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A NEW EXACT SOLUTION TO THE CLASSICAL EQUATIONS OF MOTION OF THE RELATIVISTIC STRING WITH MASSIVE ENDS

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1. INTRODUCTION

Interest in the string theory in hadronic physics is inspired by a striking analogy between the open string and the view of the meson as a quark-antiquark pair held together by a QCD flux tube [1]. The string model gives the description of the hadron mass spectrum which is beyond the scope of the QCD present-day formalism and at the same time it reproduces basic predictions of the field approach [2-4]. The massless relativistic string is usually considered as the simplest dynamic basis of the string model of hadrons [5].

The more realistic string model, in which the quarks carry a finite fraction of the energy-momentum of the hadron, led to the study of dynamics of the relativistic string with masses at the ends (for a review, see [1]). The equations of motion for the quarks yield nonlinear boundary conditions, therefore, even, the investigation of classical motions of the massive string proved to be an extremely complicated mathematical problem when the quark masses are different from zero. Until now no general solutions of this problem have been derived. In the three-dimensional Minkowski space E^1_2 only the exact solution to the equations of motion of the relativistic string with masses at the ends is known [6,7]. The solution decribes the rotation in the given plane of the straight-line string with massive ends moving along the helices. There are no transverse string excitations in this case, hence that solution gives no corrections to the linearly rising potential between quarks. It is thus natural to consider other approaches to the investigation of the dynamics of relativistic string with point masses at the ends which may be more fruitful in finding new exact solutions.

In papers [8,9] a new formulation for the examination of classical histories of the relativistic string with massive ends was proposed in terms of geometric invariants of both the string world surface and world lines of the point masses m_a, $(a = 1,2)$ placed at the string ends. It was shown that the string variables in three dimensions are completely defined by constant curvatures $k_a = \gamma/m_a$, where γ is the string tension, and the torsions $\kappa_a(\tau)$, $(a = 1,2)$ of the endpoint trajectories which are subjected to a system of differential equations of the second order with a delayed argument that represents retardation effects of the interaction of two point masses through the string. The above example of the straight-line string with massive ends rotating in a given plane corresponds to a particular solution of this system with constant torsions $\kappa_a(\tau) = \kappa_{a0}$, $(a = 1,2)$ when the string ends are moving along the helices. In this case the string history is a helicoid in the space E_2^1.

In the present paper a new exact solution is found in the framework of the geometric formulation for periodic torsions $\kappa_a(\tau + 2\pi) = \kappa_a(\tau)$; $(a = 1,2)$ which are given by the Weierstrass function with a real period proportional to 2π and a purely imaginary period $2\omega'$. The string coordinates are expressed in terms of normal elliptic integrals and describe a more intricate motion than rotation of a stretched string in the given plane including its transverse vibrations. Just such motions ought to be considered in the string model of hadrons for the calculation of the contributions to the linear behaviour of the static interquark potential at long distances [11].

2. PRELIMINARIES

In refs. [8,9] for the coordinates $x^\mu(\tau,\sigma)$ of the world surface of the relativistic string with masses at the endpoints $x^\mu(\tau, \sigma_a)$, $\sigma_a = (0, \pi)$ $(a = 1,2)$ and with the tension γ in the Minkowski space E_2^1 the following representation

$$x_\mu(\tau,\sigma) = \frac{\psi_{+\mu}(u^+) + \psi_{-\mu}(u^-)}{2} \qquad u^\pm = \tau \pm \sigma \tag{2.1}$$

was found. Here the isotropic vectors $\psi_\pm'^2(u^\pm) = 0$ are expanded in a constant basis which is formed by two isotropic vectors e_0^μ and e_1^μ, $e_0^2 = e_1^2 = 0$, $(e_0 e_1) = 1$ and by the space-like vector e_2^μ, $e_2^2 = -1$, where $(e_0 e_2) = 0 = (e_1 e_2)$, as follows:

$$\psi_+'^\mu(u^+) = \frac{A}{f'(u^+)}\left[e_0^\mu + e_1^\mu \frac{f^2(u^+)}{2} + e_2^\mu f(u^+)\right],$$
$$\psi_-'^\mu(u^-) = \frac{A}{g'(u^-)}\left[e_0^\mu + e_1^\mu \frac{g^2(u^-)}{2} + e_2^\mu g(u^-)\right], \tag{2.2}$$

