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QUANTUM DISCRETE GAUGE MODELS WITH BOSONIC AND FERMIONIC DEGREES OF FREEDOM

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Квантовые дискретные калибровочные модели с бозонными и фермионными степенями свободы

В рамках локализации линейных канонических симметрий представлен общий подход к квантованию дискретных (имеющих конечное число бозонных и фермионных степеней свободы) моделей с квадратичными связями первого рода. Показано, что в данном подходе естественно возникают суперрасширения матричных теорий (например "нульмерные" ортосимплектические матричные теории поля).

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Quantum Discrete Gauge Models with Bosonic and Fermionic Degrees of Freedom

A general approach to quantizing discrete models (i.e. having a finite number of coordinates) with quadratic first-class constraints is presented in the framework of gauging linear canonical symmetries. It is also proposed how a natural superextension of matrix field theories (viz. orthosymplectic "zero dimensional" matrix field theories) might emerge in this approach.

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1. Introduction

Discrete models with quadratic Hamiltonians and constraints are being actively investigated at present (e.g. in context of quantization of symplectic orbits or as some discrete approximations of the string models) and there already exists considerable literature on the subject[1-4]. From the aspect of string theory the interest lies in the fact that all string theories can be considered as infinite dimensional Hamiltonian systems with quadratic constraints. For example, on quantizing the Green-Schwarz superstring in the light-cone gauge one is led to a system with quadratic constraints, where the infinite number of constraints corresponds to an infinite number of ghosts.

The motive of this paper is to present a general approach to discrete models with quadratic constraints in the framework of gauging of linear canonical symmetries[5]. We will also show how some natural super-extension of the ordinary "zero-dimensional" and one-dimensional matrix field theories can be realized in this approach. Accordingly, the plan of the paper is as follows. In section 2 the classical Hamiltonian formulation is set up for fermionic discrete systems with first-class quadratic constraints. Section 3 deals with the quantization problem incorporating ghosts in the usual BRST approach. Gauge models of fermionic discrete "strings" are considered in section 4 and it is shown how a super-extension of the matrix field theories can be obtained in the framework of our approach. Section 5 comprises of our conclusions.

2. Classical Hamiltonian Formulation of Fermionic Discrete Systems

Consider a system described by co-ordinates $z_A = (z_a, z_\alpha)$ and its conjugate momenta $\bar{z}_A = (\bar{z}_a, \bar{z}_\alpha)$ where $(z_a, \bar{z}_a)$ and $(z_\alpha, \bar{z}_\alpha)$ are even and odd variables respectively ($a = 1,2,..,N$; $\alpha = 1,2,..,K$). Introducing a compact notation for sign factors:

$$(-)^A = +1 \quad \text{if } A = a, \qquad (-)^A = -1 \quad \text{if } A = \alpha,$$
$$(-)^{AB} = -1 \quad \text{if } A = \alpha,\ B = \beta, \qquad (-)^{AB} = +1 \quad \text{otherwise},$$

we can write the commutation relations for these variables as

$$z_A z_B = (-)^{AB} z_B z_A, \quad z_A \bar{z}_B = (-)^{AB}\, \bar{z}_B z_A, \quad \bar{z}_A \bar{z}_B = (-)^{AB}\, \bar{z}_B \bar{z}_A . \qquad (1)$$

We remark that $z_a = q_a$, $\bar{z}_a = p^a$ are the standard real coordinates and momenta while $z_\alpha, \bar{z}_\alpha$ are non-hermitian Grassmann variables, see Refs.6. In terms of these variables the action has the form

$$S = \frac{1}{2}\int_0^T dt\ \{\ \bar{z}_A(t)\dot{z}_A(t) - \dot{\bar{z}}_A(t) z_A(t) - H(z,\bar{z})\} \qquad (2)$$

where H is the Hamiltonian and the dot always denotes the t-derivative. The corresponding Poisson superbrackets have the standard form

$$\{X,Y\} = X \frac{\overleftarrow{\partial}}{\partial z_A} \frac{\overrightarrow{\partial}}{\partial \bar{z}_A} Y - (-)^A X \frac{\overleftarrow{\partial}}{\partial \bar{z}_A} \frac{\overrightarrow{\partial}}{\partial z_A} Y . \qquad (3)$$

It is well known that the kinematical part of this action ("truncated" action) is invariant with respect to the rigid linear supercanonical transformations belonging to the supergroup $Osp(2N|2K,\mathbb{R})$ (the even subgroup of which is $O(2K) \otimes Sp(2N)$).

Let us discuss this in more detail. Combining the coordinates and momenta into one supervector, $Z_A = (z_A, \bar{z}_A)$, one can write the infinitesimal linear transformation in the form

$$\delta Z_A = F_{AB} Z_B , \quad \text{i.e.} \qquad (4)$$

$$\delta z_A = F^{11}_{AB}\, z_B + F^{12}_{AB}\, \bar{z}_B , \quad \delta \bar{z}_A = F^{21}_{AB}\, z_B + F^{22}_{AB}\, \bar{z}_B$$

where F_{AB} and F^{ij}_{AB} are supermatrices. The kinematical part of the action (2) is represented as

$$S_0 = \frac{1}{2}\int dt\ Z_A\, C_{AB}\, \dot{Z}_B \qquad (5)$$

with C_{AB} a super skew-symmetric matrix i.e. $C_{AB} = (-)^{AB+1} C_{BA}$. Under these circumstances (4) will be the symmetry transformation of the truncated action (5) if the supermatrix F_{AB} satisfies

$$F^T C + C F = 0 \qquad (6a)$$

with the symbol "T" denoting the supermatrix transposition defined as follows:

$$(F^T)_{AB} = (-)^{(B+A)A} F_{BA} \ ; \quad ((F^T)^T)_{AB} = (-)^{A+B} F_{AB} . \qquad (6b)$$

Using (4) we can write (6a) in the form

$$F^{12}_{AB} = (-)^{A+B+BA} F^{12}_{BA} \ ; \quad F^{21}_{AB} = (-)^{AB} F^{21}_{BA} \ ; \quad F^{11} + (F^{22})^T = 0. \qquad (6c)$$

It is usually convenient to write F in terms of independent c-number matrices $(T^M)_{AB}$ ($M = 1,2,..,\dim Osp(2N|2K) = 2(N+K)^2+N-K$): $F_{AB} = f_M (T^M)_{AB}$. From Eqs.(6) we obtain that T^M obeys the requirement of invariance of the symplectic form defined by the supermatrix C, i.e.

$$(-)^{MA}\, T^M_{AC}\, C_{CB} + C_{AC}\, T^M_{CB} = 0.$$

These matrices are the generators of the $Osp(2N|2K,\mathbb{R})$ supergroup.

To construct gauge models from the action S_0 we choose some subalgebra h of $osp(2N|2K,\mathbb{R})$ in a reducible representation with the generators T^M satisfying

$$[\ T^M, T^N\ \} = T^M T^N - (-)^{MN} T^N T^M = t^{MN}_K\, T^K. \qquad (7)$$

Now considering time-dependent parameters $f_M \to f_M(t)$, introducing "gauge potentials" $A(t)_{AB} = l_M(t)(T^M)_{AB}$, and replacing the t-derivatives by the covariant derivative $\nabla = \partial_t - A$ we obtain the new action

$$S_1 = \frac{1}{2}\int dt\ Z_A\, C_{AB}\, \left(\frac{\partial}{\partial t} \delta_{BB'} - A(t)_{BB'} \right) Z_{B'} \qquad (8)$$

in which the rigid symmetries of the action (5) are localized. The lagrangian in Eq.(8) is invariant under the gauge transformations

$$\delta Z = F(t)Z \; ; \qquad \delta A = \dot{F} + [\, F \, , \, A \,] \tag{9}$$

or, in component notation,

$$\delta Z_{A} = f_{M}(t)(T^{M})_{AB}\, Z_{B} \; ; \; \delta l_{M}(t) = \dot{f}_{M}(t) + f_{L}(t) l_{N}(t) t^{LN}_{M} \tag{10}$$

where $F(t)$ and T^{M} are elements of Lie superalgebra $h \subset g$, g is a subalgebra of $osp(2N|2K,\mathbb{R})$ corresponding to the gauge group G. For possible application to strings we have to consider only maximal subalgebras h i.e. those having the same rank as $osp(2|2K,\mathbb{R})$.

