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PARASUPERSYMMETRIC SPINNING PARTICLE AND TOPOLOGICAL GAUGE FIELDS

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1. Introduction

Recently it was recognized [1] that the generalization of the notion of supersymmetry called para-supersymmetry can be achieved by using nontrivial polynomial relations between generators as the dynamical symmetry algebras. The manifestations of parasupersymmetry in different physical systems were studied [1, 2, 3]. It was shown [1], in particular, that the hamiltonian of one-dimensional nonrelativistic particle with spin J moving in oscillator or Morse potential and magnetic field related to these potentials possesses parasupersymmetry of order $2J$. The special case $J = 1/2$ corresponds to ordinary supersymmetry quantum mechanics for one half spin particles.

In the present paper we search for parasupersymmetry in the relativistic theory by considering first-quantized relativistic particles. It is well known [4, 5, 6, 7] that there is close relation between relativistic quantum mechanics of point-like spinning or spinless particles and quantum field theory. In particular, the wave function of spinless particle whose action is invariant under worldline reparametrizations is described after quantization by quantum scalar field whereas Dirac fermion field appears as a wave function of spinning particle [4, 5]. In the last case, the action of massless spinning particle in the D-dimensional euclidean space-time is invariant under reparametrizations and has local supersymmetry. The generators of these transformations coincide with the Klein-Gordon and Dirac operators, respectively:

$$\mathcal{H} = \tfrac{1}{2}p_\mu^2, \quad \mathcal{Q} = p^\mu\psi_\mu \equiv (p\cdot\psi) \tag{1.1}$$

Here, p_μ is momentum ($\mu = 1, \ldots, D$) and ψ_μ is "spinning" coordinate of a particle obeying after quantization the following commutation relations:

$$\{\psi_\mu, \psi_\nu\} \equiv \psi_\mu\psi_\nu + \psi_\nu\psi_\mu = \tfrac{1}{2}g_{\mu\nu} \tag{1.2}$$

The supersymmetry algebra has the form

$$\mathcal{Q}^2 = \tfrac{1}{2}\mathcal{H} \tag{1.3}$$

The action of a spinning particle is invariant under Lorentz rotations and the corresponding integral of motion is the total angular momentum [6]

$$\Sigma_{\mu\nu} = p_\mu x_\nu - p_\nu x_\mu + i[\psi_\mu, \psi_\nu] \tag{1.4}$$

equal to the sum of orbital and spin parts.

The generalization of the above correspondence to higher spin fields was achieved [8] by quantizing the action of spinning particles with extended local worldline supersymmetry. It was found [8, 9, 10] that the resulting physical space of particle is described by massless antisymmetric quantum fields.

However, the imposing of the extended local supersymmetry is not the only way of enlarging symmetry. In the present paper, we propose the action for spinning particles which possesses nontrivial dynamical symmetry different from extended supersymmetry. The corresponding symmetry algebra is not Lie algebra and is known [1, 3] as polynomial algebra. It contains ordinary supersymmetry algebra (1.3) as a special case and it is this property which allows us to refer to the resulting symmetry algebra as to parasupersymmetric one.

In sect.2 we define commutation relations between dynamical fields and parasupersymmetry algebra. Here, the hamiltonian and action of the spinning particle are specified and the definition of the physical subspace is given. The representations of the symmetry algebra and the relation to the supersymmetric particle are investigated in sect.3. The physical subspace of the massless particle possessing parasupersymmetry of order $R=2$ is determined in sect.4. The generalizations to massive particles and higher orders R are discussed in sect.5.

2. Parasupersymmetric spinning particle

In the $D-$ dimensional euclidean space-time the parasupersymmetric spinning particle is described by D bosonic coordinates x_μ and by D real parafermionic variables ψ_μ.

2.1. Commutation relations

Instead of imposing commutation relations (1.2) between the operators ψ_μ we *postulate* that the total angular momentum of the particle has to be equal to (1.4). This condition allows us to establish commutation relations for parafermi operators ψ_μ as follows. The operators ψ_μ are transformed as vectors under rotations of the $D-$dimensional space

$$\delta\psi_\mu = \omega_{\mu\nu}\psi_\nu \equiv -i[\psi_\mu, \tfrac{1}{2}\omega^{\alpha\beta}\Sigma_{\alpha\beta}]$$

with $\omega_{\alpha\beta}$ and $\Sigma_{\alpha\beta}$ being angle of rotation and angular momentum, respectively. After substitution of (1.4) into this relation one finds the parafermionic commutation relations [11]

$$[\psi_\mu,[\psi_\nu,\psi_\rho]] = g_{\mu\nu}\psi_\rho - g_{\mu\rho}\psi_\nu \tag{2.1}$$

where the remaining commutation relations

$$[x_\mu, p_\nu] = ig_{\mu\nu}, \quad [x_\mu,\psi_\nu]=0, \quad [p_\mu,\psi_\nu]=0 \tag{2.2}$$

are imposed.

The resulting commutation relations being trilinear contain (1.2) as a special case. In the algebra (2.1) and (2.2) the operators ψ_μ and x_μ can be thought of as odd and even elements, respectively.

It is well known [12] that algebra (2.1) is isomorphic to the $SO(D+1)$ algebra and, hence, matrices ψ_μ belong, in general, to the reducible representation of $SO(D+1)$. If one chooses ψ_μ in the irreducible representation (irreps) of this algebra, then for an arbitrary $D-$dimensional vector p_μ the matrix $(p\cdot\psi)/|p|$ has eigenvalues $-\frac{1}{2}R, -\frac{1}{2}R+1, \ldots, \frac{1}{2}R-1, \frac{1}{2}R$ with R being an integer positive number and satisfies the following characteristic equation [12]:

$$((p\cdot\psi)-\tfrac{1}{2}R|p|)((p\cdot\psi)-(\tfrac{1}{2}R-1)|p|)\cdots((p\cdot\psi)+\tfrac{1}{2}R|p|)=0 \tag{2.3}$$

For $R=1$ Dirac matrices are the solutions of this equation: $\psi_\mu = \frac{1}{2}\gamma_\mu$. For $R=2$ it follows from (2.1) and (2.3) that matrices ψ_μ belong to the Duffin-Kemmer algebra [13, 14]

$$\psi_\mu\psi_\nu\psi_\rho + \psi_\rho\psi_\nu\psi_\mu = g_{\mu\nu}\psi_\rho + g_{\nu\rho}\psi_\mu \tag{2.4}$$

So Duffin-Kemmer algebra is the simplest nontrivial example of trilinear parafermionic commutation relations. For $R>1$ among its solutions one can find matrices satisfying the analogous equation for smaller R as well as the set of irreps parametrized by the same number R.[1]

[1]Only irreps of the $SO(3)$ algebra are uniquely fixed by the parameter R.

Instead of fixing "by hand" the representation of the $SO(D+1)$ algebra we assume henceforth that the matrix $(p\cdot\psi)$ obeys equation (2.3) for an arbitrary vector p_μ and a fixed number R. This condition is not a consequence of (2.1) and expresses an additional constraint on the matrices ψ_μ.

The wave function of a spinning particle has to be invariant under the action of local parasupersymmetric transformations and worldline reparametrizations. One chooses generators of these transformations in the same form (1.1) as for supersymmetric spinning particle. Then, the operators $\mathcal{H}$ and $\mathcal{Q}$ form the parasupersymmetric algebra

$$[\mathcal{Q},\mathcal{H}]=0, \quad \mathcal{Q}(\mathcal{Q}^2-2\mathcal{H})\cdots(\mathcal{Q}^2-\tfrac{1}{2}R^2\mathcal{H})=0, \quad \text{for even } R \tag{2.5a}$$

and

$$[\mathcal{Q},\mathcal{H}]=0, \quad (\mathcal{Q}^2-\tfrac{1}{2}\mathcal{H})\cdots(\mathcal{Q}^2-\tfrac{1}{2}R^2\mathcal{H})=0, \quad \text{for odd } R, \tag{2.5b}$$

with positive integer number R called [11] the order of parasupersymmetry. For $R=1$ the parasupersymmetry algebra reduces to supersymmetry algebra (1.3).

The transformations of dynamical fields are generated by the following operators:

$$\delta_\lambda = \tfrac{1}{2}[\ ,[\lambda,\mathcal{Q}]] = \tfrac{1}{2}[\ ,[\lambda,(p\cdot\psi)]], \quad \delta_a = -i[\ ,a\mathcal{H}] = -\tfrac{i}{2}[\ ,ap^2] \tag{2.6}$$

Here, parameters of parasupersymmetric transformations λ are the generalized grassman numbers [11] and have nontrivial commutation relations with the operators ψ_μ

$$[\psi_\mu,[\psi_\nu,\lambda]] = g_{\mu\nu}\lambda, \quad [\lambda,[\psi_\mu,\psi_\nu]]=0, \quad [\lambda,[\lambda',\psi_\mu]]=0, \quad [\psi_\mu,[\lambda,\lambda']]=0, \quad [\lambda,[\lambda',\lambda'']]=0 \tag{2.7}$$

Some of these relations are a consequence of the Jacobi identity. The reason for the choice (2.6) originates from the fact that the operators δ_λ and δ_a form the Lie algebra:

$$[\delta_\lambda,\delta_{\lambda'}] = \delta_{a=-\frac{i}{2}[\lambda,\lambda']}, \quad [\delta_\lambda,\delta_a]=0, \quad [\delta_a,\delta_{a'}]=0, \tag{2.8}$$

where the relations $[\lambda,[\lambda',\mathcal{Q}]]=0$ and $[\mathcal{Q},[\lambda,\mathcal{Q}]]=-2\lambda\mathcal{H}$ were used. So under reparametrizations and local parasupersymmetry transformations states of a spinning particle are changed as

$$|0\rangle \xrightarrow{\mathcal{H}} \exp(ia\mathcal{H})|0\rangle, \qquad |0\rangle \xrightarrow{\mathcal{Q}} \exp(-\tfrac{1}{2}[\lambda,\mathcal{Q}])|0\rangle$$

The invariance of the wave function of a spinning particle under these transformations implies that

$$\mathcal{H}|0\rangle = 0, \qquad [\lambda,\mathcal{Q}]|0\rangle = 0 \tag{2.9}$$

for arbitrary λ. This is the definition of the physical space of a spinning particle. One notices that as distinct from the standard supersymmetric case the parameters λ cannot be eliminated from the above equations.