where A is a constant and the functions $f(\tau)$ and $g(\tau)$ obey the system of two ordinary differential equations with a delayed argument:

$$m_1 \frac{d}{d\tau} \ln \frac{g'(\tau)}{f'(\tau)} + 2\,\frac{f'(\tau) + g'(\tau)}{f(\tau) - g(\tau)} = \gamma |A| \frac{|f(\tau) - g(\tau)|}{\sqrt{f'(\tau) g'(\tau)}} \tag{2.3}$$

and

$$m_2 \frac{d}{d\tau} \ln \frac{g'(\tau - \pi)}{f'(\tau + \pi)} + 2\,\frac{f'(\tau + \pi) + g'(\tau - \pi)}{f(\tau + \pi) - g(\tau - \pi)} = - \gamma|A| \frac{|f(\tau+\pi) - g(\tau - \pi)|}{\sqrt{f'(\tau + \pi) g'(\tau - \pi)}}, \tag{2.4}$$

or replacing in (2.4) argument τ by $\tau - \pi$ the latter becomes

$$m_2 \frac{d}{d\tau} \ln \frac{g'(\tau - 2\pi)}{f'(\tau)} + 2\,\frac{f'(\tau) + g'(\tau - 2\pi)}{f(\tau) - g(\tau - 2\pi)} = - \gamma|A| \frac{|f(\tau) - g(\tau - 2\pi)|}{\sqrt{f'(\tau) g'(\tau - 2\pi)}}. \tag{2.4'}$$

The system (2.3) and (2.4) is invariant under the same Moebuis transformation of the functions $f(\tau)$ and $g(\tau)$

$$f \to \frac{\alpha f + \beta}{\gamma f + \delta}, \quad g \to \frac{\alpha g + \beta}{\gamma g + \delta}, \quad \alpha\delta - \beta\gamma = 1. \tag{2.5}$$

This invariance corresponds to the Lorentz invariance of the underlying string since the Lorentz transformations of vectors $\psi_\pm'^\mu(u^\pm)$ and also, according to (2.2), the vectors of the isotropic basis e_0^μ, e_1^μ, e_2^μ induce the transformations (2.5) of the functions $f(u^+)$ and $g(u^-)$.

In the space E_2^1 we shall describe the world trajectories of the massive string endpoints in terms of two geometric invariants, curvature k_a and torsion κ_a, $(a = 1,2)$. As is known [10] these characteristics uniquely define a curve in a three-dimensional space up to its position. In refs. [8,9] we demonstrated that the world lines $x^\mu(\tau, \sigma_a)$, $(a = 1,2)$ have constant curvatures

$$k_a = \gamma / m_a \tag{2.6}$$

and their torsions are given by the expressions

$$\kappa_1(\tau) = \frac{4 f'(\tau) g'(\tau)}{A[f(\tau) - g(\tau)]^2},$$
$$\kappa_2(\tau) = \frac{4 f'(\tau+\pi) g'(\tau - \pi)}{A[f(\tau+\pi) - g(\tau - \pi)]^2}, \tag{2.7}$$

or

$$\kappa_2(\tau-\pi) = \frac{4f'(\tau)\,g'(\tau-2\pi)}{A[\,f(\tau) - g(\tau-2\pi)]^2} \tag{2.8}$$

invariant under the transformations (2.5). By using these formulas together with equations (2.3) and (2.4´), the functions $f(\tau)$ and $g(\tau)$ can be expressed in terms of the torsions $\kappa_a(\tau)$, $(a = 1,2)$ as [8,9]

$$\begin{aligned} D[f(\tau)] &= D[\int^{\tau} \sqrt{A\kappa_1(\eta)}\,d\eta] + \frac{\kappa_1(\tau)}{2}\Big(1 - \frac{k_1^2}{\kappa_1^2(\tau)}\Big) - 2k_1\frac{d}{d\tau}\sqrt{\frac{A}{\kappa_1(\tau)}} = \\ &= D[\int^{\tau}\sqrt{A\kappa_2(\eta-\pi)}\,d\eta] + \frac{\kappa_2(\tau-\pi)}{2}\Big(1 - \frac{k_2^2}{\kappa_2^2(\tau-\pi)}\Big) + 2k_2\frac{d}{d\tau}\sqrt{\frac{A}{\kappa_2(\tau-\pi)}}\,. \end{aligned} \tag{2.9}$$