Now the following question is in order: what kind of dynamical system is described by the action (8)? A general variation of the action (8) may be represented as

$$\delta S_1 = \frac{1}{2}\int_0^T dt \; [2(\delta Z \; C \; \nabla Z) - Z \; C \; \delta A \; Z] + \frac{1}{2}[Z \; C \; \delta Z]_{t=0}^{t=T}. \tag{11}$$

The first two terms give the equations of motion

$$\nabla Z = (\, \partial_t - A \,)Z = 0 \tag{12}$$

and the constraints which we discuss later. The last two terms in Eq.(11) determine the boundary conditions, namely the variables $Z(0)$ and $Z(T)$ have to be fixed. These conditions are of course unphysical and the action (8) should be accordingly modified by adding boundary terms so as to give reasonable boundary conditions as discussed below.

In our case, the most natural boundary conditions fix the bosonic canonical coordinates z_a while for fermionic variables one has to fix initial (i) coordinates z_α and final (f) "momenta" $\bar{z}_\alpha$:

$$z_a(0) = z_a^i \, , \qquad z_a(T) = z_a^f \tag{13a}$$

$$z_\alpha(0) = z_\alpha^i \, , \qquad \bar{z}_\alpha(T) = \bar{z}_\alpha^f \tag{13b}$$

The conditions (13b) for the fermionic variables are necessary for

the correct definition of the path-integral quantization[7]; in the context of string theory they have recently been discussed in Ref.8. To accommodate the boundary conditions (13) into the variational principle, we add to the action (8) the boundary terms, thereby defining the following new action

$$S_2 = S_1 + \tfrac{1}{2}[Z_A(T)\bar{Z}_A(T) - \bar{Z}_A(0)Z_A(0)]$$

$$= S_1 + \tfrac{1}{2}[z_a^f \bar{z}_a(T) - z_a \bar{z}_a^i(0)] + \tfrac{1}{2}[z_\alpha(T)\bar{z}_\alpha^f - \bar{z}_\alpha(0) z_\alpha^i]. \tag{14}$$

The variational principle $\delta S_2 = 0$ now gives the equations of motion (12), the constraints, and the boundary conditions (13). The new action can be written in the form

$$S_2 = \int dt \; [\bar{z}_a(t)\dot{z}_a(t) + \tfrac{1}{2}\{\bar{z}_\alpha(t)\dot{z}_\alpha(t) - \dot{\bar{z}}_\alpha(t) z_\alpha(t)\} - l_M T^M] -$$

$$- \tfrac{1}{2}[\bar{z}_\alpha^f z_\alpha(T) + \bar{z}_\alpha(0) z_\alpha^i] \tag{15}$$

where the l_M may be considered as Lagrange multipliers and the related constraints,

$$T^M = \tfrac{1}{2}\, Z_A \, \Gamma^M_{AB} \, Z_B \, , \tag{16}$$

are expressed in terms of the new matrices Γ^M

$$\Gamma^M_{AB} = (-)^{A\,M} \, C_{AB'}(T^M)_{B'B} = - \, T^M_{AB'} \, C_{B'B} = (-)^{AB} \, \Gamma^M_{BA} \, . \tag{17}$$

From the Grassmann parity of the action S_2 it is evident that $\Gamma^M_{AB} = 0$ if $(M) \neq (A) + (B)$. The Poisson superbrackets for the variables Z_A can be written as

$$\{Z_A \, , \, Z_B\} = C^{-1}_{AB}, \qquad (C^{-1}C = 1) \tag{18}$$

Using these we obtain the following commutation relations for the constraints (16)

$$\{T^M, T^N\} = \tfrac{1}{2}\, Z[\Gamma^M \, C^{-1}\Gamma^N - (-)^{MN}\, \Gamma^N \, C^{-1}\Gamma^M]Z = - \, t^{MN}_K \, T^K. \tag{19}$$

We see that the algebra (19) is isomorphic to the algebra (7)

$(T^M \leftrightarrow -T^M)$.

In passing, let us mention that to describe relativistic systems we simply define the relativistic phase superspace by extending $(Z_A, \bar{Z}_A)$ to $(Z^\mu_A, \bar{Z}^\mu_A)$, where μ is the D-dimensional space-time index, $\mu = 0,1,\ldots,D-1$. By contracting these indices one trivially obtains the Lorentz invariant discrete systems with a gauge supergroup which is some subgroup of $Osp(2N|2K,\mathbb{R})$.

Returning to the action S_2, (15), we stress that it differs from S_1 (8) by boundary terms. Therefore S_2 will be invariant under the gauge transformations (9) only if certain boundary conditions on the gauge transformations are fulfilled. To obtain these conditions we make the gauge variation of (14) using Eqs.(4) and (13). From the condition $\delta S_2 = 0$ we obtain

$$F^{12}_{AB}(T) + (-)^{BA} F^{12}_{BA}(T) = F^{21}_{AB}(T) + (-)^{AB+A+B} F^{21}_{BA}(T) = 0;$$

$$(F^{11}(T))^T + F^{22}(T) = 0; \tag{20a}$$

$$F^{21}_{AB}(0) + (-)^{BA} F^{21}_{BA}(0) = F^{12}_{AB}(0) + (-)^{AB+A+B} F^{12}_{BA}(0) = 0;$$

$$(F^{22}(0))^T + F^{11}(0) = 0. \tag{20b}$$

The solution of the equations (6c), (20a) and (20b) can be rewritten as the following conditions on the elements of the matrix F_{AB}

$$F^{21}_{AB}(0) = F^{12}_{AB}(0) = 0;\ (F^{22}(0))^T + F^{11}(0) = 0;\ F^{21}(T) + ((F^{21}(T))^T)^T$$

$$= F^{12}(T) + ((F^{12}(T))^T)^T = F^{JJ}(T) - ((F^{JJ}(T))^T)^T = 0. \tag{21a}$$

i.e. $$F^{21}_{AA}(T) = F^{12}_{AA}(T) = F^{JJ}_{\alpha a}(T) = F^{JJ}_{\alpha a}(T) = 0 \tag{21b}$$

The equations (21b) mean the following: matrices $F^{JJ}_{AB}(T)$ are block diagonal while the matrices $F^{12}_{AB}(T)$ and $F^{21}_{AB}(T)$ contain only

non-vanishing off-diagonal blocks.

These conditions for $t = 0$, $t = T$ reduce the gauge supergroup G to the two supergroups G_i and G_f which rotate the boundaries $(z_a(0), z_\alpha(0)), (z_a(T), \bar{z}_\alpha(T))$ and the coordinates orthogonal to them $(\bar{z}_a(0), \bar{z}_\alpha(0))$, $(\bar{z}_a(T), z_\alpha(T))$ separately. We can interpret these rotations as reparametrizations of the boundary conditions. On the other hand, the conditions (21a,b) are equivalent to certain conditions on the gauge parameters $f_M(t)$ $(t = 0,T)$. It is evidently clear also that G_i and G_f are subgroups of $Osp(N|K,\mathbb{R})$.

The complete system of equations of motion is given by the evolution equations (12) and by the constraints $T^M = 0$. The Cauchy problem for Eq.(12) can be solved as

$$Z(t) = V(t,t_0)\, Z(t_0) \tag{22}$$

$$V(t,t_0) = P \exp\{\int_{t_0}^{t} dt'\, l_M(t') T^M\} \tag{23}$$

From Eq.(19) it is now clear that the existence of the constraints $T^M = 0$ is consistent with (22) because

$$T^M(t) = \tfrac{1}{2} Z(t)\, \Gamma^M Z(t) = \tfrac{1}{2} Z(t_0)\, V^T \Gamma^M V\, Z(t_0) =$$

$$= (\tilde{V}(t,t_0))^M_N\, T^N(t_0) \tag{24}$$

where

$$\tilde{V}(t,t_0) = P \exp\{\int_{t_0}^{t} dt'\, l_M(t')\, \tilde{T}^M\} \tag{25a}$$

$$(\tilde{T}^M)^N_K = t^{MN}_K \tag{25b}$$

The matrix $\tilde{V}$ is an element of the group $G \in Osp(2N|2K)$ in the adjoint representation. For completeness let us note that the finite gauge transformations corresponding to Eqs.(9) and (10) can be represented as

$$\mathcal{Z}(t) \to U(t)\mathcal{Z}(t); \quad \nabla \to U(t)\,\nabla\, U^{-1}(t);$$
$$V(t,t_0) \to U(t)\, V(t,t_0)\, U^{-1}(t) \qquad (26)$$

where the gauge transformation matrix is

$$U(t) = \exp\{f_M(t)T^M\}$$

3.Quantization, Ghosts and BRST

In this section we generalize the results of Ref.2 to the supersymmetric case.