2.2. Hamiltonian of the spinning particle

The hamiltonian of the spinning particle whose wave functions satisfy (2.9) is equal to

$$H = e\mathcal{H} + \tfrac{i}{2}[\lambda,\mathcal{Q}] \equiv \tfrac{1}{2}ep_\mu^2 + \tfrac{i}{2}[\lambda,(p\cdot\psi)]$$

where worldline parasupergravity gauge fields e and λ play a role of the Lagrange multipliers. Using commutation relations (2.1) and (2.2) we find equations of motion for the spinning particle as

$$\begin{aligned} \dot{x}_\mu &= ep_\mu + \tfrac{i}{2}[\lambda, \psi_\mu] \\ \dot{p}_\mu &= 0 \\ \dot{\psi}_\mu &= -\tfrac{1}{2}\lambda p_\mu \end{aligned} \tag{2.10}$$

where dot denotes the derivative over proper time. Let us demonstrate that these equations are invariant under reparametrizations and have parasupersymmetry.

Under unitary transformations $\phi \to \phi' = U^\dagger \phi U$ of an arbitrary field ϕ the equation of motion changes as $i\dot{\phi}' = [\phi', H']$ where $H' = U^\dagger H U + iU^\dagger \dot{U}$. Now we don't put the restriction $U^\dagger H U = H$ which will imply conservation of the charge corresponding to the transformation on the full space of states. It is enough to require $U^\dagger H U - H \approx 0$ only on the physical subspace (2.9). The remaining term $iU^\dagger \dot{U}$ takes into account nonconservation of the charge and the dependence of parameters of transformations on proper time. For reparametrizations and local parasupersymmetry transformations this term can be compensated by proper transformations of gauge fields e and λ. Indeed, under reparametrizations when $U = \exp(ia\mathcal{H})$ and dynamical fields transform as $\delta_a \phi = -i[\phi, a\mathcal{H}]$ we get $H = U^\dagger H U$ and $iU^\dagger \dot{U} = -\dot{a}\mathcal{H}$. The shift of the hamiltonian $H' = (e - \dot{a})\mathcal{H} + \frac{i}{2}[\lambda, \mathcal{Q}]$ is compensated by transformation of the gauge field e. Thus, equations of motion (2.10) are invariant under worldline reparametrizations provided that

$$\begin{aligned} \delta_a e &= \dot{a} \\ \delta_a x_\mu &= ap_\mu \\ \delta_a \lambda &= \delta_a p_\mu = \delta_a \psi_\mu = 0 \end{aligned} \tag{2.11}$$

We notice that unlike diffeomorphisms, reparametrizations in the hamiltonian approach form the abelian group [7].

It turns out that the parasupersymmetry charge $\mathcal{Q}$ satisfies the equation

$$\dot{\mathcal{Q}} = -i[\mathcal{Q}, H] = \tfrac{i}{2}[\mathcal{Q}, [\lambda, \mathcal{Q}]] = -\lambda\mathcal{H}$$

and is conserved only on the physical space (2.9) where $\mathcal{H} \approx 0$. Then, under local infinitesimal transformations with $U = 1 - \frac{i}{2}[\varepsilon, \mathcal{Q}] + \mathcal{O}(\varepsilon^2)$ and $\delta_\varepsilon \phi = \frac{i}{2}[\phi, [\varepsilon, \mathcal{Q}]]$ one gets

$$H' \approx H + \tfrac{i}{2}[\varepsilon, \lambda]\mathcal{H} - \tfrac{i}{2}[\dot{\varepsilon}, \mathcal{Q}].$$

The shift of the hamiltonian is compensated by transformations of both gauge fields e and λ. Thus, the equations of motion are invariant under the following parasupersymmetric transformations:

$$\begin{aligned} \delta_\varepsilon e &= -\tfrac{i}{2}[\varepsilon, \lambda] \\ \delta_\varepsilon \lambda &= \dot{\varepsilon} \\ \delta_\varepsilon x_\mu &= \tfrac{i}{2}[\varepsilon, \psi_\mu] \\ \delta_\varepsilon \psi_\mu &= -\tfrac{1}{2}\varepsilon p_\mu \\ \delta_\varepsilon p_\mu &= 0 \end{aligned} \tag{2.12}$$

It can easily be checked, by using (2.11) and (2.12), that the operators δ_ε and δ_a form algebra (2.8).

In eq.(2.12) the generalized grassman numbers ε are parameters of transformations. However they are not in general completely arbitrary. The restrictions on ε follow from the property that the operator $\delta_\varepsilon \psi_\mu$ has to belong to the same algebra (2.1) and (2.3) as the operator ψ_μ. In particular, commutation relations (2.1) are fulfilled provided that the parameters ε obey (2.7). Let us turn to relation (2.3). In the simplest case $R = 2$ the variations of both sides of the relation $\mathcal{Q}^3 = 2\mathcal{H}\mathcal{Q}$ with $\delta_\varepsilon \mathcal{Q} = -\varepsilon\mathcal{H}$ and $\delta_\varepsilon \mathcal{H} = 0$ lead to

$$\psi_\mu\psi_\nu\varepsilon + \varepsilon\psi_\nu\psi_\mu = g_{\mu\nu}\varepsilon, \quad \psi_\mu\varepsilon\psi_\nu + \psi_\nu\varepsilon\psi_\mu = 0, \quad \varepsilon\psi_\mu\varepsilon = 0, \quad \varepsilon^2\psi_\mu + \psi_\mu\varepsilon^2 = 0, \quad \varepsilon^3 = 0 \tag{2.13}$$

where (2.7) was used. Thus, the parameters ε turn out to be consistent with the Duffin-Kemmer algebra (2.4). Moreover, ε can be included into this algebra by introducing new additional coordinate $M = (\mu, D+1)$ and the corresponding operator ψ_{D+1} with the properties

$$\psi_{D+1} \equiv \varepsilon, \quad g_{D+1,N} = 0 \tag{2.14}$$

Then, for $R = 2$ the operators ψ_μ and parameters ε form the $(D+1)-$dimensional Duffin-Kemmer algebra

$$\psi_M\psi_N\psi_K + \psi_K\psi_N\psi_M = g_{MN}\psi_K + g_{NK}\psi_M \tag{2.15}$$

These relations admit natural generalization to higher orders $R \geq 2$. Indeed, the inclusion of the additional coordinate (2.14) into (2.1) allows us to satisfy simultaneously the commutation relations (2.1) and (2.7) but relations (2.3) preserve their form under parasupersymmetry transformations $\mathcal{Q} \xrightarrow{\mathcal{Q}} \mathcal{Q} + \delta_\varepsilon\mathcal{Q} = (p \cdot \psi) - \varepsilon\mathcal{H} \equiv p^M\psi_M$ and $\mathcal{H} \xrightarrow{\mathcal{Q}} \mathcal{H} + \delta_\varepsilon\mathcal{H} = g_{MN}p^Mp^N = p_\mu^2$ provided that one identifies $\psi_{D+1} = \varepsilon$ and $p^{D+1} = -\mathcal{H} = -\frac{1}{2}p^2$. Thus, for an arbitrary order R parameters ε are additional elements of the algebra of the ψ_μ operators. In particular, for $p_\mu = 0$ and $p_{D+1} \neq 0$ we get from (2.3) that $\varepsilon^{R+1} = 0$. The same restrictions are to be put also on the gauge field λ since under transformations (2.12) they are shifted by $\dot{\varepsilon}$.

2.3. Action of the spinning particle

Let us define the action of the spinning particle whose quantization will reproduce trilinear commutation relations (2.1) and definition (2.9) of the physical space as

$$S = \int_0^1 dt(p_\mu\dot{x}_\mu + \tfrac{i}{2}[\psi_\mu, \dot{\psi}_\mu] - H) \equiv \int_0^1 dt(p_\mu\dot{x}_\mu + \tfrac{i}{2}[\psi_\mu, \dot{\psi}_\mu] - \tfrac{1}{2}ep_\mu^2 - \tfrac{i}{2}[\lambda, (p \cdot \psi)]) \tag{2.16}$$

Here x_μ and p_μ are independent variables, and before quantization ψ_μ and λ are generalized grassman numbers: $[\psi_\mu, [\psi_\nu, \psi_\rho]] = 0$, $[\psi_\mu, [\psi_\nu, \lambda]] = 0$ and $[\lambda, [\psi_\mu, \lambda]] = 0$.

The variation of the action over x_μ, p_μ and over e, λ leads to the equations of motion (2.10) and to definition (2.9) of the physical space, respectively. The integral of motion corresponding to the invariance of the action under rotations of the vectors x_μ, p_μ and ψ_μ coincides with the angular momentum (1.4). The invariance of the equations of motion (2.10) under reparametrizations and local parasupersymmetry transformations implies the invariance of the action (2.16) only on-shell.