$$\begin{aligned} D[g(\tau)] &= D[\int^{\tau} \sqrt{A\kappa_1(\eta)}\,d\eta] + \frac{\kappa_1(\tau)}{2}\Big(1 - \frac{k_1^2}{\kappa_1^2(\tau)}\Big) + 2k_1\frac{d}{d\tau}\sqrt{\frac{A}{\kappa_1(\tau)}} = \\ &= D[\int^{\tau}\sqrt{A\kappa_2(\eta+\pi)}\,d\eta] + \frac{\kappa_2(\tau+\pi)}{2}\Big(1 - \frac{k_2^2}{\kappa_2^2(\tau+\pi)}\Big) - 2k_2\frac{d}{d\tau}\sqrt{\frac{A}{\kappa_2(\tau+\pi)}}\,. \end{aligned} \tag{2.10}$$

Here $D[f(\tau)]$ stands for the Schwarz derivative [15,16]

$$D[f(\tau)] = \frac{f'''(\tau)}{f'(\tau)} - \frac{3}{2}\Big(\frac{f''(\tau)}{f'(\tau)}\Big)^2 \tag{2.11}$$

which is invariant under the Moebius transformations (2.5). According to (2.1) and (2.2), it follows from (2.9) and (2.10) that the string coordinates $x^{\mu}(\tau,\sigma)$ are completely defined

by the torsions $\kappa_a(\tau)$ of the world trajectories $x^{\mu}(\tau,\sigma_a)$ of massive string endpoints.

Thus, in the framework of the geometrical method, the classical dynamics of the relativistic string with massive ends in the three-dimensional Minkowski space is described by two differential equations of second order with delayed arguments (2.9) and (2.10). The system (2.9) and (2.10) is of fundamental importance in searching for the world surface of the relativistic string with massive ends in the ambient 3-dimensional space-time E_2^1. Specifically, it follows from these equations that inside the interval $0 < \tau < \pi$ the torsions $\kappa_1(\tau)$ and $\kappa_2(\tau)$ are arbitrary functions, and in order to uniquely specify a solution of eqs. (2.9) and (2.10), they should be fixed there as the initial data by the choice of the initial position $x^{\mu}(0,\sigma)$ and initial velocity $\dot{x}^{\mu}(0,\sigma)$ $0 < \sigma < \pi$ of the string [8,9]. The continuation of these functions outside the interval $0 < \tau < \pi$ is made by integrals of eqs.(2.9) and (2.10) so that two conditions of smoothness at the points 0 and π for the continued functions $\kappa_1(\tau)$ and $\kappa_2(\tau)$ $-\infty < \tau < \infty$ may always be fulfilled with the four arbitrary integration constants.

The simplest solutions to eqs.(2.9) and (2.10) are the constant torsions $\kappa_a(\tau) = \kappa_{0a}$, $(a = 1,2)$ when the string endpoints are moving along helices obeying the following conditions [8]

$$\kappa_{01}\Big(1 - \frac{k_1^2}{\kappa_{01}^2}\Big) = \kappa_{02}\Big(1 - \frac{k_2^2}{\kappa_{02}^2}\Big)\,. \tag{2.12}$$

In this case the functions $f(\tau)$, $g(\tau)$ and $g(\tau-2\pi)$ are related by the Moebius transformations

$$g(\tau) = \frac{\alpha_1 f(\tau) + \beta_1}{\gamma_1 f(\tau) + \delta_1} = \frac{\alpha_2 g(\tau-2\pi) + \beta_2}{\gamma_2 g(\tau-2\pi) + \delta_2} \tag{2.13}$$

with constant coefficients α_a, β_a, γ_a, δ_a that satisfy the normalization conditions $\alpha_a\delta_a - \beta_a\gamma_a = 1$, $(a = 1,2)$ and two relations originate from eqs.(2.3) and (2.4´). The world surface $x^{\mu}(\tau,\sigma)$ of the relativistic string with massive ends turns out to be a helicoid in the 3-dimensional space-time [6-8].

