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CANONICAL QUANTIZATION OF A RELATIVISTIC PARTICLE WITH CURVATURE AND TORSION

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1 Introduction

Recently interest has been aroused in the generalizations of relativistic particle model, the action of which depends not only on the length of the world curve but also on its curvature [1,3-6] or its torsion [2,7-9]. The models with curvature are used, for example, in the polymer theory [10]. And they can be treated as the one-dimensional version of the rigid string [11]. Investigation of the fermion-boson transmutations of the charged particles in an external Chern-Simons field reduced to the quantization of the relativistic particle with torsion [2,7]. So far the models containing only the curvature or torsion have been considered. For the completeness one has to investigate the model containing at the same time the curvature and torsion of the world trajectory. The present paper is devoted to this problem. The layout of the paper is as follows. In the second section the generalized Hamiltonian formalism for a relativistic particle with curvature and torsion in a D-dimensional space-time is constructed. A complete set of constraints in the phase space is found, their separation into the first class and the second-class constraints is fulfilled. The third section is devoted to the canonical quantization of this model. At first the general scheme of quantization in a D-dimensional space-time is considered. Further the case of a three-dimensional space-time is investigated in detail. In the sector with positive mass squared we obtain a spectrum determined by an equation involving the parameters of the model, the mass and the spin of a state. The possibility to describe in the framework of this model the states with integer, half-odd-integer and continuous spins is shown ($D=3$). By making use of the Casimir operators of the Poincare group we construct the wave equation and investigate it in detail in the rest frame. In the fourth section, the interaction with an external Abelian gauge field is introduced in the geometrical way. In Conclusion (Sect.5), the obtained results are briefly discussed and the problems waiting for their solutions are outlined. In Appendix some details of the calculation of the Poincare group invariant W on the constraint surface are presented.

2 Generalized Hamiltonian formalism for relativistic particle with curvature and torsion

We shall investigate the model defined by the action

$$S=-m\int ds-\alpha\int k(s)\,ds-\beta\int\kappa(s)\,ds, \tag{2.1}$$

where $k(s)$ is a curvature of the world curve of the particle, $\kappa(s)$ is a torsion of this curve, m is a constant with the mass dimension, α and β are dimensionless constants. If $x^{\mu}(\tau)$, $\mu=0,1,\ldots,D-1$ is a parametric representation of the world trajectory, then the action (2.1) can be rewritten in the form [15]

$$S=-m\int d\tau\sqrt{\dot{x}^2}-\alpha\int d\tau\frac{\sqrt{\bar{g}}}{\dot{x}^2}-\beta\int d\tau\frac{\sqrt{\dot{x}^2 d}}{\bar{g}}, \tag{2.2}$$

where

$$\dot{x}\equiv dx/d\tau,\quad \bar{g}=(\dot{x}\ddot{x})^2-\dot{x}^2\ddot{x}^2,\quad d=\det(d_{\alpha\beta}),$$

$$d_{\alpha\beta}=\overset{(\alpha)}{x}{}^{\mu}\overset{(\beta)}{x}{}_{\mu},\quad \overset{(\alpha)}{x}\equiv d^{\alpha}x/d\tau^{\alpha},\quad \alpha,\beta=1,2,3.$$

In the D-dimensional space-time the metric with the signature $(+,-,\ldots,-)$ is used.

To eliminate the superlight velocities in the model under consideration, we assume that $\dot{x}^2>0$. Putting $x^0(\tau)=\tau$ we can deduce from here the following conditions $\ddot{x}^2<0$ and $\dddot{x}^2<0$. If all these conditions are satisfied then the radicands in eq. (2.2) are positive.