Picking out in (2.16) the form quadratic over p_μ we eliminate this auxiliary field from the action

$$S = \int_0^1 dt\left(\frac{1}{2e}\dot{x}_\mu^2 + \frac{i}{2}[\psi_\mu, \dot{\psi}_\mu] - \frac{i}{2e}[\lambda, (\dot{x} \cdot \psi)] - \frac{1}{8e}[\lambda, \psi_\mu][\lambda, \psi_\mu]\right) \tag{2.17}$$

where e and λ are gauge fields.

Let us demonstrate that the quantization of the action leads to trilinear commutation relations (2.1). To this end, it is sufficient to restrict ourselves to the kinetic term for odd coordinates

$$S_0 = \frac{i}{2}\int_0^1 dt\ [\psi_\mu, \dot{\psi}_\mu]$$

and consider the following correlation function:

$$\langle \psi_1(t_1)\ldots\psi_N(t_N)\rangle \equiv \int \mathcal{D}\psi_\mu\ \psi_{\mu_1}(t_1)\ldots\psi_{\mu_N}(t_N)\ \exp\left(-\tfrac{1}{2}\int_0^1 dt[\psi_\mu, \dot{\psi}_\mu]\right)$$

where $\psi_k \equiv \psi_{\mu_k}(t_k)$, $t_k \in [0,1]$ and the integration measure over generalized grassman numbers is invariant under translations and rotations of ψ_μ in the $D-$dimensional space. Rotating the field $\psi_\mu(t) \to \Lambda_{\mu\nu}(t)\psi_\nu(t)$ with $\Lambda_{\mu\nu}(0) = \Lambda_{\mu\nu}(1) = 0$ and using invariance of the measure we find the Ward identities

$$\langle \psi_1(t_1)\ldots\psi_N(t_N)\frac{d}{dt}[\psi_\mu, \psi_\nu]\rangle + \sum_{k=1}^{N}\delta(t-t_k)\langle\psi_1(t_1)\ldots(g_{\mu\mu_k}\psi_\nu(t_k) - g_{\nu\mu_k}\psi_\mu(t_k))\ldots\psi_N(t_N)\rangle = 0$$

Integrating both sides of this relation over $\int_{t_1-0}^{t_1+0} dt$ with $[\psi_k(t_k), [\psi_\mu(t), \psi_\nu(t)]] = 0$ for $t \neq t_k$ one gets the trilinear commutation relations (2.1). In an analogous manner the Word identities corresponding to the invariance of the integration measure under the shifts $\psi_\mu(t) \to \psi_\mu(t) + p_\mu\varepsilon(t)$, $\varepsilon(0) = \varepsilon(1) = 0$ with an arbitrary vector p_μ and generalized grassman numbers ε lead to relations (2.7).

The spinning particle with the action (2.16) and (2.17) is essentially massless since the mass term $\int_0^1 dt\, eM^2$ breaks parasupersymmetry (2.12). Nevertheless, it is possible to introduce mass as follows. Note that momentum p_μ satisfies the equation of motion $\dot{p}_\mu = 0$ and is not changed under transformations (2.11) and (2.12). As a consequence, we can put the constraint

$$p_D - M \approx 0 \tag{2.18}$$

and don't spoil (2.10), (2.11) and (2.12). After replacement $p_D = M$ in (2.16) the Lorentz indices run as $\mu = 1, \ldots, D-1$ and the action acquires the addition

$$\delta S = \int_0^1 dt(\tfrac{i}{2}[\psi_D, \dot{\psi}_D] - \tfrac{1}{2}eM^2 - \tfrac{i}{2}M[\lambda, \psi_D])$$

The equation of motion and the transformation properties of the parafermionic field ψ_D are found from (2.10), (2.11) and (2.12) as

$$\dot{\psi}_D = -\tfrac{1}{2}M\lambda, \quad \delta_a\psi_D = 0, \quad \delta_\varepsilon\psi_D = -\tfrac{1}{2}M\varepsilon$$

After eliminating from $S + \delta S$ the dependence on p_μ the action of the massive spinning particle reduces to

$$S = \int_0^1 dt\left(\frac{1}{2e}\dot{x}_\mu^2 - \frac{1}{2}eM^2 + \frac{i}{2}[\psi_\mu, \dot{\psi}_\mu] + \frac{i}{2}[\psi_D, \dot{\psi}_D] - \frac{i}{2e}[\lambda, \dot{x}_\mu\psi_\mu] - \frac{i}{2}M[\lambda, \psi_D] - \frac{1}{8e}[\lambda, \psi_\mu][\lambda, \psi_\mu]\right)$$

and under its quantization the wave function of the particle obeys

$$(p_\mu^2 + M^2)|0\rangle = 0, \quad [\lambda, p^\mu\psi_\mu + M\psi_D]|0\rangle = 0, \quad \mu = 1, \ldots, D-1$$

Thus, starting with massless parasupersymmetric particle in the $D-$dimensional space we can describe massive parasupersymmetric particle in the $(D-1)$-dimensional space by imposing (2.18).

3. Physical space of the spinning particle

3.1. Representation of the parasupersymmetric algebra

The operators ψ_μ are elements of the representation of the $SO(D+1)$ algebra which satisfy additional restrictions (2.3). To solve these relations for an arbitrary order R one notices that the $D-$dimensional Dirac matrices are general solutions for $R = 1$. Then, the direct product of two spinor representations gives the general solution for $R = 2$ and so on. Thus, for arbitrary R all irreducible solutions of (2.1) and (2.3) are contained in the decomposition of the direct product of R spinor irreps. In this sense, the most general form of ψ_μ matrices is [11, 12]

$$\hat{\psi}_\mu = \frac{1}{2}\sum_{\alpha=1}^{R}\gamma_\mu^{(\alpha)}, \quad \gamma_\mu^{(\alpha)} = \underbrace{I \times \cdots \times \gamma_\mu}_{\alpha} \times \cdots \times I \tag{3.1}$$

and is called [12] the Green anzatz. By applying the standard methods this solution can be decomposed over irreducible ones. Denote projectors onto irreps as $\mathcal{P}_A$. They have the following properties:

$$I = \sum_A \mathcal{P}_A, \quad \psi_\mu\mathcal{P}_A = \mathcal{P}_A\psi_\mu, \quad \mathcal{P}_A\mathcal{P}_B = \delta_{AB}\mathcal{P}_A \tag{3.2}$$

We conclude that solutions of (2.1) and (2.3) are

$$\psi_\mu = \mathcal{P}\hat{\psi}_\mu = \hat{\psi}_\mu\mathcal{P}, \quad \mathcal{P} = \sum_{A\in S}\mathcal{P}_A \tag{3.3}$$

where the choice of the set S of irreps $A \in S$ entering into the decomposition of the projector $\mathcal{P}$ is fully arbitrary. In fact, the set S is a free parameter of the spinning particle.

Let us suppose that one fixed the explicit form of the projector $\mathcal{P}$. It turns out that the projector does not conserve

$$\dot{\mathcal{P}} = -i[\mathcal{P}, H] = \tfrac{1}{2}[\mathcal{P}, [\lambda, \mathcal{Q}]] = \tfrac{1}{2}[\mathcal{Q}, [\lambda, \mathcal{P}]] \neq 0$$

unless $\mathcal{P} = I$ and the operators ψ_μ coincide with the Green anzatz. This relation implies that if at the initial moment of time the operators ψ_μ were defined in the representation with the projector $\mathcal{P} \neq I$, then evaluating the spinning particle will turn into the representation differing from $\mathcal{P}$. To conserve the condition (3.3) under evolution of the spinning particle one is enforced to put additional conditions on the Lagrange multipliers $[\lambda, \mathcal{P}] = 0$ or

$$\lambda = \mathcal{P}\lambda = \lambda\mathcal{P} \tag{3.4}$$

An analogous condition is imposed on the parameters ε of parasupersymmetric transformations (2.12) since variation $\delta_\varepsilon\psi_\mu = -\frac{1}{2}\varepsilon p_\mu$ has to satisfy (3.3). In the special case $\mathcal{P} = I$ relation (3.4) reduces to identity but in general it is possible to define the projector $\mathcal{P}$ in such a manner that the only solution of (3.4) will be the trivial one $\lambda = 0$. It will mean that the spinning particle cannot propagate in the representation of odd coordinates ψ_μ thus defined.

3.2. Relation to the supersymmetric spinning particle

Suppose one fixed the order of parasupersymmetry and projector $\mathcal{P}$ in eq.(3.3). The physical subspace of the spinning particle is defined by equations (2.9) and (3.4) together with the condition

$$|0\rangle = \mathcal{P}\ |0\rangle \tag{3.5}$$

which picks out the subspace where the operators ψ_μ act.