3. TRAJECTORIES WITH PERIODIC TORSIONS

It turns out that the system (2.9)-(2.10) possesses smooth periodic solutions

$$\kappa_a(\tau) = \kappa_a(\tau + 2\pi), \quad a = 1, 2. \tag{3.1}$$

In fact, with the use of (3.1) we may write the sum and difference of (2.9) and (2.10) in the following from

$$D[\int^{\tau} \sqrt{A\kappa_1(\eta)}\, d\eta] + \frac{\kappa_1(\tau)}{2}(1 - \frac{k_1^2}{\kappa_1(\tau)}) =$$

$$= D[\int^{\tau} \sqrt{A\kappa(\eta + \pi)}\, d\eta] + \frac{\kappa_2(\tau + \pi)}{2}(1 - \frac{k_2^2}{\kappa_2(\tau + \pi)}), \tag{3.2}$$

$$\frac{d}{d\tau}(k_1\sqrt{\frac{A}{\kappa_1(\tau)}} + k_2\sqrt{\frac{A}{\kappa_2(\tau + \pi)}}) = 0. \tag{3.3}$$

From (3.3) one finds the integral of motion

$$\frac{k_1}{\sqrt{\kappa_1(\tau)}} + \frac{k_2}{\sqrt{\kappa_2(\tau \pm \pi)}} = k^2, \tag{3.4}$$

where k^2 is an arbitrary positive constant. Note that relations (3.4) contain only one arbitrary constant k^2 so that the smoothers of the curve $\kappa_a(\tau)$ continued outside the interval $0 < \tau < \pi$ cannot be guaranteed. In this case the equalities (3.1) and (3.4) may give rise to discontinuous solutions for $\kappa(\tau)$ over the whole real axis $-\infty < \tau < \infty$, which are not considered here.

In the class of smooth functions we find for the torsions $\kappa(\tau)$, a = 1,2, in the interval $0 < \tau < \pi$, the following representation:

$$\frac{1}{\sqrt{\kappa_1(\tau)}} = \frac{k^2}{k_1 + k_2|p(\tau)|}$$
$$\frac{1}{\sqrt{\kappa_2(\tau \pm \pi)}} = \frac{k^2|p(\tau)|}{k_1 + k_2|p(\tau)|} \tag{3.5}$$

which makes (3.4) an identity. The real-valued function $p(\tau)$ is defined by eqs.(3.1) and (3.2). Let us show that $p(\tau)$ is periodic $p(\tau) = p(\tau + 2\pi)$, and can be extended smoothly to the whole real axis τ. Inserting (3.5) into (3.2) we obtain the second-order differential equation for $p(\tau)$:

$$p(\tau)p''(\tau) - [\frac{1}{2} + \frac{k_2|p(\tau)|}{k_1 + k_2|p(\tau)|}]p'^2(\tau) +$$
$$+ \frac{1}{2}[\frac{(p^2(\tau) - 1)(k_1 + k_2|p(\tau)|)^2}{k^4} - \frac{k^4p^2(\tau)(k_1 + k_2|p(\tau)|)}{k_1 + k_2|p(\tau)|}] = 0. \tag{3.6}$$

The substitution

$$p'^2(\tau) = \phi(p) \tag{3.7}$$

changes (3.6) to a first-order equation for the function $\phi(p)$, and integrating the latter over $p(\tau)$ we obtain

$$\phi(p) = k^2p^4(\tau) - \frac{\Delta(p)}{k^4}(k_1 + k_2|p(\tau)|)^2 \equiv w^2(p), \tag{3.8}$$

where

$$\Delta(p) = p^2(\tau) - 2\rho p(\tau) + 1 \tag{3.9}$$

and ρ is the integration constant. Now the function $p(\tau)$ is defined by eq.(3.7), the r.h.s. of which, eq.(3.8), is a polynomial of the fourth degree in $p(\tau)$ with real-valued coefficients and positive $\phi(p) \equiv w^2(p) > 0$ for real $p(\tau)$. After putting $p(\tau) = 0$ eq.(3.8) becomes $w^2(0) = -k_1^2/k^4 < 0$, whence it follows that $p(\tau)$ takes values either on the half-line $p(\tau) > 0$ or on $p(\tau) < 0$. The latter in turn ensures the coefficients of polynomial (3.8) being signed.