Following the usual rules for quantizing constrained hamiltonian systems[9,10], consider the path-integral representation for the transition amplitude (propagator):

$$\mathfrak{D}[\, z^f_a, \bar{z}^f_\alpha;\ z^i_a, z^i_\alpha] = \int D\mu\, \exp[i\int_0^T dt\, \{\bar{z}_a(t)\, \dot{z}_a(t) + \qquad (27)$$

$$+ \tfrac{1}{2}[\bar{z}_\alpha(t)\dot{z}_\alpha(t) - \dot{\bar{z}}_\alpha(t) z_\alpha(t) - l_M(t)T^M(t)]\} - \tfrac{1}{2}[\bar{z}^f_\alpha z_\alpha(T) + \bar{z}_\alpha(0) z^i_\alpha]]$$

where the integration measure is

$$D\mu = \prod_{0\leq t\leq T} [D\bar{z}_A\, Dz_A\, Dl_M]\, \delta(\bar{z}_\alpha(T) - \bar{z}^f_\alpha)\, \delta(z_a(T) - z^f_a)$$
$$\delta(z_\alpha(0) - z^i_\alpha)\, \delta(z_a(0) - z^i_a)\, [\Delta_{FP}\, \Pi_{gf}] \qquad (28)$$

Here the integration is performed over all Lagrange multipliers $l_M(t)$ and all super-phase-space trajectories $\bar{z}_A(t)$, $z_A(t)$ with fixed variables at the boundaries of the evolution interval (see condition (13)). In accordance with Ref.2, we also include the Faddeev-Popov determinant Δ_{FP} and the gauge-fixing term Π_{gf} in the definition of the integration measure.

We fix the gauge by choosing $l_M(t)$ independent of t

$$l_M(t) = \frac{1}{T}\, \hat{l}_M \qquad (29)$$

In this gauge the evolution matrix $V(T,0)$ is simply $\exp(\hat{l}_M T^M)$, see Eq.(23). If the end point values of $f_M(t)$ vanished, all $\hat{l}_M$ would

be invariant under the gauge transformations (26). In fact, there are residual transformations of $\hat{l}_M$ corresponding to the reparametrization (21a) of the boundary conditions (13):

$$\exp\{\hat{l}_M\, T^M\} \to \exp\{f_M(T)T^M\}\, \exp\{\hat{l}_M T^M\}\, \exp\{f_M(0)T^M\} \qquad (30a)$$

$$\mathcal{Z}(0) \to \exp\{f_M(0)T^M\}\mathcal{Z}(0); \quad \mathcal{Z}(T) \to \exp\{f_M(T)T^M\}\mathcal{Z}(T) \qquad (30b)$$

where the parameters of the transformations $F(T) = f_M(T)T^M$ and $F(0) = f_M(0)T^M$ satisfy (20),(21) and $\exp\{F(T)\} \in G_f$ and $\exp\{F(0)\} \in G_i$. The transformations (30a) are automorphisms of the group G and generate the group $G_i \otimes G_f$. Therefore, the invariant combinations of the parameters $\hat{l}_M$ may be considered as coordinates of the different trajectories (30a). The transformations of the end-point variables are analogous to reparametrizations of the boundary conditions in the propagator of strings in string models (see Ref.11), and the invariant combinations of the parameters $\hat{l}_M$ (coordinates of the trajectories (30a)) correspond to the Teichmueller parameters.

Our gauge condition (29) is implemented by setting

$$\Pi_{gf} = \int d\mu(\hat{l}_M) \prod_{t,M} \delta(l_M(t) - \frac{1}{T}\, \hat{l}_M\,) \qquad (31)$$

where $d\mu(\hat{l}_M)$ is some measure over the Lie supergroup G. Using standard techniques[9] Δ_{FP} is presented in the form:

$$\Delta_{FP} = \mathrm{Ber}(\partial_t \delta^N_M - t^{KN}_M\, l_K) = \mathrm{Ber}(\partial_t - l_K\, \tilde{T}^K)$$
$$= \int D\mu_g\, \exp[i\int_0^T dt\, B(\partial_t - l_K\, \tilde{T}^K)C] \qquad (32)$$

where $D\mu_g$ is an integration measure for the standard ghost variables B^M and C_M, while $\tilde{T}^M$ realize the adjoint representation (25b). The Grassmann parities of B^M and C_M are opposite to those of the gauge potentials l_M, i.e.

$$B^M B^N = (-)^{(M+1)(N+1)} B^N B^M; \qquad C_M C_N = (-)^{(M+1)(N+1)} C_N C_M$$

Note that (B^M, C_M) are analogs of the (b,c) and (β,γ) systems in string theories.

According to usual practice[9,10] we extend the phase space by adding ghost variables and appropriate ghost terms to the action (15) (see equation (32))

$$S_3 = \int_0^T dt\ [\bar{z}_\alpha \dot{z}_\alpha + \tfrac{1}{2}(\bar{z}_\alpha \dot{z}_\alpha - \dot{\bar{z}}_\alpha z_\alpha) + B^M \dot{C}_M - \{l_M B^M, \Omega\}\ ] \qquad (33)$$

where $\Omega = [(-)^N T^N - (-)^N \frac{1}{2} B^L\ t_L^{MN}\ C_M] C_N$ is the standard BRST charge corresponding to the constraints T^M and the Poisson superbrackets for the ghosts are $\{C_M, B^N\} = \delta_M^N$. The action (33) can be obtained by substituting Eq.(32) in Eq.(27) and then collecting the the exponential terms.

The ghost equations of motion as derived from Eq.(33) are

$$\dot{C}_M = l_K\ t_M^{KN}\ C_N\ ; \qquad \dot{B}^M = -\ B^N\ t_N^{KM}\ l_K \qquad (34)$$

can be solved explicitly to give

$$C(t) = \tilde{V}(t,t_0)\ C(t_0); \quad B(t) = B(t_0)\ (\tilde{V}(t,t_0))^{-1} \qquad (35)$$

where $\tilde{V}(t,t_0)$ is defined in Eq.(25a) and thus the spaces $\{B^N, C_M\}$ realize the adjoint representation of the group G. Substituting Eqs.(31)-(33) in Eq.(27) we obtain the more explicit form of the propagator of our system. In all subsequent discussions it is this expression for the propagator that will be our concern.

Now a discussion of the ghost boundary conditions is in order. First note that the action S_3 (33) is invariant under the gauge transformation (10) extended by the transformations of the ghost variables

$$\delta C_M = f_K(t)\ t_M^{KN}\ C_N\ ; \qquad \delta B^M = -\ B^N\ t_N^{KM}\ f_K(t). \qquad (36)$$

Next, consider the fact that at the end-points, $t=0$, $t=T$, the gauge parameters are restricted by the conditions (21a,b). These conditions define supergroups G_i and G_f, the adjoint representations (30) of which are obviously reducible. This means that with respect to the action of the groups G_i and G_f we can extract invariant spaces $\mathcal{I}_\rho, \mathcal{I}_\pi$ in the spaces of ghosts $\mathcal{M} = (C_M(t), B^N(t))$ when $t=0$ or $t=T$ i.e.

$$\mathcal{M}_{t=0} = \mathcal{I}_\rho^i \oplus \mathcal{I}_\pi^i \quad ; \quad \mathcal{M}_{t=T} = \mathcal{I}_\rho^f \oplus \mathcal{I}_\pi^f\ . \qquad (37)$$

We have by definition $\mathcal{I}_\rho^i \xrightarrow{G_i} \mathcal{I}_\rho^i$, $\mathcal{I}_\pi^i \xrightarrow{G_i} \mathcal{I}_\pi^i$ and $\mathcal{I}_\rho^f \xrightarrow{G_f} \mathcal{I}_\rho^f$, $\mathcal{I}_\pi^f \xrightarrow{G_f} \mathcal{I}_\pi^f$. We identify $\mathcal{I}_\rho^i, \mathcal{I}_\rho^f$ with the ghosts' coordinate spaces and $\mathcal{I}_\pi^i$, $\mathcal{I}_\pi^f$ with the ghosts' momentum spaces. The fixing of the variables (at $t=0,T$) of the coordinate spaces $\mathcal{I}_\rho^i$, $\mathcal{I}_\rho^f$ necessitates adding new terms to the action S_3 (33) as was the case in Eq.(14). It is rather difficult to discuss these problems for a general group G. However, in the next section we consider the special case when $G = Osp(N|K,\mathbb{R})_+ \otimes Osp(N|K,\mathbb{R})_-$ (the so-called chiral discrete fermionic "strings"[3]). In this special case it is possible to fix the boundary conditions for ghosts explicitly and perform the further calculation of the functional integral (27).