We start with solution of (3.4) in the simplest case when the matrices ψ_μ are given by the Green anzatz and $\mathcal{P} = 1$. Relations (3.4) and (3.5) become identity and (2.7) is the only restriction on λ. The fields λ being additional elements of the algebra of the ψ_μ operators are expressed for a fixed order R as [11]

$$\hat{\lambda} = \sum_{\alpha=1}^{R} \lambda^{(\alpha)}, \quad \lambda^{(\alpha)}\lambda^{(\alpha)} = 0, \quad \{\lambda^{(\alpha)}, \gamma_\mu^{(\alpha)}\} = 0, \quad [\lambda^{(\alpha)}, \lambda^{(\beta)}] = 0, \quad [\lambda^{(\alpha)}, \gamma_\mu^{(\beta)}] = 0,\ \alpha \neq \beta$$

where $\lambda^{(\alpha)}$ are arbitrary. After substitution into (2.9) we find, using $[\hat{\lambda}, \hat{\psi}_\mu] = \sum_{\alpha=1}^{R} \lambda^{(\alpha)}\gamma_\mu^{(\alpha)}$, the following equations for the wave function $|0\rangle_{\alpha_1\ldots\alpha_R} \equiv \Psi_{\alpha_1\ldots\alpha_R}(p)$ of the spinning particle

$$p_\mu \gamma^\mu_{\alpha_k\beta_k} \Psi_{\alpha_1\ldots\beta_k\ldots\alpha_R}(p) = 0$$

well known as the Bargman-Wigner equations. The solutions of these equations describe a couple of "spins". One notices that the same relations were found [8] for the wave function of the spinning particle possessing worldline $N = R$ extended supersymmetry.[2] Therefore, the physical space of the parasupersymmetric spinning particle with the operators ψ_μ defined in the Green anzatz representation and the physical space of the spinning particle with an extended supersymmetry are equivalent. The difference between these particles appears when one puts additional restrictions on the representation of the operators ψ_μ by choosing nontrivial projector $\mathcal{P}$ in the definition (3.3).

For $\mathcal{P} \neq I$ both the parasupersymmetric parameters ε and Lagrange multipliers λ obey the condition (3.4) and, as consequence, the resulting equations (2.9) on the physical space become less restrictive. Therefore, the physical space of the spinning particle with $\mathcal{P} = I$ when one uses the Green anzatz is a subspace of the physical space of the spinning particle with $\mathcal{P} \neq I$.

Thus, one connects novel properties of parasupersymmetric particles with nontrivial choice of the projectors $\mathcal{P} \neq I$ in (3.3). To define $\mathcal{P}$ we have to determine the projectors $\mathcal{P}_A$ onto irreps of the algebra of the ψ_μ operators for arbitrary orders R.

For $R = 1$ the matrices ψ_μ obey (1.2) and coincide with the $D-$dimensional Dirac matrices. The only irreps are spinor ones[3] and the $R = 1$ parasupersymmetric particle is in fact an ordinary spinning particle.

For $R = 2$ the matrices ψ_μ belong to the Duffin-Kemmer algebra (2.4) which has a set of irreps (3.2) found in [14]. For instance, in the $D = 4-$dimensional space-time the algebra has only three irreps [14, 17]: trivial ($\psi_\mu = 0$), scalar and vector ones with dimensions 1, 5 and 10, respectively.[4]

[2]For extended supersymmetric spinning particle the internal $O(N)$ symmetry is introduced to extract from the wave function $\Psi_{\alpha_1\ldots\alpha_R}(p)$ the field with maximum spin $N/2$ in the $D = 4-$dimensional space-time. After quantization the internal symmetry turns out to be anomalous [15, 16].

[3]For odd dimensions D there are two spinor irreps differing by the sign of matrices.

[4]This notation goes back to the theory [17] of relativistic wave equations of the form $((p\cdot\psi) + M)\phi(p) = 0$. These equations with the matrices ψ_μ chosen in the scalar and vector representations describe massive scalar and vector fields,respectively.

4. Parasupersymmetric spinning particle of order $R = 2$

The physical space of parasupersymmetric spinning particle of order $R = 2$ is defined by equations (2.9) and (3.4) were matrices ψ_μ belong to the Duffin-Kemmer algebra (2.4). Before solving them we find it convenient to introduce the notion of dual matrices $\tilde{\psi}_\mu$ and dual parameters $\tilde{\lambda}$.

4.1. Dual operators

The dual matrices are defined as real solutions of the equation [18]

$$\psi_\mu\psi_\nu + \tilde{\psi}_\nu\tilde{\psi}_\mu = g_{\mu\nu}$$

Together with the identity $(p\cdot\psi)^3 = p^2(p\cdot\psi)$ this definition implies

$$(p\cdot\tilde{\psi})^3 = p^2(p\cdot\tilde{\psi}), \quad (p\cdot\psi)(p\cdot\tilde{\psi}) = 0 \tag{4.1}$$

for an arbitrary $D-$dimensional vector p_μ. Thus, dual matrices also belong to the Duffin-Kemmer algebra and satisfy

$$\psi_\mu^2 = 1 - \tilde{\psi}_\mu^2, \quad \psi_\mu\tilde{\psi}_\mu = 0, \quad \psi_\mu\psi_\nu = -\tilde{\psi}_\nu\tilde{\psi}_\mu, \quad \psi_\mu\tilde{\psi}_\nu + \psi_\nu\tilde{\psi}_\mu = 0, \quad (\mu \neq \nu) \tag{4.2}$$

To understand the meaning of dual matrices one introduces operators $\psi_\mu = \frac{1}{2}(\gamma_\mu^{(1)} + \gamma_\mu^{(2)})$, $\tilde{\psi}_\mu = \frac{1}{2}(\gamma_\mu^{(1)} - \gamma_\mu^{(2)})$ and gets from (4.2): $\{\gamma_\mu^{(1)}, \gamma_\nu^{(1)}\} = 2\delta_{\mu\nu}$, $\{\gamma_\mu^{(2)}, \gamma_\nu^{(2)}\} = 2\delta_{\mu\nu}$, $[\gamma_\mu^{(1)}, \gamma_\nu^{(2)}] = 0$. It is easy to recognize in the above expressions the Green decomposition of parafermi operators of order $R = 2$:

$$\psi_\mu = \tfrac{1}{2}(\gamma_\mu \times I + I \times \gamma_\mu), \quad \tilde{\psi}_\mu = \tfrac{1}{2}(\gamma_\mu \times I - I \times \gamma_\mu) \tag{4.3}$$

Dual parameters of parasupersymmetric transformations $\tilde{\lambda}$ are defined analogously to (4.2) as

$$\psi_\mu\lambda = -\tilde{\lambda}\tilde{\psi}_\mu, \quad \lambda\tilde{\lambda} = 0 \tag{4.4}$$

If the parameter λ is identified with the additional element ψ_{D+1} of the Duffin-Kemmer algebra, then $\tilde{\lambda}$ is an analog of the dual operator $\tilde{\psi}_{D+1}$.

4.2. Representation of the Duffin-Kemmer algebra

Expression (3.3) for the operators ψ_μ contains projectors $\mathcal{P}_A$ onto irreps of the Duffin-Kemmer algebra. We define $\mathcal{P}_A$ following [14]. Introduce the operator

$$\eta = \sum_{\mu=1}^{D} \eta_\mu, \quad \eta_\mu = 2\psi_\mu^2 - 1$$

where η_μ form a commutative subalgebra with the properties

$$[\eta_\mu, \eta_\nu] = 0, \quad \eta_\mu^2 = 1, \quad \eta_\mu\psi_\mu = \psi_\mu, \quad \eta_\mu\psi_\nu = -\psi_\nu\eta_\mu, \quad \mu \neq \nu$$

The eigenvalues of the operators η_μ and η are equal to ± 1 and $-D, -D+2, \ldots, D-2, D$, respectively. Let Π_A be projectors onto eigenstates of the operator η

$$\eta\Pi_A = \eta_A\Pi_A, \quad \eta_A = D - 2A, \quad A = 0, 1, \ldots, D$$

with

$$I = \sum_{A=0}^{D} \Pi_A, \quad \Pi_A \Pi_B = \delta_{AB} \Pi_A$$

The explicit form of Π_A in terms of the matrices ψ_μ is

$$\begin{aligned}
\Pi_0 &= \psi_1^2 \psi_2^2 \cdots \psi_D^2, \\
\Pi_1 &= \tilde{\psi}_1^2 \psi_2^2 \cdots \psi_D^2 + \cdots + \psi_1^2 \psi_2^2 \cdots \tilde{\psi}_D^2 \\
&\vdots \\
\Pi_A &= \tilde{\psi}_{\{1}^2 \cdots \tilde{\psi}_A^2 \psi_{A+1}^2 \cdots \psi_{D\}}^2
\end{aligned} \tag{4.5}$$

and $\{\ldots\}$ denotes symmetrization over the vector indices with $\psi_\mu^2 \psi_\nu^2 = \psi_\nu^2 \psi_\mu^2$. Being combined with the identities $\psi_\mu \psi_\nu^2 = \tilde{\psi}_\nu^2 \psi_\mu$ $(\mu \neq \nu)$ and $\lambda \psi_\mu^2 = \tilde{\psi}_\mu^2 \lambda$ the above expression allows us to establish that

$$\begin{aligned}
\psi_\mu \Pi_A &= \Pi_{D-1-A} \psi_\mu, \quad \psi_\mu \Pi_D = 0, \\
\tilde{\psi}_\mu \Pi_A &= \Pi_{D+1-A} \tilde{\psi}_\mu, \quad \tilde{\psi}_\mu \Pi_0 = 0 \\
\lambda \Pi_A &= \Pi_{D-A} \lambda
\end{aligned} \tag{4.6}$$

The first relation prompts to define the projectors $\mathcal{P}_A$ as

$$\mathcal{P}_{-1} = \Pi_D, \quad \mathcal{P}_A = \Pi_A + \Pi_{D-1-A}, \quad 0 \leq A \leq \tfrac{D-2}{2} \tag{4.7}$$