As is known[12], the solution of equation (3.7) can be represented in terms of elliptic functions with periods 2ω and $2\omega'$. To this end for simplicity we consider the case of equal masses at the string ends, $m_1 = m_2$ when according to (26) $k_1 = k_2$ and one puts $k^4 = k_1|q|$, where q is an arbitrary constant, $E = 2[1 + \epsilon(p)\rho]$ and $q^2/4 + E > 4$. In this case the elliptic curve (3.8) has two mutually inverse real-valued positive roots

$$p_1 = \frac{1 + \sqrt{1 - 4\lambda}}{1 - \sqrt{1 - 4\lambda}}, \qquad p_2 = p_1^{-1}, \tag{3.10}$$

where

$$\lambda = \frac{-E + \sqrt{E^2 + 4q^2}}{2q^2}, \qquad 0 < \lambda < \frac{1}{2}.$$

With the use of (3.10) the solution to eq.(3.7) may be represented as follows:

$$|p(\tau)| = p_1 + \frac{\phi'(|p|)}{4[\ (\tau) - \frac{\phi''(|p|)}{24}]}\Bigg|_{|p|=p_1} . \qquad (3.11)$$

Here $(\tau) = (\tau, g_2, g_3)$ is the Weierstrass function with real period 2ω and pure imaginary period $2\omega'$, g_2, g_3 are real-valued invariants of the polynomial (3.8), $g_2^3 - 27g_3^2 > 0$. In the interval $0 < \tau < \omega$, when $e_1 < (\tau) < \infty$, where

$$e_1 = \wp(\omega) = \frac{k}{12|q|}(q^2 + 2E) > 0$$

by virtue of $e_1 > [\phi''(|p|)/24]_{|p|=p_1}$, the function (3.11) is smooth and monotonically decreasing from a maximum $|p(0)| = p_1$ at point $\tau = 0$ down to a minimum $|p(\omega)| = p_2 = p_1^{-1}$ at point $\tau = \omega$ and has at most three points of inflection. In accordance with the properties of the Weierstrass function /12-14/, outside the interval $0 < \tau < \omega$ the function $|p(\tau)|$ is continued to the period 2ω in an even manner

$$|p(-\tau)| = |p(\tau)|. \qquad (3.12)$$

and the whole real axis τ periodically with period 2ω

$$|p(\tau + 2\omega)| = |p(\tau)|, \qquad (3.13)$$

The lines p_1 and $p_2 = p_1^{-1}$ are envelopes of curve (3.11).

Thus, formula (3.5) supplemented with (3.11), according to (3.12) and (3.31) defines the torsions $\kappa_a(\tau)$ (a = 1,2) as smooth 2ω-periodic even functions

$$\kappa_a(\tau) = \kappa_a(-\tau), \quad \kappa_a(\tau + 2\omega) = \kappa_a(\tau) \qquad (3.14)$$

for all real values of the evolution parameter on the world sheet τ. To fulfil equalities (3.1), the real half-period ω of the function (3.11) is to be fixed at π, which results in the following condition on the arbitrary constants ρ and q

$$\omega = \int_{e_1}^{\infty} \frac{dt}{\sqrt{4t^3 - g_2 t - g_3}} = \pi. \qquad (3.15)$$

The properties of torsions, (3.14) and (3.15), together with the expressions for the metric tensor component of the string world surface

$$\dot{x}^2(\tau, \sigma_a) = \frac{A}{\kappa_a(\tau)}, \ (a = 1,2) \qquad (3.16)$$

presented in refs. /8,9/ imply

$$\dot{x}^2(-\tau, \sigma_a) = \dot{x}^2(\tau, \sigma_a), \quad \dot{x}^2(\tau + 2\pi, \sigma_a) = \dot{x}^2(\tau, \sigma_a). \qquad (3.17)$$