4.Gauge Models of Fermionic Discrete "Strings".

Consider the case when the gauge group $G = Osp(N|K,\mathbb{R})_+ \oplus Osp(N|K,\mathbb{R})_-$ $(G \in Osp(2N|2K,\mathbb{R}))$ and the first $Osp(N|K,\mathbb{R})_+$ is dual to the second $Osp(N|K,\mathbb{R})_-$ in Cartan's sense. We shall discuss this later. With the above choice of the gauge group we can represent the symplectic matrix C_{AB} (5) in the form[3]

$$C_{AB} = \frac{1}{2}\begin{bmatrix} \mathcal{D}^{-1}_{AB} & 0 \\ 0 & (\mathcal{D}^{-1})^{\tilde{T}}_{AB} \end{bmatrix}; \qquad \mathcal{D}^{\tilde{T}}_{AB} = \mathcal{D}_{BA}. \tag{38}$$

We now introduce new coordinates $(z^+_A, z^-_B) = \mathcal{Z}_A$ in phase space. These chiral variables are represented as linear combinations of canonical variables

$$z_A^{\pm} = (\bar{z}_A \pm \mathcal{D}_{AB}\, z'_B)\ (\pm 1)^{A/2} \tag{39}$$

where $(-)^{A/2} = 1$ if $A = a$, and $(-)^{A/2} = -i$ if $A = \alpha$. This factor is needed to make $z_\alpha^{\pm}$ hermitian.

The chiral Poisson superbrackets for the variables (39) are

$$\{z^+_A, z^+_B\} = 2\mathcal{D}_{AB} = \{z^-_B, z^-_A\}; \qquad \{z^+_A, z^-_B\} = 0 \tag{40}$$

if the matrix $\mathcal{D}_{AB}$ satisfies $\mathcal{D}_{AB} = (-)^{AB+1}\mathcal{D}_{BA}$. It is clear that the supermatrix $\mathcal{D}_{AB}$ can be chosen in the block form

$$\mathcal{D}_{AB} = \begin{bmatrix} \partial_{ab} & 0 \\ 0 & \frac{1}{2}\partial_{\alpha\beta} \end{bmatrix}; \quad \partial_{ab} = -\partial_{ba}\ ;\ \partial_{\alpha\beta} = \partial_{\beta\alpha}. \tag{41}$$

The action of the group $G = Osp(N|K,\mathbb{R})_+ \otimes Osp(N|K,\mathbb{R})_-$ on (z^+_A, z^-_B) is such that $Osp(N|K,\mathbb{R})_+$ rotates z^+ while $Osp(N|K,\mathbb{R})_-$ rotates z^-.

With the above choice for the group G and using Eqs.(38), (39) and (41) the action S_2 (15) can be rewritten as (for the boundary conditions (13)):

$$S_2 = \int_0^T dt\ [\bar{z}_a(t)\dot{z}_a(t) + \tfrac{1}{2}\{z^+_\alpha\, \partial^{-1}_{\alpha\beta}\, \dot{z}^+_\beta + z^-_\alpha\, \partial^{-1}_{\alpha\beta}\, \dot{z}^-_\beta\} - l^+_M\, T_+{}^M - l^-_M\, T^M_-] - \tfrac{1}{2}[\bar{z}^f_\alpha z_\alpha(T) + \bar{z}_\alpha(0) z^i_\alpha]. \tag{42}$$

Here the constraints

$$T^{\ M}_{\pm} = \tfrac{1}{4} z_A^{\ \pm}\, \Gamma^{\pm\ M}_{AB}\, z_B^{\ \pm} \tag{43}$$

are expressed in terms of the matrices $\Gamma^{\pm M}$ as

$$\Gamma^{\pm\ M}_{AB} \equiv -(T^{\pm M})_{AC}\, (\mathcal{D}^{\ \pm})^{-1}_{CB}\ ; \qquad (T^{\pm\ M})_{BA} = -\,\Gamma^{\pm\ M}_{BC}\, \mathcal{D}^{\ \pm}_{CA} \tag{44}$$

where $\mathcal{D}^+ = \mathcal{D}_{AB}$, and $\mathcal{D}^- = \mathcal{D}^{\tilde{T}}_{AB}$. In this notation we can rewrite Eqs.(40) in the form

$$\{z^\pm_A, z^\pm_B\} = 2\mathcal{D}^\pm_{AB}\ ; \qquad \{z^+_A, z^-_B\} = 0.$$

The matrices $\{T^{\pm M}\}$ generate two Lie superalgebras of the group $Osp(N|K,\mathbb{R})_+ \otimes Osp(N|K,\mathbb{R})_-$:

$$[T^{\pm\ M}, T^{\pm\ N}]_{\pm} = (\pm)^{MN}\, t^{MN}_K\, T^{\pm\ K}. \tag{45}$$

We also mention that in (42) we have introduced the Lagrange multipliers $l^\pm_M$ related to the gauge fields $A^\pm = l^\pm_M T^{\pm M}$ (see section 2).

It is also seen from (45) that the algebras T^+ and T^- are distinguished only by their signs in the anticommutator. Such algebras are called dual in Cartan's sense[12] and generate the self-dual supergroup $G = Osp(N|K,\mathbb{R})_+ \otimes Osp(N|K,\mathbb{R})_-$. It is rather interesting to note that the fermionic string is based on the self-dual product of two superconformal groups while the N=2 Green-Schwarz superstring can be constructed as a σ-model on the self-dual product of two N=1 supertranslation groups[13].

One should bear in mind the following symmetric properties of the Γ-matrices

$$\Gamma^{\pm\ M}_{AB} = (-)^{AB+A+B}\, \Gamma^{\mp\ M}_{AB} = (-)^{AB}\, \Gamma^{\pm\ M}_{BA}$$
$$\Gamma^{\pm\ M}_{AB} = 0 \ \text{ iff } \ (M) \neq (A) + (B). \tag{46}$$

The Poisson brackets for $T^M_\pm$ are derived in full analogy with (19) as:

$$\{\ T^M_\pm\ ,\ T^N_\pm\ \} = -(\pm)^{MN}\, t^{MN}_K\, T^K_\pm \quad ; \quad \{\ T^M_+\ ,\ T^N_-\ \} = 0. \tag{47}$$

Returning to the Hamiltonian action S_2 (42) we emphasize that S_2 is invariant under the gauge transformations

$$\delta z_A^{\ \pm} = \{\ f_M^{\ \pm}(t) T^{\pm\ M}_{AB}\}\ z_B^{\ \pm} = F^\pm_{AB}\ z_B^{\ \pm}$$
$$\delta A^\pm = \dot{F}^\pm + [\ F^\pm\ ,\ A^\pm\ ] \tag{48}$$

if and only if the following boundary conditions are fulfilled (compare with Eqs.(20), (21))

$$f_m^{\ +}(T) - f_m^{\ -}(T) = 0 \quad ; \quad f_m^{\ +}(0) - f_m^{\ -}(0) = 0; \tag{49a}$$

$$f_\mu^{\ +}(T) + i f_\mu^{\ -}(T) = 0 \quad ; \quad f_\mu^{\ -}(0) + i f_\mu^{\ +}(0) = 0. \tag{49b}$$