Indeed, for even space-time dimensions D it can be easily verified that the operators $\mathcal{P}_A$ do satisfy (3.2). For odd dimensions there appears a problem with definition of $\mathcal{P}_A$ for $A = \frac{D-1}{2}$ when (4.7) is not valid. It is well-known [14] that for $D = 2Z + 1$ and $A = \frac{D-1}{2} = Z$ "twin algebras" appear as

$$\mathcal{P}_Z^{(+)} = \frac{1}{2}(\Pi_Z + \omega), \quad \mathcal{P}_Z^{(-)} = \frac{1}{2}(\Pi_Z - \omega), \quad \mathcal{P}_Z^{(+)} \mathcal{P}_Z^{(-)} = 0 \tag{4.8}$$

where $\mathcal{P}_Z^{(\pm)}$ are projectors onto twin algebras and the pseudoscalar element ω satisfies the conditions [14]

$$\omega = \tfrac{i^Z}{Z!(Z+1)!} \varepsilon^{\mu_1 \ldots \mu_D} \psi_{\mu_1} \cdots \psi_{\mu_D}, \quad \omega \Pi_A = \Pi_A \omega = \delta_{AZ} \omega, \quad \omega^2 = \Pi_Z, \quad \psi_\mu \omega = \omega \psi_\mu$$

Finally, the decomposition of the unity element of the Duffin-Kemmer algebra looks like

$$I = \mathcal{P}_{-1} + \mathcal{P}_0 + \mathcal{P}_1 + \cdots + \mathcal{P}_{Z-1} \tag{4.9a}$$

for even dimensions $D = 2Z$ and

$$I = \mathcal{P}_{-1} + \mathcal{P}_0 + \mathcal{P}_1 + \cdots + \mathcal{P}_{Z-1} + \mathcal{P}_Z^{(+)} + \mathcal{P}_Z^{(-)} \tag{4.9b}$$

for even dimensions $D = 2Z + 1$. The projectors $\mathcal{P}_{-1}$ correspond to the trivial representation $\psi_\mu = \mathcal{P}_{-1} \hat{\psi}_\mu = 0$.

4.3. Covariant projectors

Expression (4.5) for Π_A contains the sum of $\binom{D}{A} = \frac{D!}{A!(D-A)!}$ orthogonal projectors. Notice that the number of items coincides with the number of the independent components of a totally antisymmetric tensor of rank A or $D - A$ in the $D-$dimensional space-time. Therefore, it is natural to represent Π_A as $\Pi_A = \frac{1}{A!} P^\dagger_{\mu_1 \ldots \mu_A} P_{\mu_1 \ldots \mu_A}$ or $\Pi_A = \frac{1}{(D-A)!} P^\dagger_{\mu_1 \ldots \mu_{D-A}} P_{\mu_1 \ldots \mu_{D-A}}$ where $P_{\ldots}$ are non-hermitian operators totally antisymmetric with respect to indices. But the existence of vector indices does not mean that $P_{\ldots}$ are $D-$dimensional tensors. Nevertheless, it is possible to define $P_{\ldots}$ in such a manner that they will transform covariantly (as totally antisymmetric tensors) under rotations of space-time.

The covariant operators $P_{\ldots}$ are defined by the following recurrent relations:

$$\begin{aligned}
P &= \psi_1^2 \psi_2^2 \cdots \psi_D^2, \\
P_{\mu_1 \ldots \mu_{2N}} &= P_{\mu_1 \ldots \mu_{2N-1}} \tilde{\psi}_{\mu_{2N}} = P \psi_{\mu_1} \tilde{\psi}_{\mu_2} \cdots \psi_{\mu_{2N-1}} \tilde{\psi}_{\mu_{2N}} \\
P_{\mu_1 \ldots \mu_{2N+1}} &= P_{\mu_1 \ldots \mu_{2N}} \psi_{\mu_{2N+1}} = P \psi_{\mu_1} \tilde{\psi}_{\mu_2} \cdots \tilde{\psi}_{\mu_{2N}} \psi_{\mu_{2N+1}}
\end{aligned} \tag{4.10}$$

These operators are antisymmetric with respect to indices due to the relation $\psi_\mu \tilde{\psi}_\nu = -\psi_\nu \tilde{\psi}_\mu$. A simple calculation yields

$$P^\dagger_{1 \ldots 2N} P_{1 \ldots 2N} = \tilde{\psi}_1^2 \cdots \tilde{\psi}_{2N}^2 \psi_{2N+1}^2 \cdots \psi_D^2, \quad P^\dagger_{1 \ldots 2N-1} P_{1 \ldots 2N-1} = \psi_1^2 \cdots \psi_{2N-1}^2 \tilde{\psi}_{2N}^2 \cdots \tilde{\psi}_D^2$$

and one identifies the r.h.s. of these expressions with one of the items in (4.5). However, a more careful examination indicates that there is difference in constructing projectors Π_A in terms of the covariant operators for odd and even dimensions.

For the even dimensions $D = 2Z$ projectors are given by

$$\Pi_A = \tfrac{1}{A!} P^\dagger_{\mu_1 \ldots \mu_A} P_{\mu_1 \ldots \mu_A}, \quad \text{for even } A \tag{4.11a}$$

and by

$$\Pi_A = \tfrac{1}{(D-A)!} P^\dagger_{\mu_1 \ldots \mu_{D-A}} P_{\mu_1 \ldots \mu_{D-A}}, \quad \text{for odd } A. \tag{4.11b}$$

At the same time, for the odd dimensions $D = 2Z + 1$ it is impossible to define Π_A for odd A using the operators $P_{\ldots}$ since the r.h.s. of (4.11a) and (4.11b) coincide and

$$\Pi_A = \tfrac{1}{A!} P^\dagger_{\mu_1 \ldots \mu_A} P_{\mu_1 \ldots \mu_A} = \tfrac{1}{(D-A)!} P^\dagger_{\mu_1 \ldots \mu_{D-A}} P_{\mu_1 \ldots \mu_{D-A}}, \quad \text{only for even } A. \tag{4.12a}$$

To overcome this difficulty, one introduces dual covariant operators differing from the original operators by replacing the ψ_μ matrices by the dual ones $\tilde{\psi}_\mu$ and vice versa

$$\begin{aligned}
\tilde{P} &= \tilde{\psi}_1^2 \tilde{\psi}_2^2 \cdots \tilde{\psi}_D^2, \\
\tilde{P}_{\mu_1 \ldots \mu_{2N}} &= \tilde{P} \tilde{\psi}_{\mu_1} \psi_{\mu_2} \cdots \tilde{\psi}_{\mu_{2N-1}} \psi_{\mu_{2N}} \\
\tilde{P}_{\mu_1 \ldots \mu_{2N+1}} &= \tilde{P} \tilde{\psi}_{\mu_1} \psi_{\mu_2} \cdots \psi_{\mu_{2N}} \tilde{\psi}_{\mu_{2N+1}}
\end{aligned}$$

Then, the expression for Π_A is

$$\Pi_A = \tfrac{1}{A!} \tilde{P}^\dagger_{\mu_1 \ldots \mu_A} \tilde{P}_{\mu_1 \ldots \mu_A} = \tfrac{1}{(D-A)!} \tilde{P}^\dagger_{\mu_1 \ldots \mu_{D-A}} \tilde{P}_{\mu_1 \ldots \mu_{D-A}}, \quad \text{only for odd } A. \tag{4.12b}$$

Moreover, in the odd dimensional space-time there is a special case $A = \frac{D-1}{2} = Z$ when twin-algebras appear. To define in an analogous way the projectors $\mathcal{P}_Z^{(\pm)}$, we define the operators

$$P^{(\pm)}_{\mu_1 \ldots \mu_Z} = \tfrac{1}{2} \left(P_{\mu_1 \ldots \mu_Z} \pm \tfrac{i^Z}{(Z+1)!} \varepsilon_{\mu_1 \ldots \mu_Z \mu_{Z+1} \ldots \mu_D} P_{\mu_{Z+1} \ldots \mu_D} \right) \tag{4.12c}$$

and after tiresome calculation get for $D = 2Z + 1$ and even Z[5]

$$\mathcal{P}_Z^{(\pm)} = \tfrac{1}{Z!} \left(P^{(\pm)}_{\mu_1 \ldots \mu_Z}\right)^\dagger P^{(\pm)}_{\mu_1 \ldots \mu_Z}, \quad \left(P^{(+)}_{\mu_1 \ldots \mu_Z}\right)^\dagger P^{(-)}_{\nu_1 \ldots \nu_Z} = 0, \tag{4.12d}$$

Thus, projectors Π_A are given by (4.11a) and (4.11b) for even dimensions D and by expressions (4.12a), (4.12b) and (4.12d) for odd dimensions D of space-time.