To complete this section, we note that when $m_1 = m_2$ the motion of the string ends proceeds along similar curves with $k_1 = k_2$ and $\kappa_1(\tau) = \kappa_2(\tau)$. In fact, the function (3.11) satisfies simple rule of addition

$$|p(\tau \pm \pi)| = \frac{1}{|p(\tau)|}. \qquad (3.18)$$

Substitution of (3.18) into expression (3.5) for $\frac{1}{\kappa_2(\tau)}$ gives

$$\kappa_2(\tau) = \frac{k}{|q|}[1 + |p(\tau)|]^2 = \kappa_1(\tau), \qquad (3.19)$$

whence with (3.5), it follows that

$$\dot{x}^2(\tau, 0) = \dot{x}^2(\tau, \pi). \qquad (3.20)$$

This equality means equality of lengths of trajectories of the masses in equal intervals of τ:

$$\ell_1 = \int_{\tau_1}^{\tau_2} \sqrt{\dot{x}^2(\tau, 0)}\, d\tau = \int_{\tau_2}^{\tau_2} \sqrt{\dot{x}^2(\tau, \pi)}\, d\tau = \ell_2.$$

4. DEFINITION OF THE STRING WORLD SURFACE

We shall define the functions $f(\tau)$ and $g(\tau)$ from eqs.(2.9) and (2.10) taking into account that their r.h.s. are periodic owing to (3.1). Therefore the left-hand sides of these equations, i.e. the Schwarz derivatives of the functions $f(\tau)$ and $g(\tau)$, are periodic as well, $D(f(\tau)) = D(f(\tau + 2\pi))$, $D(g(\tau)) = D(g(\tau + 2\pi))$ whence, as explained in refs. /15,16/, it follows that:

$$f(\tau+2\pi)=\frac{af(\tau)+b}{cf(\tau)+d},\quad g(\tau+2\pi)=\frac{ag(\tau)+b}{cg(\tau)+d};\qquad ad-bc=1. \tag{4.1}$$

The coefficients of these Moebius transformations are taken the same in order that the torsions (2.7) and (2.8) obeyed the condition (3.1). Specifying $b = c = 0$ and $a = d$, from (4.1) we obtain the periodic functions $f(\tau+2\pi)=f(\tau)$ and $g(\tau+2\pi)=g(\tau)$ corresponding to the case of the massless string.

In the general case, the real-values pairs of solutions $(f(\tau),\ g(\tau))$ and $(f(\tau+2\pi),\ g(\tau+2\pi))$ for $a+d\geq 2$ have either one or two points of intersection given by the equation

$$F(f)=cf^2(\tau)+(d-a)f(\tau)-b=0 \tag{4.2}$$

whereas for $a+d<2$ they do not intersect at all. With (4.1) the expression (2.8) for $\kappa_2(\tau\pm\pi)$ assumes the form

$$\kappa_2(\tau\pm\pi)=\frac{4f'(\tau)g'(\tau)}{A[(af(\tau)+b)-g(\tau)(cf(\tau)+d)]^2}. \tag{4.3}$$

Expressing $4f'(\tau)g'(\tau)$ in terms of $\kappa_1(\tau)$ from formula (2.7) and inserting it back into (4.3) with using the notation $\kappa_1(\tau)/\kappa_2(\tau\pm\pi)=p^2(\tau)$ we arrive at the equation quadratic in $g(\tau)$:

$$p^2(\tau)[f(\tau)-g(\tau)]^2=[(af(\tau)+b)-g(\tau)(cf(\tau)+d)]^2. \tag{4.4}$$

Two roots of this equation correspond to two different choices of the sign of function $p(\tau)$ and can be written as a common expression

$$g(\tau)=\frac{[a-p(\tau)]f(\tau)+b}{cf(\tau)+[d-p(\tau)]},\quad ad-bc=1; \tag{4.5}$$

whose coefficients, in contrast to the case of constant torsions (see formula (2.13)), depend on τ and form a matrix with the determinant

$$\Delta(p)=p^2(\tau)-(a+d)p(\tau)+1. \tag{4.6}$$

Comparing (4.1) with (4.5) we get the equality

$$g(\tau)=\frac{ag(\tau-2\pi)+b}{cg(\tau-2\pi)+d}=\frac{[a-p(\tau)]f(\tau)+b}{cf(\tau)+[d-p(\tau)]},$$

whence it follows that

$$g(\tau-2\pi)=\frac{[p^{-1}(\tau)-d]f(\tau)+b}{cf(\tau)+[p^{-1}(\tau)-a]}. \tag{4.7}$$