The Lagrangian function in action (2.2) depends on $\dot{x},\ddot{x}$ and $\dddot{x}$. Therefore the resulting Euler-Lagrange equations are ordinary differential equations of the sixth oder. The action (2.2) is invariant under the Poincare transformations in the ambient space-time and under the reparametrizasion $\tilde{\tau}=f(\tau)$. As a consequence, the Lagrangian in this model is singular. Let us construct the generalized Hamiltonian description of this model following the papers[1,2,13]. To begin with we introduce the canonical variables

$$q_1=x\,,\qquad q_2=\dot{x}\,,\qquad q_3=\ddot{x}\,, \tag{2.3}$$

$$p_1 = -\frac{\partial L}{\partial \dot{x}} - \frac{dp_2}{d\tau}, \tag{2.4}$$

$$p_2 = -\frac{\partial L}{\partial \ddot{x}} - \frac{dp_3}{d\tau}, \tag{2.5}$$

$$p_3 = -\frac{\partial L}{\partial \dddot{x}}, \tag{2.6}$$

where L is the Lagrangian function in (2.2). The Lorentz indices in eqs (2.3)-(2.6) are omitted for simplicity. We shall do so further if misunderstanding does not appear.

We shall need the explicit form of the canonical momentum p_3 only. It is given by

$$p_3^\mu = \beta \frac{\sqrt{\dot{x}^2}}{g} \sqrt{d} \sum_{\alpha=1}^{3} d^{3\alpha} \overset{(\alpha)}{x^\mu}, \tag{2.7}$$

where $d^{\alpha\beta}$ is the matrix inverse to $d_{\alpha\beta}$: $d_{\alpha\beta}d^{\beta\gamma} = \delta_\alpha^\gamma$. From (2.7) we deduce three primary constraints

$$\overset{(1)}{\phi}_1 = p_3^2 + \beta^2 \frac{q_2^2}{g} = 0, \tag{2.8}$$

$$\overset{(1)}{\phi}_2 = p_3 q_2 = 0, \tag{2.9}$$

$$\overset{(1)}{\phi}_3 = p_3 q_3 = 0, \tag{2.10}$$

where $g = (q_2q_3)^2 - q_2^2q_3^2$. They have the same form as primary constraints in the theory of the relativistic particle with torsion [2].

The Poisson brackets will be defined as follows

$$\{f, g\} = \sum_{a=1}^{3} \left(\frac{\partial f}{\partial p_a^\mu} \frac{\partial g}{\partial q_{a\mu}} - \frac{\partial f}{\partial q_a^\mu} \frac{\partial g}{\partial p_{a\mu}} \right). \tag{2.11}$$

The primary constraints (2.8)-(2.10) are mutually in involution

$$\{\overset{(1)}{\phi}_1, \overset{(1)}{\phi}_2\} = 0, \quad \{\overset{(1)}{\phi}_1, \overset{(1)}{\phi}_3\} = 2\overset{(1)}{\phi}_1 \approx 0, \quad \{\overset{(1)}{\phi}_2, \overset{(1)}{\phi}_3\} = \overset{(1)}{\phi}_2 \approx 0. \tag{2.12}$$

The sign $\approx$ means a weak equality [14]. The canonical Hamiltonian is

$$H = -p_1\dot{x} - p_2\ddot{x} - p_3\dddot{x} - L = -p_1q_2 - p_2q_3 + m\sqrt{q_2^2} + \alpha \frac{\sqrt{g}}{q_2^2}. \tag{2.13}$$

The equations of motion in the phase space are written as follows

$$\frac{df}{d\tau} = \frac{\partial f}{\partial \tau} + \{f, H\} + \sum_{a=1}^{3} \lambda_a \{f, \overset{(1)}{\phi}_a\}, \tag{2.14}$$

where f is a function of the canonical variables and evolution parameter τ.

We now proceed to find the secondary constraints by making use of the Dirac method [1,14]. Demanding the stationarity of the primary constraints

$$\frac{d \overset{(1)}{\phi}_a}{d\tau} = \{\overset{(1)}{\phi}_a, H_T\} \approx 0, \quad a = 1,2,3, \tag{2.15}$$

where $H_T = H + \sum_{b=1}^{3} \lambda_b \overset{(1)}{\phi}_b$ we obtain three new constraints

$$\overset{(2)}{\phi}_1 = p_2 p_3 - \beta^2 \frac{q_2 q_3}{g} = 0, \tag{2.16}$$

$$\overset{(2)}{\phi}_2 = p_2 q_2 = 0, \tag{2.17}$$

$$\overset{(2)}{\phi}_3 = p_2 q_3 - \frac{\alpha}{q_2^2} \sqrt{g} = 0. \tag{2.18}$$