To find transformation properties of the operators $P_{\ldots}$ and to prove that they indeed are covariant we notice that under rotations of the $D-$dimensional space with the angle $\omega_{\alpha\beta}$, the operator $P_{\mu_1 \ldots \mu_N}$ is transformed as

$$\delta P_{\mu_1 \ldots \mu_N} = -i[P_{\mu_1 \ldots \mu_N}, \tfrac{1}{2}\omega_{\alpha\beta}\Sigma_{\alpha\beta}] = \tfrac{1}{2}\omega_{\alpha\beta}[P_{\mu_1 \ldots \mu_N}, [\psi_\alpha, \psi_\beta]] \tag{4.13}$$

where angular momentum $\Sigma_{\alpha\beta}$ was postulated in (1.4). Let us examine the action of $P_{\mu_1 \ldots \mu_N}$ on the matrices ψ_μ. The evaluation using (4.2) gives

$$\begin{aligned} P\psi_{\mu_1} &= P_{\mu_1} \\ P_{\mu_1 \ldots \mu_{2N-1}}\psi_{\mu_{2N}} &= \delta_{\mu_{2N}[\mu_1} P_{\mu_2 \ldots \mu_{2N-1}]} \\ P_{\mu_1 \ldots \mu_{2N}}\psi_{\mu_{2N+1}} &= P_{\mu_1 \ldots \mu_{2N}\mu_{2N+1}} \end{aligned} \tag{4.14}$$

where $[\cdots]$ denotes antisymmetrization over indices and $P_{\mu_1 \ldots \mu_{D+1}} \equiv 0$. The action of $P_{\mu_1 \ldots \mu_N}$ on the dual matrices $\tilde{\psi}_\mu$ is defined analogously

$$\begin{aligned} P\tilde{\psi}_{\mu_1} &= 0 \\ P_{\mu_1 \ldots \mu_{2N-1}}\tilde{\psi}_{\mu_{2N}} &= P_{\mu_1 \ldots \mu_{2N-1}\mu_{2N}} \\ P_{\mu_1 \ldots \mu_{2N}}\tilde{\psi}_{\mu_{2N+1}} &= -\delta_{\mu_{2N+1}[\mu_1} P_{\mu_2 \ldots \mu_{2N}]} \end{aligned} \tag{4.15}$$

So the matrix ψ_μ acting on the operator $P_{\mu_1 \ldots \mu_N}$ with an even (or odd) number N of indices increases (or decreases) by one this number. The action of dual matrices on $P_{\mu_1 \ldots \mu_N}$ has an opposite effect. The above relations can be generalized for the dual operators as

$$\tilde{P}_{\mu_1 \cdots \mu_N}\psi_{\mu_{N+1}} = \left(\widetilde{P_{\mu_1 \cdots \mu_N}\tilde{\psi}_{\mu_{N+1}}}\right), \quad \tilde{P}_{\mu_1 \cdots \mu_N}\tilde{\psi}_{\mu_{N+1}} = \left(\widetilde{P_{\mu_1 \cdots \mu_N}\psi_{\mu_{N+1}}}\right)$$

Using properties (4.14) together with the identity $\psi_\mu\psi_\nu\psi_\mu = 0$ for $\mu \neq \nu$ we obtain

$$[\psi_\alpha, \psi_\beta]P_{\mu_1 \ldots \mu_N} = 0, \quad P_{\mu_1 \ldots \mu_N}[\psi_\alpha, \psi_\beta] = \sum_{i=1}^{N}(\delta_{\alpha\mu_i}\delta_{\beta\hat{\mu}_i} - \delta_{\alpha\hat{\mu}_i}\delta_{\beta\mu_i})P_{\mu_1 \ldots \hat{\mu}_i \ldots \mu_N}$$

and after substitution into (4.13) we conclude that the operators $P_{\mu_1 \ldots \mu_N}$ possess transformation properties of a totally antisymmetric tensor of rank N

$$\delta P_{\mu_1 \ldots \mu_N} = \sum_{i=1}^{N} \omega_{\mu_i\hat{\mu}_i} P_{\mu_1 \ldots \hat{\mu}_i \ldots \mu_N} \tag{4.16}$$

It is evident that the same statement is valid for the dual operators $\tilde{P}_{\mu_1 \ldots \mu_N}$.

[5]For odd Z operators $P_{\ldots}$ are to be replaced by the dual operators $\tilde{P}_{\ldots}$

4.4. Physical subspace

The physical space of the spinning particle is described by equations (2.9), (3.4) and (3.5). Let $|0\rangle$ be an arbitrary vector from this space. After insertion of identity decompositions (4.9a) and (4.9b) as $|0\rangle = \sum_A \mathcal{P}_A|0\rangle$ and substitution of the explicit form (4.7) and (4.8) of projectors expressed in terms of the covariant operators $P_{\ldots}$ and $\tilde{P}_{\ldots}$, we find using (4.11a), (4.11b), (4.12a), (4.12b) and (4.12d) that in the subspace of irreps of the Duffin-Kemmer algebra the states $|0\rangle$ are described by the set of orthogonal vectors

$$|A_{\mu_1 \ldots \mu_N}\rangle = P_{\mu_1 \ldots \mu_N}|0\rangle \tag{4.17}$$

and by the dual vectors $|\tilde{A}_{\mu_1 \ldots \mu_N}\rangle = \tilde{P}_{\mu_1 \ldots \mu_N}|0\rangle$ for both odd D and N. The properties of covariant operators imply that $A_{\mu_1 \ldots \mu_N}$ are totally antisymmetric tensors of rank N.

Thus, the physical space of the parasupersymmetric spinning particle is described by the antisymmetric tensor fields $A_{\mu_1 \ldots \mu_N}$. Eqs.(2.9) and (3.5) impose certain restrictions on these fields. For instance, the condition (2.9) of worldline reparametrization invariance turns into the massless Klein-Gordon equation for the tensor fields

$$p^2 A_{\mu_1 \ldots \mu_N}(p) = 0$$

As to the second equation (2.9), it can be rewritten using dual matrices and dual parameters (4.4) as

$$[\lambda, \mathcal{Q}]|0\rangle = [\tilde{\lambda}, \tilde{\mathcal{Q}}]|0\rangle = 0$$

where $\tilde{\mathcal{Q}} = p_\mu\tilde{\psi}_\mu$. Multiplying these relations by λ and $\tilde{\lambda}$ and taking into account the identities $\lambda\tilde{\lambda} = 0$, $\lambda\psi_\mu\lambda =$ and $\tilde{\lambda}\tilde{\psi}_\mu\tilde{\lambda} = 0$ one gets

$$\lambda^2\mathcal{Q}|0\rangle = \tilde{\lambda}^2\tilde{\mathcal{Q}}|0\rangle = 0 \tag{4.18}$$

with λ and $\tilde{\lambda}$ being generalized grassman numbers. It follows from (2.13) that λ^2 and $\tilde{\lambda}^2$ anticommute with $\mathcal{Q}$ and $\tilde{\mathcal{Q}}$ and can be eliminated at first sight from (4.18). However, the parameters λ and $\tilde{\lambda}$ are not really arbitrary and are restricted by condition (3.4) as only the projector $\mathcal{P}$ in (3.3) differs from the unit operator. Suppose one has fixed the representation of the operators ψ_μ or, equivalently, specified the set S of irreps entering into decomposition of projector $\mathcal{P}$ in (3.3)

$$\mathcal{P} = \sum_{A\in S}\mathcal{P}_A = \sum_{A\in S}(\Pi_A + \Pi_{D-1-A}) \equiv \sum_{A\in S'}\Pi_A \tag{4.19}$$

Additional terms have to be added to the last expression if twin-algebras enter into the set S. Then, the parameters λ obey

$$\lambda = \mathcal{P}\lambda = \lambda\mathcal{P} = \tfrac{1}{2}\sum_{\alpha\in s}(\Pi_\alpha + \Pi_{D-\alpha})\,\lambda = \lambda\,\tfrac{1}{2}\sum_{\alpha\in s}(\Pi_\alpha + \Pi_{D-\alpha}) \tag{4.20}$$

where the identity $\Pi_A\lambda = \lambda\Pi_{D-A}$ and reality condition $\lambda^\dagger = \lambda$ were used. Here, summation is performed over such indices $\alpha \in s \subset S'$ that $\mathcal{P}\Pi_\alpha \neq 0$ and $\mathcal{P}\Pi_{D-\alpha} \neq 0$ simultaneously. In other words, for a number α to appear in the set s both numbers α and $D-\alpha$ have to belong to the set S'. If the set s turns out be empty, then $\lambda = 0$. The factor $\frac{1}{2}$ was added in (4.20) to avoid double counting of indices α and $\alpha' = D - \alpha$.

After substitution of (4.20) into (4.18) the first equation is replaced by

$$\lambda^2\sum_{\alpha\in s}(\Pi_\alpha + \Pi_{D-\alpha})\mathcal{Q}|0\rangle = \lambda^2\mathcal{Q}\sum_{\alpha\in s}(\Pi_{D-1-\alpha} + \Pi_{\alpha-1})|0\rangle = 0$$

and the second one in (4.18) is transformed as

$$\tilde{\lambda}\tilde{Q}^2|0\rangle = -\tilde{\lambda}Q\lambda|0\rangle = -\tfrac{1}{2}\tilde{\lambda}Q\lambda\sum_{\alpha\in s}(\Pi_\alpha + \Pi_{D-\alpha})|0\rangle = \tfrac{1}{2}\tilde{\lambda}^2\tilde{Q}\sum_{\alpha\in s}(\Pi_\alpha + \Pi_{D-\alpha})|0\rangle = 0$$

These relations take into account all restrictions on λ and $\tilde{\lambda}$ and therefore parameters λ^2 and $\tilde{\lambda}^2$ can be eliminated from them. The resulting equations on the wave function of the spinning particle have the form

$$Q\sum_{\alpha\in s}(\Pi_{D-1-\alpha} + \Pi_{\alpha-1})|0\rangle = \tilde{Q}\sum_{\alpha\in s}(\Pi_\alpha + \Pi_{D-\alpha})|0\rangle = 0 \tag{4.21}$$

or, using (4.6)

$$\sum_{\alpha\in s}(\Pi_\alpha + \Pi_{D-\alpha})Q|0\rangle = \sum_{\alpha\in s}(\Pi_{D+1-\alpha} + \Pi_{\alpha+1})\tilde{Q}|0\rangle = 0$$

Here, the set s is uniquely fixed as soon as the set S is specified in (4.19) and vice versa, for any set s one can reconstruct the corresponding sets S. The set S has the sense of a free parameter of the parasupersymmetric spinning particle.