The conditions (49a) are identical to the boundary conditions in the bosonic discrete string models[2], and are analogous to the corresponding conditions in the bosonic string theory[14]. The complete system of equations of motion is given by the evolution equations (12) (rewritten for the present case, $G = \mathrm{Osp}(N|K,\mathbb{R})_+ \otimes \mathrm{Osp}(N|K,\mathbb{R})_-$) and by the constraints $T_\pm^M = 0$. As in the general case (22), (23) the Cauchy problem can be formally solved as

$$z^{\ \pm}(t) = V_\pm(t,t_0)\, z^{\ \pm}(t_0)$$

$$V_\pm(t,t_0) = \mathrm{Pexp}\{\int_{t_0}^{t} dt'\, l_M^\pm(t')T^{\pm M}\}. \tag{50}$$

The finite gauge transformations corresponding to Eqs.(26) can be represented as

$$z^{\ \pm}(t) \to U_\pm(t)\, z^{\ \pm}(t)$$

$$V_\pm(t,t_0) \to U_\pm(t)V_\pm(t,t_0)U_\pm^{-1}(t_0); \quad U_\pm(t) = \exp\{f_M^{\ \pm}(t)T^{\pm M}\}. \tag{51}$$

Consider now the quantization of the above described model. We shall be using essentially the results of section 3. The functional integral is

$$\mathfrak{D}_{fi} = \int d\hat{l}_M^+ d\hat{l}_M^- \int Dl_M^+ Dl_M^- \prod_{t,M} \delta(l_M^+(t) - (1/T)\hat{l}_M^+)\ \delta(l_M^-(t) - (1/T)\hat{l}_M^-) K_{fi}, \tag{52a}$$

$$K_{fi} = \int D\tilde{\mu}\ \exp\ (i\ S_3). \tag{52b}$$

Here K_{fi} is the relevant heat kernel and the measure $D\tilde{\mu}$ corresponds to integration over all paths $X(t)$ in the extended phase space $X = (z_A, \bar{z}_A, C_M^\pm, B_\pm^M)$ with fixed end-points in the

coordinate space (see Eqs.(13)) and in the ghost coordinate space.

The action S_3 in (52b) is the ghost extension of the action S_2 in Eq.(42). Hence with the help of the expression (33) we can rewrite our action in the form (up to a ghost boundary term) as

$$\tilde{S}_3 = \int_0^T dt\ [\bar{z}_a(t)\dot{z}_a(t) + \tfrac{1}{4}[z_\alpha^+\ D_{\alpha\beta}^{-1}\ \dot{z}_\beta^+ + z_\alpha^-\ D_{\alpha\beta}^{-1}\ \dot{z}_\beta^-] + (B_+^M\dot{C}_M^+ + B_-^M\dot{C}_M^-)$$

$$- \{l_M^+ B_+^M,\ \Omega^+\} - \{l_M^- B_-^M,\ \Omega^-\}] - \tfrac{1}{2}[\bar{z}_\alpha^{\ f} z_\alpha(T) + \bar{z}_\alpha(0) z_\alpha^{\ i}] \tag{53}$$

where $\Omega^\pm = (-)^N[T_\pm^N - (\pm)^{MN}\ \tfrac{1}{2}B_\pm^L\ t_L^{MN}\ C_M^\pm]\ C_N^\pm$ are the standard BRST charges corresponding to our constraints $T_\pm^N$ and the Poisson superbrackets for the ghosts are

$$\{C_N^\pm,\ B_\pm^M\} = \delta_N^M \quad ; \quad \{C_N^\pm,\ B_\mp^M\} = 0. \tag{54}$$

The ghost equations of motion

$$\dot{C}_M^\pm = (\pm)^{NL} l_N^\pm\ t_M^{NL} C_L^\pm \quad ; \quad \dot{B}_\pm^M = -(\pm)^{MN} B_\pm^L\ l_N^\pm\ t_L^{NM} \tag{55}$$

can be solved explicitly to give

$$C^\pm(t) = \tilde{V}_\pm(t,t_0)\ C^\pm(t_0); \quad B_\pm(t) = B_\pm(t_0)\ [\tilde{V}_\pm(t,t_0)]^{-1}$$

$$\tilde{V}_\pm(t,t_0) = \mathrm{Pexp}\{\int_{t_0}^{t} dt'\, l_M^\pm(t')\ \tilde{T}^{\pm\ M}\} \tag{56}$$

where $(\tilde{T}^{\pm M})_N^L = (\pm)^{ML} t_N^{ML}$ are the generators of the two dual gauge supergroups in the adjoint representation. It is worth mentioning here that the action $\tilde{S}_3$ (see Eq.(53)) is invariant under the gauge transformations (48) and (compare with Eq.(55))

$$\delta C_M^\pm = (\pm)^{NL} f_N^{\ \pm}\ t_M^{NL}\ C_L^\pm \quad ; \quad \delta B_\pm^M = -(\pm)^{MN}\ B_\pm^L\ f_N^{\ \pm}\ t_L^{NM} \tag{57}$$

In order to fix the boundary conditions for ghosts it is appropriate now to change the chiral ghost variables $B_\pm^M$ and $C_M^\pm$ to standard canonical coordinates $(\rho_M, \bar{\rho}^M)$ and momenta $(\pi^M, \bar{\pi}_M)$ by using the following linear canonical supertransformations:

$$B_\pm^M = \mp\ a_\pm^M\ \bar{\rho}^M + b_\pm^M\ \pi^M; \quad C_M^\pm = a_\mp^M\ \rho_M \mp (-)^M\ b_\mp^M\ \bar{\pi}_M \tag{58}$$

(there is no summation over M). It is necessary to have

$$a_+^M\, b_-^M + b_+^M\, a_-^M = 1 \tag{59a}$$

so as to obtain the canonical Poisson superbrackets related to (54) as:

$$\{\rho_M\ ,\ \pi^N\} = \{\bar\rho^{\ N},\ \bar\pi_M\} = \delta_M^N\ . \tag{59b}$$

In accordance with the boundary conditions (49a,b) the spaces $\mathcal{I}^{i,f}_{(\rho,\bar\rho)}$ and $\mathcal{I}^{i,f}_{(\pi,\bar\pi)}$ must be invariant under the gauge transformations G_i and G_f. Then, consider the transformations (57) for coordinates $\rho,\bar\rho$ and momenta $\pi,\bar\pi$:

$$\begin{aligned}\delta\bar\rho^M &= \delta(-\,b_-^M\, B_+^M + b_+^M\, B_-^M\ )\\ &= -\ (\ b_-^M\, a_+^L\, f_N^{\ +} + (-)^{MN}\, b_+^M\, a_-^L\, f_N^{\ -}\ )\ \bar\rho^L\ t_L^{NM}\\ &\quad +\ (\ b_-^M\, b_+^L\, f_N^{\ +} - (-)^{MN}\, b_+^M\, b_-^L\, f_N^{\ -}\ )\ \pi^L\ t_L^{NM}\end{aligned} \tag{60a}$$

$$\begin{aligned}\delta\rho_M &= \delta(\ b_+^M\, C_M^+ + b_-^M\, C_M^-\ )\\ &= (\ b_+^M\, a_-^L\, f_N^{\ +} + (-)^{NL}\, b_-^M\, a_+^L\, f_N^{\ -}\ )\ \rho_L\ t_M^{NL}\\ &\quad +((-)^{L+1} b_+^M\, b_-^L\, f_N^{\ +} + (-)^{NL+L} b_-^M\, b_+^L\, f_N^{\ -})\ \bar\pi_L\ t_M^{NL}\end{aligned} \tag{60b}$$

$$\begin{aligned}\delta\pi^M &= \delta(\ a_-^M\, B_+^M + a_+^M\, B_-^M\ )\\ &= (\ a_-^M\, a_+^L\, f_N^{\ +} - (-)^{MN}\, a_+^M\, a_-^L\, f_N^{\ -}\ )\ \bar\rho^L\ t_L^{NM}\\ &\quad -(\ a_-^M\, b_+^L\, f_N^{\ +} + (-)^{MN}\, a_+^M\, b_-^L\, f_N^{\ -}\ )\ \pi^L\ t_L^{NM}\end{aligned} \tag{60c}$$

$$\begin{aligned}\delta\bar\pi_M &= (-)^M\, \delta(-\,a_+^M\, C_M^+ + a_-^M\, C_M^-\ )\\ &= ((-)^{M+1} a_+^M\, a_-^L\, f_N^{\ +} + (-)^{M+NL} a_-^M\, a_+^L\, f_N^{\ -})\ \rho^L\ t_M^{NL}\\ &\quad +\ ((-)^{M+L} a_+^M\, b_-^L\, f_N^{\ +} + (-)^{M+L+NL} a_-^M\, b_+^L\, f_N^{\ -})\,\bar\pi_L\ t_M^{NL}\end{aligned} \tag{60d}$$