The requirement of the stationarity of the constraints (2.16)–(2.18) on the equations of motion

$$\frac{d \overset{(2)}{\phi}_a}{d\tau} = \{\overset{(2)}{\phi}_a, H\} + \sum_{b=1}^{3} \lambda_b \{\overset{(2)}{\phi}_a, \overset{(1)}{\phi}_b\} \approx 0, \quad a = 1,2,3 \tag{2.19}$$

results in three additional constraints with the canonical Hamiltonian (2.[illegible]) between them

$$\overset{(3)}{\phi}_1 = p_1 p_3 + p_2^2 + \alpha \frac{p_2 q_3}{\sqrt{g}} - \beta^2 \frac{q_3^2}{g} = 0, \tag{2.20}$$

$$\overset{(3)}{\phi}_2 = H - p_1 q_2 - p_2 q_3 + m \sqrt{q_2^2} + \frac{\alpha}{q_2^2} \sqrt{g} = 0, \tag{2.21}$$

$$\overset{(3)}{\phi}_3 = -p_1 q_3 + m \frac{q_2 q_3}{\sqrt{q_2^2}} = 0. \tag{2.22}$$

At this stage the process of generation of constraints is stopped. The requirement of the stationarity of the last constraints (2.20)–(2.22) enables us to determine the Lagrange multipliers λ_1 and λ_2 in the total Hamiltonian

$$\lambda_1 = m \frac{(q_2 q_3)^2 - q_2^2 q_3^2}{2(p_1 p_3) q_2^2 \sqrt{q_2^2}}, \quad \lambda_2 = 3 \frac{p_1 p_2}{p_1 p_3}. \tag{2.23}$$

Now we have to separate all the constraints into the first and second class constraints. For this purpose we construct the matrix C with the elements

$$[illegible] = \{[illegible], [illegible]\} \quad a, b [illegible]. \tag{2.24}$$

where $\theta_{3(b-1)+a} = \overset{(b)}{\phi}_a$, $a,b = 1,2,3$. The matrix Ω can be rewritten in a block form

$$\Omega = \begin{vmatrix} 0 & 0 & A \\ 0 & B & C \\ -A^t & -C^t & D \end{vmatrix}, \tag{2.25}$$

where 0 is the (3×3)-zero matrix and

$$A = \begin{vmatrix} 0 & 0 & -2p_1p_3 \\ 0 & 0 & 0 \\ -p_1p_3 & 0 & 0 \end{vmatrix}, \qquad B = \begin{vmatrix} 0 & 0 & -p_1p_3 \\ 0 & 0 & 0 \\ p_1p_3 & 0 & 0 \end{vmatrix},$$

$$C = \begin{vmatrix} c & 0 & -p_1p_2 \\ -2p_1p_3 & 0 & 0 \\ -2p_1p_2 & 0 & \frac{m\beta^2}{\sqrt{q_3^2p_3^2}} \end{vmatrix}, \quad D = \begin{vmatrix} 0 & -3p_1p_2 & d \\ 3p_1p_2 & 0 & \frac{mg}{q_2^2\sqrt{q_2^2}} \\ -d & -\frac{mg}{q_2^2\sqrt{q_2^2}} & 0 \end{vmatrix}, \tag{2.26}$$

where

$$c = \beta^2\,\frac{p_1q_2}{g} - \alpha\,\frac{p_1p_3}{\sqrt{g}} + 2\alpha\,\frac{p_2^3q_3^2}{q_2^2\sqrt{g}},$$

$$d = m^2 - p_1^2 + \alpha\,\frac{m\sqrt{g}}{(q_2^2)^{3/2}}.$$

The sign ^{t} means a transposition As known [15], the number of the first class constraints equals Dim ker Ω. If the vector $\xi \in$ ker Ω, $\xi = \{\xi_1, \ldots, \xi_9\}$ then