Substituting (4.5) and (4.7) into formulae (2.7) and (2.8), respectively, and changing $\kappa_1(\tau)$ and $\kappa_2(\tau\pm\pi)$ by the expressions (3.5) we obtain the equation

$$\Delta(p)\cdot\left(\frac{f'(\tau)}{F(f)}\right)^2-p'(\tau)\left(\frac{f'(\tau)}{F(f)}\right)-\frac{(k_1+k_2|p(\tau)|)^2}{4k^4}=0, \tag{4.8}$$

that defines the two-valued function $f'(\tau)/F(f(\tau))$ in terms of $p(\tau)$ and $p'(\tau)$ as follows

$$\frac{f'(\tau)}{F(f(\tau))}=\frac{p'(\tau)\quad\sqrt{p'^2(\tau)+\frac{\Delta(p)}{k^4}(k_1+k_2|p(\tau)|)^2}}{2\Delta(p)}. \tag{4.9}$$

Using (4.5), (4.7) and (4.9), it is easy to show that the boundary conditions (2.3) and (2.4) reproduce eqs.(3.7) and (3.8) with the constant ρ expressed in terms of the coefficients of transformation (4.1). Inserting (4.5) and (4.7) into eqs.(2.3) and (2.4) we represent their sum and difference in the form

$$2\frac{d}{d\tau}\ln\left[\Delta(p)-p'(\tau)\frac{F(f)}{f'(\tau)}\right]-4(p-p^{-1})\frac{f'(\tau)}{F(f)}-2\frac{p'(\tau)}{p(\tau)}=$$
$$=(k_1-k_2)/p(\tau)|\left[\Delta(p)\frac{f'^2(\tau)}{F^2(f)}-p'\frac{f'(\tau)}{F(f)}\right]^{-1/2}, \tag{4.10}$$

$$\frac{2}{p(\tau)}\left[p'(\tau)-2\Delta(p)\frac{f'(\tau)}{F(f)}\right]=\frac{k_1+k_2|p(\tau)|}{\sqrt{\Delta(p)\frac{f'^2(\tau)}{F^2(\tau)}-p'(\tau)\frac{f'(\tau)}{F(f)}}}. \tag{4.11}$$

Substitution of (4.9) into (4.11) gives the equation

$$\pm\sqrt{p'^2(\tau)+\frac{\Delta(p)}{k^4}(k_1+k_2|p(\tau)|)^2}=-k^2p(\tau), \tag{4.12}$$

where the sign of the root is determined by that of the function $p(\tau)$. After comparing (4.6) with (3.9) and identifying

$$2\rho=a+d \tag{4.13}$$

the above is easily recognized as eq.(3.7). Upon substitution of (4.9) into eq.(4.10) the latter takes the form (3.6) and is also reduced to (3.7) and (3.8). Thus, the function $p(\tau)$ with (4.13) is defined by the representation (3.11).

Using (3.11) we now determine the functions $f(\tau)$ and $g(\tau)$. Owing to (4.12) the expression (4.9) assumes the form

$$\frac{f'(\tau)}{F(f)} = \frac{p'(\tau) - k^2 p(\tau)}{2\Delta(p)}. \tag{4.14}$$

To express the function $g(\tau)$ in terms of $p(\tau)$ we consider the relationship

$$\frac{g'(\tau)}{G(g)} = \frac{g'(\tau)}{cg^2(\tau) + (d-a)g(\tau) - b}. \tag{4.15}$$

Substitution of expressions (4.5) and (4.14) into (4.15) gives

$$\frac{g'(\tau)}{G(g)} = -\frac{p'(\tau) + k\, p(\tau)}{2\Delta(p)}. \tag{4.16}$$

Integrating (4.14) and (4.16) we get

$$\int^{f(\tau)} \frac{df}{F(f)} = c_1 + \frac{1}{2}\int^{p(\tau)} \frac{dp}{\Delta(p)} - \frac{k^2}{2} J(\tau),$$

$$\int^{g(\tau)} \frac{dg}{G(g)} = c_2 - \frac{1}{2}\int^{p(\tau)} \frac{dp}{\Delta(p)} - \frac{k^2}{2} J(\tau). \tag{4.17}$$