When t=0 and t=T, the transformations (60) must leave the spaces $\{\rho,\bar\rho\}$ and $\{\pi,\bar\pi\}$ as invariant i.e. $\delta\bar\rho \propto \bar\rho$, $\delta\rho \propto \rho$, $\delta\pi \propto \pi$, $\delta\bar\pi \propto \bar\pi$. This statement when combined with (49a,b) leads to the following conditions:

$$\begin{aligned} t = 0,\ T :&\quad b_-^m b_+^n = b_-^n b_+^m\ ;\quad b_-^\mu b_+^\lambda = b_+^\mu b_-^\lambda\ ;\\ &\quad a_-^m a_+^n = a_+^m a_-^n\ ;\quad a_-^\mu a_+^\nu = a_-^\nu a_+^\mu\ ;\\ t = 0 :&\quad b_-^\mu b_+^l = i\ b_+^\mu b_-^l\ ;\quad a_+^m a_-^\lambda = i\ a_-^m a_+^\lambda\ ;\\ t = T :&\quad b_+^\mu b_-^l = i\ b_-^\mu b_+^l\ ;\quad a_-^m a_+^\lambda = i\ a_+^m a_-^\lambda\ .\end{aligned} \tag{61}$$

Note that the constant vectors $a_\pm^M$ and $b_\pm^L$ in (58) are different for the t=0 and for the t=T cases. Taking into account the condition (59a) we solve (61) and obtain the general solution

$$\begin{aligned} t= 0,T &: a_\pm^m = \varkappa^m a_\pm\ ;\ b_\pm^m = (1/\varkappa^m) b_\pm\ ;\ a_+b_- + b_+a_- = 1\\ t= 0 &: a_\pm^\lambda = (\mp)^{1/2}\,\varkappa^\lambda\ a_\pm\ ;\quad b_\pm^\lambda = [(\pm)^{1/2}\varkappa^\lambda]^{-1} b_\pm\\ t= T &: a_\pm^\lambda = (\pm)^{1/2}\,\varkappa^\lambda\ a_\pm\ ;\quad b_\pm^\lambda = [(\mp)^{1/2}\varkappa^\lambda]^{-1} b_\pm\end{aligned} \tag{62}$$

where $\varkappa^M$ is an arbitrary constant vector and we have used the convention $(-1)^{1/2}=-i$. The solution (62) may be equivalently written as:

$$\begin{aligned} t = 0 &: a_\pm^M = (\mp)^{M/2}\ \varkappa^M\ a_\pm\ ;\quad b_\pm^M = [(\pm)^{M/2}\varkappa^M]^{-1} b_\pm\\ t = T &: a_\pm^M = (\pm)^{M/2}\ \varkappa^M\ a_\pm\ ;\quad b_\pm^M = [(\mp)^{M/2}\varkappa^M]^{-1} b_\pm\end{aligned} \tag{63a}$$

with $\varkappa^M$ some arbitrary constant vector and $a_+b_- + b_+a_- = 1$. Note that we are free to redefine the ghost coordinates and momenta so that $\varkappa^M$=1. The boundary conditions for the ghosts are chosen in the form:

$$\bar\rho^M(T) = \bar\rho^{fM};\quad \bar\rho^M(0) = \bar\rho^{iM};\quad \rho_M(T) = \rho_M^f;\quad \rho_M(0) = \rho_M^i \tag{63b}$$

Thus, we arrive at the correct action $\tilde{\tilde S}_3$ with the appropriate ghost boundary conditions according to the equations (58), (63):

$$\tilde{\tilde S}_3 = \tilde S_3 + [\ \bar\pi_M(T)\ \bar\rho^M(T) - \bar\pi_M(0)\ \bar\rho^M(0)\ ]. \tag{64}$$

Substituting the solutions (50),(56) of the equations of motion in the action (64) we obtain the classical action

$$S^{cl}= \tfrac{1}{2}[\ \bar z_\alpha(T)z^f_\alpha - \bar z_\alpha(0)z^i_\alpha\ ] - \tfrac{1}{2}[\ \bar z^f_\alpha z_\alpha(T) + \bar z_\alpha(0)z^i_\alpha\ ]$$
$$+ [\ \bar\pi_M(T)\ \bar\rho^{fM} - \bar\pi_M(0)\ \bar\rho^{iM}\ ]. \qquad (65)$$

It now becomes necessary to express the coordinates $\bar z_\alpha(T)$, $\bar z_\alpha(0)$, $z_\alpha(T)$, $\bar z_\alpha(0)$, $\bar\pi_M(T)$ and $\bar\pi_M(0)$ into Eq.(65) in terms of boundary variables (13),(63b). To achieve this it is convenient to introduce new variables

$$Y_A = (z_\alpha\ ,\ D^{-1}_{\alpha\beta}\ \bar z_\beta)\ ; \qquad \bar Y_A = (\bar z_A\ ,\ D_{\alpha\beta}\ z_\beta) \qquad (66)$$

and we have in terms of these variables

$$z^+_A = \bar Y_A + D_{AB}\ Y_B\ ; \qquad z^-_A = (i)^A\ (\bar Y_A - D_{AB}\ Y_B);$$
$$Y_A(T) = Y^f_A \qquad \text{(see eq.(13))}. \qquad (67)$$

Then (65) can be rewritten in the form

$$S^{cl} = \tfrac{1}{2}[\bar Y_A(T)Y^f_A - \bar z_A(0)z^i_A] + [\bar\pi_M(T)\bar\rho^{fM} - \bar\pi_M(0)\bar\rho^{iM}]. \qquad (68)$$

The Cauchy solutions (50) can be recast as

$$\bar Y(T) + D\ Y^f = V_+\ (\ \bar z(0) + D\ z^i\ ),$$
$$I^{-1}(\ \bar Y(T) - D\ Y^f) = V_-\ I(\ \bar z(0) - D\ z^i\ ), \qquad (69)$$

where $I_{AB} = (-i)^A\delta_{AB} = (-1)^{A/2}\delta_{AB}$, $V_\pm = \mathrm{Pexp}\{\int_0^T dt' f_M^{\ \pm}(t')T^M\}$. We also use $J_{AB} = (-)^A\delta_{AB}$ so that $I^{-1} = IJ$ and $J^2 = 1$. Matrices I and J are useful in the algebra of antioommuting variables (see the first reference in Refs.12).

Accordingly with Eqs.(63a) the solutions (56) now become:

$$a_-\ \tilde I\ \rho^f - b_-\ \tilde J\ \bar\pi(T) = \tilde V_+\ (a_-\ \rho^i - b_-\ \tilde I\ \bar\pi(0)),$$
$$a_+\ \rho^f + b_+\ \tilde I\ \bar\pi(T) = \tilde V_-\ (a_+\ \tilde I\ \rho^i + b_+\ \tilde J\ \bar\pi(0)) \qquad (70)$$

with $\tilde I_{MN} = (-i)^M\delta_{MN}$, $\tilde J_{MN} = (-)^M\delta_{MN}$, $\tilde V_\pm = \mathrm{Pexp}\{\int_0^T dt' f^\pm_M(t')\tilde T^{\pm\ M}\}$.