4.5. Antisymmetric tensor fields as wave functions of the spinning particle

There is a variety of sets S leading to different physical models. In the following we will consider in detail only two extreme cases:

(i) $\mathcal{P} = I$ and set S contains all irreps of the $SO(D+1)$ algebra labelled by the order R;
(ii) the set S contains the only irreps.

4.5.1. Fully reducible representation

In the first case, the matrices ψ_μ are given by (4.3) and the wave function satisfies

$$Q|0\rangle = \tilde{Q}|0\rangle = 0 \tag{4.22}$$

The wave function is described by the antisymmetric tensor fields (4.17). To get equations on the fields $A_{\mu_1\cdots\mu_N}(p)$ following from (4.22) we multiply its both sides by projectors Π_A and apply relations (4.14) and (4.15). There appears a difference between odd and even dimensions.

For even dimensions after multiplication of (4.22) by the covariant operator $P_{\mu_1\cdots\mu_{2N}}$ one uses (4.14) and (4.15) to get

$$p_{\mu_{2N+1}}P_{\mu_1\cdots\mu_{2N}\mu_{2N+1}}|0\rangle = -p_{[\mu_1}P_{\mu_2\cdots\mu_{2N}]}|0\rangle = 0$$

and after multiplication by $P_{\mu_1\cdots\mu_{2N-1}}$

$$p_{[\mu_1}P_{\mu_2\cdots\mu_{2N-1}]}|0\rangle = p_{\mu_{2N}}P_{\mu_1\cdots\mu_{2N-1}\mu_{2N}}|0\rangle = 0$$

Recalling the definition of fields (4.17) we rewrite the above relations as

$$p_{\mu_1}A_{\mu_1\cdots\mu_N}(p) = 0, \quad p_{[\mu_{N+1}}A_{\mu_1\cdots\mu_N]}(p) = 0 \tag{4.23a}$$

The same equations in the coordinate representation have a simple form after introduction of the external differential form

$$A_N(x) = \tfrac{1}{N!}A_{\mu_1\cdots\mu_N}(x)dx^{\mu_1}\cdots dx^{\mu_N}, \quad d\,A_N(x) = 0, \quad d * A_N(x) = 0, \quad 0 \le N \le D \tag{4.23b}$$

The solutions of these equations for $1 \le N \le D-1$ are $A_N(x) = d\omega_{N-1}(x)$ and describe massless $U(1)$ gauge antisymmetric tensor fields $\omega_{\mu_1\cdots\mu_{N-1}}$ of rank $N-1$. In the special cases $N = D$ and $N = 0$ the equations have trivial solutions $A_0(x) = const$ and $A_D(x) = dx^1\cdots dx^D \times const$.

For odd dimensions the projectors (4.12a) and (4.12b) have a different form and one has to multiply both sides of (4.22) by the covariant operators $P_{\dots}$ and $\tilde{P}_{\dots}$. The resulting equations on the differential forms $A_N(x)$ and $\tilde{A}_N(x)$ are completely analogous to (4.23a) and (4.23b). The difference from the above case is manifested as $P^\dagger_{\mu_1\cdots\mu_N}P_{\mu_{N+1}\cdots\mu_D} \neq 0$ or, equivalently, $\langle A_{\mu_1\cdots\mu_N}|A_{\mu_{N+1}\cdots\mu_D}\rangle \neq 0$. Nevertheless, the difference disappears after one defines the forms $B_N^{(\pm)}(x) = A_N \pm i^{\frac{D(D-1)}{2}} * A_{D-N}$ for even N and $B_N^{(\pm)}(x) = \tilde{A}_N \pm i^{\frac{D(D-1)}{2}} * \tilde{A}_{D-N}$ for odd N and gets the equations

$$\langle B_N^{(\pm)}|B_M^{(\pm)}\rangle = 0, \quad \text{for } N \neq M$$

and

$$dB_N^{(\pm)} = 0, \quad d * B_N^{(\pm)} = 0, \quad 0 \le N \le D \tag{4.24}$$

In the special case $N = Z$, the appearance of two fields $B_Z^{(\pm)}(x)$ corresponds to two projectors (4.12d) onto twin-algebras. Under parity transformations $x_\mu \to -x_\mu$ the fields $B_N^{(\pm)}(x)$ turn into $B_N^{(\mp)}(x)$. We conclude that equations (4.23b) and (4.24) coincide and have the same solutions.

Thus, for $\mathcal{P} = I$ and matrices ψ_μ given by the Green anzatz the physical space of the $R = 2$ parasupersymmetric particle is described by massless totally antisymmetric tensor $U(1)$ gauge fields.

4.5.2. Irreducible representation

The second case considered below corresponds to the opposite limit when the operators ψ_μ belong to the irreps of the Duffin-Kemmer algebra and projector $\mathcal{P}$ is defined as

$$\mathcal{P} = \mathcal{P}_A = \Pi_A + \Pi_{D-1-A} \tag{4.25}$$

where A labels one of the irreps in (4.9a) and (4.9b). The condition on the Lagrange multipliers (3.4) looks now like

$$\lambda = (\Pi_A + \Pi_{D-1-A})\lambda\mathcal{P}_A = \lambda(\Pi_{D-A} + \Pi_{1+A})(\Pi_A + \Pi_{D-1-A})$$

We conclude that $\lambda \neq 0$ for $D - A = A$ or $1 + A = D - 1 - A$ and in both cases

$$\lambda = \Pi_{D/2}\lambda = \lambda\Pi_{D/2} \tag{4.26}$$

Two consequences follow from this relation. At first, for odd coordinates ψ_μ of the spinning particle to be defined in the irreps of the Duffin-Kemmer algebra the dimension of space-time must be even[6] and, at second, the irreps is uniquely fixed to be $A = \frac{D}{2}$.

[6]This is why the twin-algebras was not included in (4.25)

Comparing (4.20) and (4.26) we find that the set s in (4.21) contains the only element $s = \{\alpha = \frac{D}{2}\}$, and equations (4.21) and (3.5) on the wave function reduce to

$$\mathcal{Q}\Pi_{\frac{D}{2}-1}|0\rangle = \tilde{\mathcal{Q}}\Pi_{\frac{D}{2}}|0\rangle = 0, \quad |0\rangle = (\Pi_{\frac{D}{2}-1} + \Pi_{\frac{D}{2}})|0\rangle \tag{4.27}$$

For $D = 2Z$ we use identities $\mathcal{Q}\Pi_{\frac{D}{2}-1} = \Pi_{\frac{D}{2}}\mathcal{Q}$ and $\tilde{\mathcal{Q}}\Pi_{\frac{D}{2}} = \Pi_{\frac{D}{2}+1}\tilde{\mathcal{Q}}$, the explicit form of the projectors (4.11a) and (4.11b) and charges ($\mathcal{Q} = (p\cdot\psi)$ and $\tilde{\mathcal{Q}} = (p\cdot\tilde{\psi})$) to rewrite the above relations as

$$P_{\mu_1\cdots\mu_Z}(p\cdot\psi)|0\rangle = \tilde{P}_{\mu_1\cdots\mu_{Z-1}}(p\cdot\tilde{\psi})|0\rangle = 0, \quad \text{for even } Z$$

and

$$P_{\mu_1\cdots\mu_Z}(p\cdot\psi)|0\rangle = \tilde{P}_{\mu_1\cdots\mu_{Z+1}}(p\cdot\tilde{\psi})|0\rangle = 0, \quad \text{for odd } Z$$

Moreover, the last relation in (4.27) implies that the wave function $|0\rangle$ is described by two antisymmetric tensor fields satisfying

$$p_{\mu_{Z+1}}A_{\mu_1\ldots\mu_Z\mu_{Z+1}}(p) = 0, \quad p_{\mu_Z}A_{\mu_1\ldots\mu_Z}(p) = 0, \quad \text{for even } Z$$

and

$$p_{[\mu_Z}A_{\mu_1\ldots\mu_{Z-2}\mu_{Z-1}]}(p) = 0, \quad p_{[\mu_{Z+1}}A_{\mu_1\ldots\mu_Z]}(p) = 0, \quad \text{for odd } Z$$

In the coordinate representation these two sets of equations have the same form if for even Z one replaces the fields $A_{\mu_1\ldots\mu_N}(p)$ by the dual fields

$$d\,A_Z(x) = 0, \quad d\,A_{Z-1}(x) = 0 \tag{4.28}$$

where $A_N(x) \equiv \frac{1}{N!}A_{\mu_1\ldots\mu_N}(x)dx^{\mu_1}\cdots dx^{\mu_N}$ are differential $N-$forms.

Equations of motion (4.28) possess gauge invariance. They are unchanged under the abelian transformations of fields

$$A_N(x) \to A_N(x) + d\chi_{N-1}(x), \quad \text{or} \quad A_{\mu_1\ldots\mu_N}(x) \to A_{\mu_1\ldots\mu_N}(x) + \partial_{[\mu_1}\chi_{\mu_2\ldots\mu_N]}(x) \tag{4.29a}$$

where $N = Z$ or $N = Z-1$ and $\chi_{\mu_2\ldots\mu_N}(x)$ is an arbitrary totally antisymmetric field. To understand the origin of gauge invariance, recall the property (4.1) of dual operators $\mathcal{Q}\tilde{\mathcal{Q}} = 0$. Then, it can be easily seen that the equations of motion (4.27) are invariant under transformations

$$|0\rangle \to |0\rangle + (\Pi_Z\mathcal{Q} + \Pi_{Z-1}\tilde{\mathcal{Q}})|\chi\rangle \tag{4.29b}$$

with $|\chi\rangle$ being an arbitrary state. The relations (4.29a) and (4.29b) turn into each other after decomposition of the state $|\chi\rangle$ over the antisymmetric tensor fields $\chi_{Z-1}(x)$ and $\chi_{Z-2}(x)$.