Here the integrals are depending on the condition $|a+d| \gtreqless 2$ performed in terms of the same elementary functions since the discriminants of polynomials (4.2), (4.15), and (4.6) coincide, and the elliptic integral

$$J(\tau) = -\int_{p_1}^{|p(\tau)|} \frac{dp\cdot p}{\Delta(p)\, W(p)} \tag{4.18}$$

with the use of (3.11), is split into a sum of normal elliptic integrals of the first and third kind. Solutions (4.17) should be periodic up to the Moebius transformations (4.1). The latter may, depending on whether $|a+d| > 2$, $|a+d| = 2$ or $|a+d| < 2$, always be reduced either to the hyperbolic, or parabolic, or elliptic form, respectively, by Moebius transformations (2.5) (see, e.g.,[15,16]). Then insertion of (4.17) into (4.1) with (3.13) leads to the constraint on arbitrary

constants $(a + d)$ and q:

$$\epsilon(p)\, k^2\, \Omega\left(\frac{a+d}{2}, q\right) = R(a+d) \tag{4.19}$$

in addition to (3.15). Here Ω is a real period of the integral (4.18) and a function R depends on the choice of parametrization of the coefficients in (4.1). Finally, the coordinates $x^{\mu}(\tau, \sigma)$ of the minimal surfaces of the relativistic string with massive ends are given via expressions (4.17) for the functions $f(\tau)$ and $g(\tau)$ by formulas (2.1) and (2.2).

5. CONCLUSION

In this paper it has been shown that the geometric approach to the dynamics of the relativistic string with massive ends when the string world sheet is completely defined by the geometric invariants of the world trajectories of the massive ends of the string in the Minkowski space E_2^1 [8,9] allows us to derive both the well-known exact solution describing the rotation in a given plane of the straight-line string with quarks at its ends[6-8] and a new exact solution that describes a more intricate motion of this system. In the first case the trajectories of motion of the massive endpoints turn out to be helices with the constant torsions and define the surface which is a helicoid in E_2^1. It is worth mentioning that the helicoid is the only nontrivial minimal surface belonging to the class of rules surfaces generated by a moving straight lines in a space. A solution of that sort does not describe transverse excitations of the string and hence does not contribute to the linear behaviour of the static potential between quarks at long distances.

On the contrary, the new exact solution we have found here for massive string endpoints moving along the same world trajectories with the constant curvatures and periodic torsions describes a surface that is not a helicoid and therefore does not belong to the class of rules surfaces. Thus, the solution in question describes transverse excitations of the string and, according to ref.[17] , radial motions of the massive points.

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Новое точное решение классических уравнений движения релятивистской струны с массивными концами

Классическая динамика релятивистской струны с массивными концами формулируется в терминах геометрических инвариантов мировых траекторий точечных масс на концах струны. Рассматривается трехмерное пространство Минковского E_2^1. В этой формулировке переменные струны определяются кривизной и кручением мировых траекторий масс, для которых выведены дифференциальные уравнения с запаздывающими аргументами, учитывающие эффекты запаздывания взаимодействия масс, связанных струной. Хорошо известный пример вращающейся прямолинейной струны с массивными концами соответствует частному решению этих уравнений, когда кручения траекторий постоянны. Найдено новое точное решение с периодическими кручениями мировых траекторий масс, координаты струны в этом случае выражаются эллиптическими интегралами, зависящими от кручения и представляют движение струны с поперечными колебаниями.

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A New Exact Solution to the Classical Equations of Motion of the Relativistic String with Massive Ends

The classical histories of the relativistic string with massive ends in space-time E_2^1 are examined in terms of geometric invariants of both the string world surface and world-lines of the point masses at the string ends. In our formulation the string variables are completely defined by means of the constant curvatures and torsions of the endpoint trajectories which are subjected to a system of differential equations with a delayed argument that incorporates retardation effects of the interaction of two point masses through the string. The well-known example of the rotating straight-line string with massive ends corresponds to a particular solution of this system for the constant torsions. A new exact solution for the periodic torsions of the world trajectories of the massive string ends is found. In this case the string coordinates are represented in terms of normal elliptic integrals and describe a more intricate motion including its transverse vibrations than rotation of a stretched string in a given plane.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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