Equations (69) and (70) give the following relations

$$\bar z(0) = \frac{2}{V\ -\ IV\ I}[D\ Y^f - \tfrac{1}{2}(\ V_+ + IV_-I\ )\ D\ z^i],$$

$$\bar Y(T) = [V_+ + IV_-I]\frac{1}{V_+ - IV_-I}\ DY^f - V_+\ \frac{2}{V_+ - IV_-I}\ IV_-I\ Dz^i, \qquad (71)$$

$$\bar\pi(0) = \frac{1}{\tilde I^{-1}\tilde V_+\tilde I - \tilde V_-\tilde J}\ [(C_-\tilde I^{-1}\tilde V_+ + C_+\tilde V_-\tilde I)\rho^i - (C_+ + C_-)\rho^f\ ],$$

$$\bar\pi(T) = \frac{1}{\tilde I^{-1}\tilde V_-^{-1}\tilde I - \tilde V_+^{-1}\tilde J}\ [\rho^i(C_+ + C_-) - (C_-\tilde V_+^{-1}\tilde I + C_+\tilde I^{-1}\tilde V_-^{-1})\rho^f],$$

where $C_\pm = a_\pm/b_\pm$. Using (71) in (68) one has for S^{cl}

$$S^{cl} = \tfrac{1}{2}[\ Y^fJ\ V_+ + IV_-I)\ \frac{1}{V_+ - IV_-I}\ D\ Y^f$$
$$-\ Y^fJ\ \frac{2}{(IV_-I)^{-1} - V_+^{-1}}\ D\ z^i - z^iJ\ \frac{2}{V_+ - IV_-I}\ D\ Y^f$$
$$+\ z^iJ\ \frac{1}{V_+ - I\ V_-I}\ (V_+ + IV_-I)\ D\ z^i\ ]$$
$$+\ [\ \bar\rho^f\ (C_-\ \tilde I\ \tilde V_-\tilde I + C_+\ \tilde V_+)\ \frac{1}{\tilde V_+ - \tilde I\ \tilde V_-\tilde I}\ \tilde I\ \rho^f \qquad (72)$$
$$-\ \bar\rho^f\ \frac{(\ C_+ + C_-\ )}{(\tilde I\tilde V_-\tilde I)^{-1} - \tilde V_+^{-1}}\ \rho^i - \bar\rho^i\ \tilde I\ \frac{(\ C_+ + C_-\ )}{\tilde V_+ - \tilde I\ \tilde V_-\ \tilde I}\ \tilde I\ \rho^f$$
$$+\ \bar\rho^i\ \tilde I\ \frac{1}{\tilde V_+ - \tilde I\ \tilde V_-\tilde I}\ (C_-\tilde V_+ + C_+\tilde I\tilde V_-\tilde I)\ \rho^i\ ].$$

In order to simplify this expression one can redefine $\bar\rho^i\tilde I \to \rho^i$ and $\tilde I\rho^f \to \rho^f$. Then, using the identity $YJDY = 0$ we can rewrite (72) in the concise form

$$S^{cl} = (Y^f I^{-1} V\_I - z^i J) \frac{1}{V_+ - IV\_I} (DY^f - V_+ D z^i)$$

$$+ [C_+ (\bar{\rho}^f \tilde{I} \rho^f - \bar{\rho}^i \tilde{I} \rho^i) \tag{72a}$$

$$+ (C_+ + C_-)(\bar{\rho}^f \tilde{I}\tilde{V}\_\tilde{I} - \bar{\rho}^i \tilde{I}) \frac{1}{\tilde{V}_+ - \tilde{I}\tilde{V}\_\tilde{I}} (\tilde{I} \rho^f - \tilde{V} \rho^i).$$

The heat kernel K_{fi} in (52) is now rewritten in the form:

$$K_{fi}(Y^i_A, \bar{\rho}^{fM}, \rho^f_M, z^i_A, \bar{\rho}^{iM}, \rho^i_M) = Z\exp(i\, S^{cl}) \tag{73}$$

To find the factor Z we use the differential equations on K_{fi}. To derive these equations first note that

$$K_{fi} = \langle Y^f, \bar{\rho}^f, \rho^f | \mathrm{P} \exp\{-i\int_0^T dt' \mathcal{H}(t')\} | z^i, \bar{\rho}^i, \rho^i \rangle \tag{74}$$

where the operator $\mathcal{H}(t')$ has the form

$$\mathcal{H}(t') = l^+_M T^M_+ + l^-_M T^M_- + B^M_+ l^+_N t^{NL}_M C^+_L - B^M_- l^-_N t^{LN}_M C^-_L . \tag{75}$$

The momentum variables $\bar{z}_A$, π^N, $\bar{\pi}_M$, are operators here and using the standard rule for quantizing dynamical variables i.e. $[\ ,\ ] = i\{\ ,\ \}$, we can write in the Schroedinger representation

$$\bar{z}_A = (-)^{A+1} i \frac{\vec{\partial}}{\partial z_A} ; \quad \pi^N = (-)^N i \frac{\vec{\partial}}{\partial \rho_N} ; \quad \bar{\pi}_M = (-)^M i \frac{\vec{\partial}}{\partial \bar{\rho}^M} . \tag{76}$$

We derive from Eq.(74)

$$\frac{\partial}{\partial T} K_{fi} = - i\, \mathcal{H}(t)\, K_{fi}. \tag{77}$$

From Eq.(77) one finds that modulo a constant

$$Z = \mathrm{Ber}^{d/2} \left[\frac{1}{V_+ - IV\_I} D \right] \mathrm{Ber}'[\tilde{V}_+ - \tilde{I}\tilde{V}\_\tilde{I}] \tag{78}$$

The expressions (72), (73) and (78) for Z gives the supersymmetric generalization of the corresponding result for the bosonic case[2,4]. It should, however, be noted that the operator $(\tilde{V}_+ - \tilde{I}\tilde{V}\_\tilde{I})$ has zero eigenstates. The prime in Eq.(78) means that the corresponding zero eigenvalues must be removed from the superdeterminant in the standard way. Then, it is necessary to multiply K_{fi} by the δ-function $\delta(\bar{\rho}^i_0 - \bar{\rho}^f_0)\delta(\rho^i_0 - \rho^f_0)$ where $\bar{\rho}_0, \rho_0$ are ghost zero modes. The appropriate procedure for bosonic discrete "strings" was considered in Ref.4.

Note that considering the infinite dimensional case we substitute instead of the algebra $Osp(N|K,\mathbb{R})$ the infinite dimensional $N = 1$ superconformal algebra Superconf.(S^1). One can show, after tedious calculations, that the functional Z (78) is equal to the partition function for the fermionic string theory[14].

In order to obtain the propagator, $\mathfrak{D}_{fi}$, one has to perform the integration over the Lagrange multipliers $l^{\pm}$ in Eq.(52a) i.e:

$$\mathfrak{D}_{fi} = \int d\hat{l}^+ d\hat{l}^- \mathrm{Ber}^{d/2}\left[\frac{1}{V_+ - IV\_I} D\right] \mathrm{Ber}[\tilde{V}_+ - I\tilde{V}\_I] \exp(iS^{cl}) \tag{79}$$

where $V_{\pm} = \exp(\hat{l}^{\pm}_M T^{M\pm})$, $\tilde{V}_{\pm} = \exp(\hat{l}^{\pm}_M \tilde{T}^{M\pm})$ are elements of the gauge group $G \in Osp(N|K,\mathbb{R})_+ \otimes Osp(N|K,\mathbb{R})_-$.

Let us now consider the case when the operator ∂_{ab} has eigenstates z^0_a with zero eigenvalue. The presence of such states enables the introduction in our system of conserved total momentum[4]. In this case it is possible to impose the further condition that in (79) all boundaries reduce to points. This means that

$$Y^f_A = (Q^f z^0_a, 0); \quad z^i_a = (Q^i z^0_a, 0); \quad \rho \sim \rho_0; \quad \bar{\rho} \sim \bar{\rho}_0. \tag{80}$$

The propagator now takes the form:

$$\mathfrak{D}_{fi} = \delta(\bar{\rho}^i_0 - \bar{\rho}^f_0)\, \delta(\rho^i_0 - \rho^f_0) \int d\hat{l}_+ d\hat{l}_- \exp [S_{matrix}] \tag{81}$$

with

$$S_{matrix} = (d/2)\, \mathrm{str} \ln \left[\frac{1}{V_+ - IV\_I} D \right] + \mathrm{str}_{adj.} \ln [\tilde{V}_+ - I\tilde{V}\_I]$$
$$+ S^{cl}(Q^f, Q^i) \tag{82}$$

where (compare with Eq.(3.18) of Ref.4)

$$S^{cl}(Q^f, Q^i) = (Q^f - Q^i)^2 / (2l^0)$$

and l^0 is some functional of the Lagrange multipliers $\hat{l}^M_\pm$. This functional l^0 is invariant under the gauge transformations (51).