$$\xi_4 = \xi_6 = \xi_7 = \xi_9 = 0,$$

$$(p_1p_3)\xi_3 + 2(p_1p_3)\xi_5 - 3(p_1p_2)\xi_8 = 0, \tag{2.27}$$

$$2(p_1p_3)\xi_1 - \frac{mg}{q_2^2\sqrt{q_2^2}}\xi_8 = 0.$$

Thus we get

$$\text{Dim ker}\,\Omega = 3.$$

Therefore in the model under consideration there are three first-class constraints and six second-class constraints. The number of physical degrees of freedom equals obviously $3D - 6$. The first-class constraints can be separated by the formula [15]

$$\Phi_a = \sum_{i=1}^{9} \overset{(a)}{\xi}_i\,\theta_i, \qquad a = 1,2,3, \tag{2.28}$$

where $\overset{(a)}{\xi}_i$, $a = 1,2,3$ are the basis vectors of ker Ω. By making use of eq.(2.27) one can easily construct these vectors up to an arbitrary factor for each $\overset{(a)}{\xi}$. They have the following nonzero components

$$\overset{(1)}{\xi}_2 = 1; \quad \overset{(2)}{\xi}_3 = -2, \quad \overset{(2)}{\xi}_5 = 1,$$

$$\overset{(3)}{\xi}_1 = \frac{mg}{2q_2^2\sqrt{q_2^2 p_1 p_3}} , \quad \overset{(3)}{\xi}_3 = 2\,\frac{p_1 p_2}{p_1 p_3} , \quad \overset{(3)}{\xi}_8 = 1 . \tag{2.29}$$

Taking into account (2.28) and (2.29) we obtain the first-class constraints

$$\Phi_1 = p_3 q_2 = 0 , \tag{2.30}$$

$$\Phi_2 = p_2 q_2 - 2 p_3 q_3 = 0 , \tag{2.31}$$

$$\Phi_3 = \frac{mg}{2q_2^2\sqrt{q_2^2 p_1 p_3}} \overset{(1)}{\phi}_1 + 2\,\frac{p_1 p_2}{p_1 p_3} \overset{(1)}{\phi}_3 + H = 0 . \tag{2.32}$$

As the second-class constraints ω_s , $s = 1, \ldots ,6$ one can take six arbitrary constraints from the set $\{\theta_i , \ i = 1, \ldots ,9\}$ with $\det \| \{\omega_s, \omega_{s'}\} \| \not\approx 0$, $s, s' = 1, \ldots , 6$. This can be done in many ways. For example, one may put

$$\omega_a = \overset{(2)}{\phi}_a = 0, \quad \omega_{3+a} = \overset{(3)}{\phi}_a = 0 , \quad a = 1,2,3. \tag{2.33}$$

In this case the Hamiltonian (2.13) is considered to be the second-class constraint. However we can substitute $H = \omega_6$ in (2.33) by $\overset{(1)}{\phi}_1$. At the quantum level we shall consider both these possibilities.

3 Quantum theory

At first we consider the general scheme of the canonical quantization of this model in the D -dimensional space-time. We are dealing with a generalized Hamiltonian system in the $6D$-dimensional phase space with three first-class constraints (2.30)-(2.32) and six second-class constraints (2.33). The state vectors will be defined by the conditions

$$\Phi_a \, |\psi> = 0 , \quad a = 1,2,3. \tag{3.1}$$

The commutators of the operators q_a and p_a , $a = 1,2,3$ should be determined by the Dirac brackets constructed by means of the second-class constrants ω_s . After this the constraints ω_s will vanish at the quantum level identically. Therefore they can be omitted in conditions (3.1). As a result, the wave equations (3.1) can be rewritten only in terms of the primary constraints

$$\overset{(1)}{\phi}_a \, |\psi> = 0 , \quad a = 1,2,3. \tag{3.2}$$

If the canonical Hamiltonian is substituted in the set of the second-class constraints by $\overset{(1)}{\phi}_1$, then the same substitution will take place in the wave equations (3.2) .