The gauge ambiguity of the wave function analogous to (4.29a) and (4.29b) did not appear in the previous case. This effect is one of the manifestation of the restrictions one puts on the representation space of the operators ψ_μ.

Thus, the physical subspace of the parasupersymmetric spinning particle whose spinning coordinates belong to the irreps of the Duffin-Kemmer algebra is described by two antisymmetric tensor fields satisfying (4.28). In the performed consideration relativistic quantum mechanics of the spinning particle was the starting point. After the second quantization one deals with quantum fields and treats the wave function of spinning particle as asymptotic state of quantum fields. The natural question arises: what kind of action for the fields $A_{\mu_1\ldots\mu_Z}(x)$ and $A_{\mu_1\ldots\mu_{Z-1}}(x)$ allows

one to justify equations of motion (4.28) and gauge invariance (4.29a). It is remarkable that such an action exists [19, 20]

$$S = \int d^Dx\ \varepsilon^{\mu_1\cdots\mu_Z\mu_{Z+1}\mu_{Z+2}}A_{\mu_1\ldots\mu_Z}(x)\partial_{\mu_{Z+1}}A_{\mu_{Z+2}\ldots\mu_D}(x) \equiv \int_{R^D} A_Z(x)\,d\,A_{Z-1}(x) \tag{4.30}$$

It is unique and coincides with the action of the $D = 2Z-$dimensional topological field theory [21]. Of course, it is possible to propose another form of actions by adding the corresponding Lagrange multipliers to ensure equations of motion (4.28). But in that case the multipliers will be in thier turn by additional antisymmetric tensor fields. The action (4.30) does not require introduction of any auxiliary fields and is minimal and unique in that sense.

We conclude that after quantization of the parasupersymmetric spinning particle propagating in the representation space of the irreps of the Duffin-Kemmer algebra the physical subspace is described by topological gauge fields.

4.5.3. Generalizations to arbitrary representations

We have considered above only two extreme cases of the representation space of the Duffin-Kemmer algebra: $\mathcal{P} = I$ and $\mathcal{P} = \mathcal{P}_A$. To understand what will occur for an arbitrary choice of projector $\mathcal{P}$ in (3.3), one compares equations of motion (4.23b) and (4.28) for the fields $A_Z(x)$ and $A_{Z-1}(x)$. Note that after transition from $\mathcal{P} = \mathcal{P}_A$ to $\mathcal{P} = I$ when the representation space of matrices ψ_μ is enlarged, the number of restrictions on these fields increases. Two additional equations $d * A_Z(x) = d * A_{Z-1}(x) = 0$ appear that can be considered as conditions fixing gauge ambiguity (4.29a). It is natural that for an arbitrary projector $\mathcal{P} \neq I$ one encounters intermediate situation when the resulting set of equations on the antisymmetric tensor fields is a subset of equations (4.23b). Among them one can find pairs of equations $d\,A_N(x) = d * A_N(x) = 0$ and single ones $d\,A_N(x) = 0$ (or $d * A_N(x) = 0$). In the former case $A_N(x)$ is the strength of $U(1)$ connection $A_N(x) = d\omega_{N-1}(x)$ but in the latter $A_N(x)$ is a topological gauge field. The explicit realization of this scheme depends on the specific form of the projector $\mathcal{P}$ or, equivalently, on the set S of irreps.

Consider as an example the $D = 3-$dimensional spinning particle with the projector $\mathcal{P}$ defined as

$$\mathcal{P} = \mathcal{P}_0 + \mathcal{P}_1^{(+)} + \mathcal{P}_1^{(-)} = \Pi_0 + \Pi_1 + \Pi_2$$

or $S = \{0, 1^+, 1^-\}$, $S' = \{0,1,2\}$ and $s = \{1,2\}$ in (4.19) and (4.20). Solving (4.21) we get that the wave function of particle obeys the equations

$$P_{\mu_1\mu_2}(p\cdot\psi)|0\rangle = \tilde{P}_{\mu_1}(p\cdot\psi)|0\rangle = 0, \quad \tilde{P}_{\mu_1\mu_2\mu_3}(p\cdot\tilde{\psi})|0\rangle = P_{\mu_1\mu_2}(p\cdot\tilde{\psi})|0\rangle = 0$$

or, applying (4.14) and (4.15)

$$p_{\mu_3}A_{\mu_1\mu_2\mu_3}(p) = 0, \quad p_{\mu_2}\tilde{A}_{\mu_1\mu_2}(p) = p_{[\mu_3}\tilde{A}_{\mu_1\mu_2]}(p) = 0, \quad p_{[\mu_2}\tilde{A}_{\mu_1]}(p) = 0$$

Thus, the physical subspace of the spinning particle is described by the strength tensor $\tilde{A}_{\mu_1\mu_2}(p)$ and abelian Chern-Simons gauge field $A_{\mu_1}(p)$.

5. Conclusions

We have considered the relativistic spinning particle with the action invariant under reparametrizations and local worldline parasupersymmetric transformations. The corresponding symmetry algebra being polynomial contains a set of nontrivial irreps. It is the property that allows for parasupersymmetric particle to have properties different from supersymmetric particle.

The physical space of the massless spinning particle possessing parasupersymmetry of order R was defined in (2.9), (3.4) and (3.5). For $R = 2$ the symmetry algebra reduces to the Duffin-Kemmer algebra. Using properties of the irreps of this algebra we found that the wave function of the massless particle is described by the strength tensors of $U(1)$ antisymmetric gauge fields and topological gauge fields. Generalization to the $(D-1)-$dimensional massive particle is achieved by putting the additional constraint (2.18) on the wave function of $D-$dimensional massless particle, or in the coordinate representation $\partial/\partial x_D|0\rangle = iM|0\rangle$. It is convenient to decompose the antisymmetric tensor field $A_{\mu_1 \cdots \mu_N}(x)$ with $\mu_i = 1, \ldots, D-1, D$ into two antisymmetric fields $B_{\alpha_1 \cdots \alpha_N}(x) \equiv A_{\alpha_1 \cdots \alpha_N}$ and $B_{\alpha_1 \cdots \alpha_{N-1}}(x) \equiv A_{D\alpha_1 \cdots \alpha_{N-1}}$ with $\alpha_i = 1, \ldots, D-1$. If the field $A_{\mu_1 \cdots \mu_N}(x)$ obeys $dA_N = d * A_N = 0$, then for massive particle the above relations after imposing of (2.18) are replaced by $dB_{N-1} - iMB_N = 0$, $d * B_N - (-)^N iM * B_{N-1} = 0$ and $B_N(x)$ is the strength of the massive antisymmetric tensor field $B_{N-1}(x)$. The topological gauge field $A_N(x)$ satisfying $dA_N(x) = 0$ turns into $dB_{N-1}(x) - iMB_N(x) = 0$.

To define the wave function of the spinning particle for an arbitrary order R one has to specify the projector onto representation of parasupersymmetry algebra in (3.3). For $R > 2$ we have no detailed description of the irreps analogous to that for the Duffin-Kemmer algebra. If one chooses the trivial projector $\mathcal{P} = I$ or uses the Green anzatz (3.1), then the wave function coincides with the wave function of $N = R$ extended supersymmetric particle found in [8, 9, 10]. Nevertheless, there is a possibility to define the nontrivial projector as follows. The Green anzatz originates from the direct product of R spinor irreps. We can define the "modified" Green anzatz by forming the direct product of $R/2$ reducible representations of the Duffin-Kemmer algebra. The replacement of $\gamma_\mu^{(\alpha)}$ operators in (3.1) by operators ψ_μ from (3.3) corresponds to the nontrivial projector $\mathcal{P} \neq I$. The resulting wave function of the parasupersymmetric spinning particle of order R is equal to the product of $R/2$ wave functions of $R = 2$ parasupersymmetric particles.

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Парасуперсимметричная спиновая частица и топологические калибровочные поля

Проводится первичное квантование D-мерной релятивистской спиновой частицы, действие которой инвариантно относительно репараметризаций и локальных парасуперсимметричных преобразований. Соответствующая алгебра симметрии не является алгеброй Ли и известна как полиномиальная алгебра. Установлено, что физическое подпространство частицы описывается тензорами напряженности абелевых антисимметричных калибровочных полей и топологическими калибровочными полями. Абелево поле Черна-Саймонса появляется как волновая функция частицы в D=3-мерном пространстве. Полученные результаты обобщаются на случай массивной парасуперсимметричной спиновой частицы.

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Parasupersymmetric Spinning Particle and Topological Gauge Fields

The first quantization of the D-dimensional relativistic spinning particle with the action invariant under reparametrizations and local worldline parasupersymmetric transformations is performed. The corresponding symmetry algebra is not of the Lie kind and is known as a polynomial algebra. It is found that the physical space of the massless particle is described by the strength tensors of abelian antisymmetric fields and topological gauge fields. In the special case D = 3 the abelian Chern-Simons gauge field appears as wave function of the particle. The generalization to a massive parasupersymmetric spinning particle are presented.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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