The functional integral of our system is represented by the expression (81). We thus arrive at the conclusion that our system is equivalent to some "zero-dimensional" matrix field model with the functional integral (81) and with the action (82). We emphasize that our model as described by (81) and (82) can be considered as the super extension of "zero-dimensional" matrix field theories[15].

We have to consider matrix field theories with two ortho-symplectic matrices. The parameters in our theory are the dimensions (N,K) of the gauge group $Osp(N|K)$, space-time dimension d and the arbitrary parameters defining the symplectic matrix D. It would be interesting to investigate the double scaling limit[16] in our model.

Finally, note that Eq.(77) can be rewritten in the form:

$$[\, \mathrm{str}\,\{ T^{\pm\, M}\, V_\pm \frac{\vec{\partial}}{\vec{\partial}\, V_\pm} \} + \mathrm{str}_{adj.}\{ \tilde{T}^{\pm\, M}\, \tilde{V}_\pm \frac{\vec{\partial}}{\vec{\partial}\, \tilde{V}_\pm} \}\,]\, K_{fi}$$

$$= (-i)\,[\, T^M_\pm + T^M_{\pm\, ghosts}\,]\, K_{fi} \qquad (83)$$

where $T^M_\pm$ are defined in eqs.(43) and

$$T^M_{\pm\, ghosts} = (\pm)^{NL} B^M_\pm\, l^\pm_N\, t^{NL}_M\, C^\pm_L$$

The extra terms connected with conformal anomalies will appear in (83) in the infinite-dimensional case G=Supconf(S^1) ⊗ Supconf(S^1), as it would be necessary to make normal ordering prescriptions for the operators $T^M_\pm$ and $T^M_{\pm ghosts}$ and perform regularization procedure for the divergent expression for Z (78).

At the end of this section we would like to discuss briefly

the possible interpretation of Eq.(83). Let us rewrite Eq.(83) in the form:

$$T^M_{\pm\, tot}\, K_{fi} = [\, T^M_{V_\pm} + T^M_{\tilde{V}_\pm} + T^M_\pm + T^M_{\pm\, ghosts}\,]\, K_{fi} = 0 \qquad (84)$$

where we introduce

$$T^M_{V\pm} = \mathrm{str}\,\{ T^{\pm\, M}\, V_\pm\, \Pi^\pm \}, \qquad \Pi^\pm = -i \frac{\vec{\partial}}{\vec{\partial}\, V_\pm},$$

$$T^M_{\tilde{V}\pm} = \mathrm{str}_{adj.}\{ \tilde{T}^{\pm\, M}\, \tilde{V}_\pm\, \tilde{\Pi}^\pm \}, \qquad \tilde{\Pi}^\pm = -i \frac{\vec{\partial}}{\vec{\partial}\, \tilde{V}_\pm}. \qquad (85)$$

The variables $\Pi^\pm$ and $\tilde{\Pi}^\pm$ can be considered as momenta conjugate to $V_\pm(t) = V_\pm(t,0)$ and $\tilde{V}_\pm(t) = \tilde{V}(t,0)$. The solution of (84) can be written in the form

$$K_{fi} = \langle f|\,[\int \prod_t \{dl^+_M dl^-_M\}\, \mathrm{Texp}\{ i\int dt\, (l^+_M(t) T^M_{+tot} + l^-_M(t) T^M_{-tot})\}]\,|i\rangle$$

$$= \int D\mu \prod_t \{dl^+_M\, dl^-_M\}\, \exp\{ i\, S_{tot}\} \qquad (86)$$

where $\prod_t\{dl^+_M dl^-_M\}$ is a product of the left invariant measures over the group $G_+ \otimes G_-$ generated by the operators $\{T^M_{\pm tot}\}$, $D\mu$ is a measure over the trajectories $X(t)$ in the phase space with coordinates $X = \{z, \bar{z}, \pi, \rho, \bar{\pi}, \bar{\rho}, V, \tilde{V}, \Pi, \tilde{\Pi}\}$ satisfying the boundary conditions (13a,b), (63b), and

$$V_\pm(0) = \tilde{V}_\pm(0) = 1; \qquad (87a)$$

$$V_\pm(T) = V_\pm\,, \quad \tilde{V}_\pm(T) = \tilde{V}_\pm\,. \qquad (87b)$$

The action S_{tot} in Eq.(86) has the form:

$$S_{tot} = \tilde{\tilde{S}}_3 + S_{grav}$$

where $\tilde{\tilde{S}}_3$ is defined in Eq.(64) while S_{grav} is the action of "one-dimensional supergravity" (one-dimensional orthosymplectic matrix field theory).

$$S_{grav} = \int_0^T dt[\cdot \mathrm{str}\{(\dot{V}_+\Pi^+ + \dot{V}_-\Pi^-)-(l_M^+ \, T^{+M} \, V_+ \, \Pi^+ + l_M^- \, T^{-M} \, V_- \, \Pi^-)\}$$

$$+ \mathrm{str}_{adj}\{(\dot{\tilde{V}}_+\tilde{\Pi}^+ + \dot{\tilde{V}}_-\tilde{\Pi}^-)-(l_M^+ \, \tilde{T}^{+M} \, \tilde{V}_+ \, \tilde{\Pi}^+ + l_M^- \, \tilde{T}^{-M} \, \tilde{V}_- \, \tilde{\Pi}^-)\}]. \quad (88)$$

It should be noted that the variation of S_{grav} over momenta $\Pi^\pm$ and $\tilde{\Pi}^\pm$ gives rise to the equations

$$\dot{V}_\pm(t) = [l_M^\pm(t) \, T^{\pm \, M}] \, V_\pm(t); \quad \dot{\tilde{V}}_\pm(t) = [l_M^\pm(t) \, \tilde{T}^{\pm \, M}] \, \tilde{V}_\pm(t) \quad (89)$$

Now taking into account the initial conditions (87), the solutions of (89) are represented in the form (50) and (56).

In the infinite dimensional case of the closed fermionic string, when the gauge group $G_+ \otimes G_-$ will be isomorphic to Superconf(S^1) ⊗ Superconf(S^1), additional terms related to the conformal anomaly terms in Eq.(83) will appear in the action (88). Our view is that in this case we would obtain, instead of (88), the action describing two-dimensional N=1 superconformal gravity considered in Ref.17. We hope to study this conjecture in our subsequent publications.

5. Conclusions

In this paper we have tried to set up a general approach to gauge systems with quadratic constraints. For simplicity, we consider dynamical (super)systems with a finite number of degrees of freedom. We can interpret these systems as models of the bound states of a collection of relativistic particles. It is rather interesting to note that for investigations of physical models of bound states it is possible to use techniques and methods developed for string theories.

Our basic results may be summarized as follows:

1. Specifically, we have solved the quantization problem for a chiral fermionic discrete system with the gauge group $G = Osp(N|K,\mathbb{R})_+ \otimes Osp(N|K,\mathbb{R})_-$. In this case, it is possible to fix the boundary conditions for ghosts unambiguously and find the explicit form of the propagator. With this, the efficacy and validity of the approach as propounded in Refs.2-4 is thereby established and the supersymmetric generalization of Refs.2 and 4 obtained.

2. We have also shown that when zero-mode states are present (so that a conserved total momentum exists), it is possible to impose point boundary conditions on the functional integral and the resulting theory is then a natural super-extension of the ordinary "zero-dimensional" matrix field theory[15]. The parameters of this theory are the dimensions (N,K) of the gauge group $Osp(N,K)$, space-time dimension d and the arbitrary parameters defining the symplectic matrix D.

3. Using the differential equation for the propagator, we have constructed the one-dimensional orthosymplectic matrix field theory which can be interpreted as "one dimensional supergravity".

On the other hand, the chiral fermionic discrete models considered in this paper are very similar to the fermionic string models if we use the gauge group Superconf(S^1) ⊗ Superconf(S^1) (specific to fermionic strings) instead of the finite dimensional gauge group $Osp(N|K) \otimes Osp(N|K)$ considered here. When this aspect is taken into account, there arises the interesting prospect of trying to construct the interacting field theory (i.e. 3-vertex operator) for our discrete models invariant under the gauge group $Osp(N|K) \otimes Osp(N|K)$. We think that this would be very useful for exploring interacting string field theory from an algebraic point of view.

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