The number of the wave equations (3.2) can be reduced by introducing the gauge conditions. For example, the condition

$$\chi_1 = q_2 q_3 = 0 \tag{3.3}$$

entails considerable simplification. From (3.3) it follows that

$$q_2^2 = \text{const} . \tag{3.4}$$

Thus eq (3.3) is, in fact, the proper time gauge. This gauge eliminates completely the functional freedom in the equations of motion (2.14), and the last Lagrange multiplier turns out to be

$$\lambda_2 = q_3^2/q_2^2. \tag{3.5}$$

In principle, we can impose one or two gauge conditions in addition to (3.3)

$$\chi_c(q_a,\ p_a,\ \tau) = 0\ , \quad c = 2,3$$

demanding that

$$\det \|\{\chi_a,\ \Phi_b\}\| \not\approx 0\ , \quad a,b = 1,2,3, \tag{3.6}$$

$$\frac{\partial\chi_c}{\partial\tau} + \{\chi_c,\ H\} + \sum_{a=1}^{3} \lambda_a\, \{\chi_c,\ \overset{(1)}{\phi}_a\} \approx 0\ , \quad c = 2,3, \tag{3.7}$$

where λ_a, $a = 1,2,3$ are determined in (2.33) and (3.5).

Further simplification is achieved when $D = 3$. *In this case three vectors* q_2, q_3, p_3 form, by virtue of the constraints (2.8)-(2.10) and gauge condition (3.3),(3.4), a complete orthogonal basis. The velocity q_2^μ is a time-like vector while the acceleration q_3^μ and the momentum p_3^μ are the space-like vectors. The constraints $\overset{(2)}{\phi}_a = 0$, $a = 1,2,3$ enables us to obtain in this basis the expansion for the momentum p_2^μ in the form

$$p_2^\mu = \alpha\, \frac{q_3^\mu}{\sqrt{-q_2^2 q_3^2}}\ , \qquad \mu = 0,1,2. \tag{3.8}$$

The constraints $\overset{(3)}{\phi}_a = 0$, $a = 1,2,3$ and gauge condition (3.3) determine the projections of the momentum p_1^μ on the basis vectors q_2^μ, q_3^μ and p_3^μ

$$p_1 q_2 = m\sqrt{q_2^2}\ , \qquad p_1 q_3 = 0\ , \qquad p_1 p_3 = \beta^2/q_2^2\ . \tag{3.9}$$

From here we deduce

$$p_1^\mu = q_2^\mu\, \frac{m}{\sqrt{q_2^2}} + p_3^\mu\, \frac{q_3^2}{q_2^2}\ . \tag{3.10}$$

Squaring eq. (3.10) we obtain

$$p_1^2 = M^2 = m^2 + \beta^2\, \frac{q_3^2}{(q_2^2)^2}\ , \tag{3.11}$$

where M^2 is the mass of the particle with the action (2.1). From (3.11) it follows that

$$M^2 < m^2 \tag{3.12}$$

and M^2 is not positive definite because $q_3^2 < 0$. Thus in the model under consideration the squared mass is determined by the initial conditions (by the Cauchy data) for variables q_3^μ and it can be either positive, or negative, or it can vanish. [1] Further we shall confine our consideration to the sector in this model, where $p_1^2 = M^2 > 0$.

Let us examine the angular momentum in this model

$$M_{\mu\nu} = \sum_{a=1}^{3} (q_{a\mu}p_{a\nu} - q_{a\nu}p_{a\mu}) \, . \tag{3.13}$$

At the quantum level the algebra of the operators $M_{\mu\nu}$ should be determined by the commutators of the operators q_a and p_a, $a = 1,2,3$. In their turn these commutators are defined, as mentioned above, by the corresponding Dirac brackets. But the requirement of the Poincare invariance of the theory under consideration determines the algebra of the operators $p_{1\mu}$ and $M_{\mu\nu}$ completely. This algebra must be the same as the algebra of the Poincare group. Without calculating the correspondig Dirac brackets we assume that the Poincare invariance takes place. As the scalar Casimir operators of the Poincare group we take the following ones [16]

$$p_1^2 = p_1^\mu p_{1\mu} \, , \tag{3.14}$$

$$W = \frac{1}{2} M_{\mu\nu} M^{\mu\nu} p_1^2 - (M_{\mu\sigma} p_1^\mu)^2 \, . \tag{3.15}$$

In the four-dimensional space-time the invariant W is the squared Pauli-Lubanski vector with sing minus, $W = -w_\mu w^\mu$, where $w_\mu = (1/2)\epsilon_{\mu\nu\rho\sigma} M^{\nu\rho} p_1^\sigma$. The obvious advantage of the definition (3.15) is the possibility to use it for arbitrary D.

