding electromagnetic distributions on a particular test function calculates from (4.8). The usual gauge invariance of electrodynamics requires now invariance of all these observables under the transformations (4.3). Since in our approach A_μ is a physical field and not an artificially introduced potential, the gauge invariance of electromagnetism is now more natural than in the standard approach.

5. Conclusions

We have presented a general scheme for classical electrodynamics that is free from any troubles with distribution-valued sources encountered by the usual approach. The size of the paper does not allow us expound all advantages of our approach; this will be done in a series of subsequent papers started by the present paper. In particular, we shall show that our approach perfectly works for point charges, it gives a new insight into the long-standing problem of the energy-momentum tensor of the electromagnetic field in a medium and has a nice extension to the theory of gravity.

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