The relativistic invariance requires that the physical state vectors $|\psi>$ should be the eigenvectors of the operators (3.14) and (3.15)

$$p_1^2 \, |\psi> = M^2 \, |\psi> \, . \tag{3.16}$$

As mentioned above, we suppose that $M^2 > 0$. In this case we can go to the rest frame where $p_1^\mu = (p_1^0 = M, \;\; \mathbf{p}_1 = 0)$. Here we have

$$W = (p_1^0)^2 \, M_{12} M^{12} = \frac{M^2}{2} \, C_2(SO(2)) \, , \tag{3.17}$$

[1]In the ordinary theory of the relativistic particle with the action $S = -m \int d\tau \sqrt{|\dot{x}^2|}$ the squared mass of the particle p^2 is positive only for the initial data obeying the inequality $\dot{x}^2 > 0$. It is important that this condition, satisfied at the initial moment will be fulfilled always. In the general case for arbitrary sing of $\dot{x}^2$ we have

$$p_\mu = -\frac{\partial L}{\partial \dot{x}^\mu} = m \frac{\dot{x}_\mu}{\sqrt{|\dot{x}^2|}} \, \mathrm{sign}\,(\dot{x}^2) \, .$$

Squaring of this equation gives

$$p^2 = m^2 \, \mathrm{sign}(\dot{x}^2) \, .$$

where $C_2(SO(2))$ is the squared Casimir operator of the $SO(2)$ group (see for example [18]). As known [17] the group of rotations on the plane $SO(2)$ has three different representations with integer, half-odd-integer and continuous values for spin j. In all these cases we have [18]

$$C_2(SO(2)) = 2\, j^2 . \tag{3.18}$$

It should be noted here that in the model under consideration we have to deal with the tensor representations of the $SO(2)$ group but not with spinor ones because the initial action (2.1) contains no spin degrees of freedom Thus the eigenvalues of the Casimir operator W are

$$W = M^2 j^2 , \quad M^2 > 0 , \quad j \geq 0 . \tag{3.19}$$

We assume, in addition to (3.16), that the state vector $|\psi>$ is the eigenvector of the Casimir operator W

$$W\,|\psi> = M^2\, j^2\, |\psi> , \quad M^2 > 0 , \quad j \geq 0 . \tag{3.20}$$

Let us proceed to the consideration of the wave equation. First we take as the second-class constraints the set (2.33). In addition to the proper time gauge (3.3) we introduce one more gauge condition that transforms the constraints $\overset{(1)}{\phi}_3$ into the second-class constraint. As a result, only one wave equation survives in (3.2)

$$\overset{(1)}{\phi}_1\, |\psi> = (\, p_3^2 q_3^2 - \beta^2\,)\, |\psi> = 0 . \tag{3.21}$$

If $D = 3$, one can express the left-hand side of (3.21) in terms of the Casimir operators p_1^2 and W. For this purpose we have to calculate the invariant W on the surface in the phase space determined by all the constraints and gauge conditions except for $\overset{(1)}{\phi}_1$. After a rather cumbersome calculation (see Appendix) we obtain

$$1 - x = \frac{W - (\alpha\,\sqrt{m^2 - p_1^2} + |\,\beta\,|\, m\, \sqrt{x})^2}{\beta^2 p_1^2}\, \frac{x}{1 + x} , \tag{3.22}$$

where $x = \beta^2/(p_3^2 q_3^2)$. Thus the condition (3.21)

$$(1 - x)\, |\psi> = 0 \tag{3.23}$$

is equivalent to

$$[\, W - (\alpha\, \sqrt{m^2 - p_1^2} + |\,\beta\,|\, m\,)^2\,]\, |\psi> = 0 . \tag{3.24}$$

From (3.16),(3.20) and (3.24) we obtain immediately the mass spectrum

$$M^2\, j^2 = (\, \alpha\sqrt{m^2 - M^2} + |\,\beta\,|\, m\,)^2 \tag{3.25}$$

By virtue of this equation M^2 should be less than m^2 in accordance with (3.12).

If $\alpha \neq |\beta|$, then eq (3.25) can be solved with respect to the ratio M/m

$$\frac{M}{m} = \frac{|\beta| j}{\alpha^2 + j^2}\left(1 + \sqrt{1 + \frac{\alpha^2 + j^2}{\beta^2 j^2}(\alpha^2 - \beta^2)}\right) . \tag{3.26}$$

When

$$-|\beta| \leq \alpha \leq |\beta| \tag{3.27}$$

the ratio M/m takes real values only if the spin of the state j satisfies the condition

$$1 - \frac{\alpha^2}{\beta^2} < \frac{j^2}{\alpha^2 + j^2} . \tag{3.28}$$

It is interesting to note that under the conditions (3.27) and (3.28) the ratio M/m turns out to be a double-valued function of the model parameters α, β and the spin j of the state

$$\frac{M}{m} = \frac{|\beta| j}{\alpha^2 + j^2}\left(1 \pm \sqrt{1 + \left(\frac{\alpha^2}{j^2} + 1\right)\left(\frac{\alpha^2}{\beta^2} - 1\right)}\right) . \tag{3.29}$$

The same situation takes place in some infinite component wave equations [18]. For $\alpha = \pm|\beta|$ it is not success to resolve eq. (3.18) with respect to M/m. Putting in (3.26) in turn $\alpha = 0$ and $\beta = 0$ we obtain the mass spectra in the theory of the relativistic particle with torsion ($\alpha \neq 0$) or with curvature ($\beta \neq 0$) derived in [illegible].

Now we proceed to discuss the realization of the invariants p_μ^2 and W as the differential operators. Most easily it can be done in the rest frame, where by virtue of (3.17) we have

$$W = M^2 M_{12} M^{12} , \tag{3.30}$$

and p_μ^2 reduces to the multiplication by M^2. The operator M_{12} which describes the rotations on the plane can be taken in the form

$$M_{12} = -i\frac{\partial}{\partial\varphi} + c , \tag{3.31}$$

where φ is some angular variable and c is a constant to be determined below. As the wave function we shall use 2π-periodic functions

$$\psi(\varphi) = \sum_{l \in Z} e^{il\varphi} a_l . \tag{3.32}$$

Substituting of (3.29)-(3.31) into (3.20) gives

$$j^2 = (l + c)^2 , \quad l \ni Z . \tag{3.33}$$

Without loss of generality we can regard l as an integer part of j and c as its fractional part. Thus we are dealing with the usual 2π-periodic wave functions and nevertheless we can describe integral, half-odd-integral and continuous values of the spin in the model under consideration.

4 Interaction with external Abelian gauge field

As shown in the preceding Section, in the model with the action (2.1) there appears a nonvanishing intrinsic angular momentum, i.e. spin. It is worthwhile to investigate the dynamics of the spin variables in the framework of this model. However, when we are dealing with the free action (2.1), this dynamics turns out to be trivial: the spin squared ($\sim W$) and its components ($\sim M_{\mu\nu}$) are conserved.

Here we consider the introduction of the interaction with an external Abelian gauge field in the model (2.1). From the geometrical point of view it can be done in the following way

$$L_{\text{int}} = -\sum_{a=0}^{D-1} g_a n_a^\mu A_\mu(x) - \sum_{a\neq b} g_{ab}\, n_a^\mu\, n_b^\nu\, F_{\mu\nu}(x)\,, \tag{4.1}$$

where n_a^μ, $a = 0, 1, \ldots, D-1$ are the unit vectors forming the moving basis on the world trajectory, $A_\mu(x)$ is the vector potential of the external electromagnetic field and $F_{\mu\nu}$ is its strength tensor, g_a and g_{ab} are the interaction constants. This Lagrangian obviously retains the reparametrization invariance of the whole action. In order to remove the superlight velocities we have to impose the following conditions

$$n_0^2 = \left(\frac{dx^\mu}{ds}\right)^2 = 1\,,\quad n_i^2 = -1\,,\quad i = 1, 2, \ldots,\quad ds^2 = dx^\mu\, dx_\mu\,. \tag{4.2}$$

The basis vectors n_a^μ, $a = 0, 1, \ldots, D-1$ can be represented in terms of the derivatives of the radius-vector x^μ [20]

$$\begin{aligned}
\frac{dx^\mu}{ds} &= n_0^\mu\,,\\
\frac{d^2x^\mu}{ds^2} &= k_1\, n_1^\mu\,,\\
\frac{d^3x^\mu}{ds^3} &= k_1\, k_2\, n_2^\mu - k_1^2\, n_0^\mu + \frac{dk_1}{ds}\, n_1^\mu\,,\\
\cdots &\quad \cdots\cdots\cdots\\
\frac{d^Dx^\mu}{ds^D} &= k_1\, k_2 \cdots k_{D-1}\; n_{D-1}^\mu + \cdots\,,
\end{aligned} \tag{4.3}$$

where $k_1(s)$, $k_2(s)$, ..., $k_{D-1}(s)$ are the curvatures of the world trajectory. If $D = 3$, then k_1 is called the curvature and k_2 is called the torsion.

The free action (2.1) can be generalized in the D-dimensional space-time by the formula

$$S_o = -m\int ds - \sum_{i=1}^{D-1} \alpha_i \int k_i(s)\, ds\,. \tag{4.4}$$

If we restrict for simplicity the summation over i in eq.(4.4), then the analogous restriction should be made in (4.1) so that the coordinate derivatives of the same order enter into (4.1) and (4.4). The canonical quantization of the simplest model of this kind is accomplished in [21].

5 Conclusion

The results obtained in this paper show that the point like Lagrangians with higher derivatives of the form (2.1) describe in the general case the point particles with non-vanishing spin. This conclusion relays on the fact that the Casimir operator of the Poincaré group W proportional to spin squared does not vanish on the physical submanifold of the phase space defined by the constraint equations and the gauge conditions. For $D = 3$ we have obtained the exact expression for the Regge-trajectory in the sector of the theory without tachyonic states. When the spin of the state increases, its mass decreases. There is an upper bound on the squared mass of the state. If $D = 3$, we have here the possibility to describe the states with integer, half-odd-integer or continuous values of the spin. It is important to note that in the framework of this model we dealing with the coordinate representation for the wave function even in the case of the half-odd-integral spin instead of the spinor one. And there remains an open question whether one can here obtain the Dirac equation for such spin values. Further investigations are required also in order to elucidate the role of the tachyonic states in the model.

Appendix

Here we present some details of the calculation of the invariant W on the submanifold of the phase space defined by the constraint equations and gauge conditions except for $\overset{(1)}{\phi_1} = 0$. At first we take into account eqs. (2.9), (2.10), (2.16), (2.17), (2.22), (3.3) as well as the condition $p_1 p_2 = 0$ valid in the three dimensional space time. As a result, one obtains

$$W = p_1^2 (q_2^2 p_2^2 + q_3^2 p_3^2) - p_2^2 (p_1 q_2)^2 + \\ + 2 (p_1 p_3)(p_1 q_2)(p_2 q_3) - q_3^2 (p_1 p_3)^2 \tag{A.1}$$

Now we use the rest of the constrains (see also eqs. (3.8) and (3.9)) from which it follows that

$$(p_1 q_2)^2 = m^2 q_2^2, \quad p_2 q_3 = \alpha \frac{q_3^2}{\sqrt{-q_2^2 q_3^2}} \quad p_1 p_3 = \frac{\beta^2}{q_2^2}. \tag{A.2}$$

It is convenient to introduce the notation

$$\frac{\beta^2}{p_3^2\, q_3^2} \;=\; x\,. \tag{A.3}$$

By making use of (A.2) and (A.3) we transform eq (A.1) in the form

$$W \;-\; \left(\alpha \sqrt{m^2 \,-\, p_1^2} \;+\; |\,\beta\,|\, m \sqrt{x} \right)^2 \;=\; \beta^2\, p_1^2\, \frac{(1 \,-\, x\,)\,(1 \,+\, x\,)}{x}\,. \tag{A.4}$$

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