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1. In paper [1] a representation series of the Wess and Zumino conformal superalgebra[2,3] has been obtained by the induced representation method. Each representation of this series is characterized by two (in general complex) numbers (d,z) and the algebra basis is realized as differential operators acting in the space of functions

$$\Phi^{(A)}_{(d,z)}(x_\mu, \theta^+_\alpha, \theta^-_\beta), \tag{1.1}$$

where x_μ are the coordinate of the radius-vector in the Minkovsky space (the metric tensor $g_{\mu\nu}$ is chosen in the form $-g_{00}=g_{11}=g_{22}=g_{33}=1$);); and $\theta^+_\alpha, \theta^-_\beta$ $(\alpha,\beta=1,2,3,4)$ are Dirac spinors and at the same time Grassman variables, i.e.,

$$\{\theta^+_\alpha, \theta^+_\beta\} = \{\theta^+_\alpha, \theta^-_\beta\} = \{\theta^-_\alpha, \theta^-_\beta\} = 0 . \tag{1.2}$$

Besides, θ^+_α and θ^-_α satisfy the following relations:

$$(1 - i\gamma_5)\,\theta^+ = (1 + i\gamma_5)\,\theta^- = 0 , \tag{1.3}$$

$$(1 + i\gamma_5)\,\theta^+ = 2\theta^+,\ (1 - i\gamma_5)\,\theta^- = 2\theta^-,\ \gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3. \tag{1.4}$$

Hereafter the γ_μ-matrices are taken in the Majorana representation.

The index (A) of the functions (1.1) denotes the set of the discrete indices, characterizing the representation of the stability subalgebra. In particular, if the series (1.1) is a scalar one, (A) is the empty set; if it is a spinor series, the set (A) consists of one spinor index, etc.

In paper [1] the scalar representation series was discussed in more detail in that particular case when the numbers d and z are related in the following way:

$$z = \frac{2}{3} d . \tag{1.5}$$

It then turned out that in the space (1.1) there is an invariant subspace, composed of the functions

$$\Phi_{(d)}(x_\mu, \theta^+_\alpha) \tag{1.6}$$

independent of θ^-_α. The conformal superalgebra representations acting in (1.6) have been separated (they have been obtained earlier in paper [4]) and the two- and three-point functions of the superfields transforming according to these representations have been calculated.

In order to formulate the aim of the present paper we recall how the representations (1.6) have been separated. For this purpose we rewrite the generators of the conformal superalgebra, obtained in paper [1]:

$$\begin{aligned}
P_\mu &= -i \frac{\partial}{\partial x^\mu} , \\
K_\mu &= i\{(2x_\mu x_\nu - g_{\mu\nu} x^2)\frac{\partial}{\partial x_\nu} + 2x_\mu (d + \frac{1}{2}\theta^+ \frac{\partial}{\partial \theta^+} - \frac{1}{2}\theta^- \frac{\partial}{\partial \theta^-}) + \\
&\quad + 2x^\nu [\frac{\partial}{\partial \theta^+}\sigma_{\mu\nu}\theta^+ + \frac{\partial}{\partial \theta^-}\sigma_{\mu\nu}\theta^- - i\Sigma_{\mu\nu}] + \frac{\partial}{\partial \theta^-}\gamma_\mu \theta^+\} , \\
D &= i(-d - x_\mu \frac{\partial}{\partial x_\mu} - \frac{1}{2}\theta^+ \frac{\partial}{\partial \theta^+} + \frac{1}{2}\theta^- \frac{\partial}{\partial \theta^-}) , \\
M_{\mu\nu} &= i[x_\mu \frac{\partial}{\partial x^\nu} - x_\nu \frac{\partial}{\partial x^\mu} + \frac{\partial}{\partial \theta^+}\sigma_{\mu\nu}\theta^+ + \frac{\partial}{\partial \theta^-}\sigma_{\mu\nu}\theta^-] + \Sigma_{\mu\nu}, \\
S_\alpha &= -8i \Sigma_{\mu\nu}(\sigma^{\mu\nu}\theta^-)_\alpha - 4(2d - 3z)\theta^-_\alpha + 8(\gamma^\nu \theta^+)_\alpha \frac{\partial}{\partial x^\nu} + \\
&\quad + i(\gamma^\circ \frac{\partial}{\partial \theta^+})_\alpha - 8(\theta^- \gamma^\circ \theta^-)(\gamma^\circ \frac{\partial}{\partial \theta^-})_\alpha , \\
T_\alpha &= -8i \Sigma_{\mu\nu}(\sigma^{\mu\nu}\theta^+)_\alpha + \frac{1}{2} x_\mu (\gamma^\mu (1 - i\gamma_5) S)_\alpha + 4(2d + 3z)\theta^+_\alpha + \\
&\quad + i(\gamma^\circ \frac{\partial}{\partial \theta^-})_\alpha - 16\theta^+_\alpha(\theta^- \frac{\partial}{\partial \theta^-}) + i x_\mu (\gamma^\mu \gamma^\circ \frac{\partial}{\partial \theta^+})_\alpha - 8(\theta^+ \gamma^\circ \theta^+)(\gamma^\circ \frac{\partial}{\partial \theta^+})_\alpha , \\
\Pi &= -z + \theta^+ \frac{\partial}{\partial \theta^+} + \theta^- \frac{\partial}{\partial \theta^-} ,
\end{aligned} \tag{1.7}$$

where $\sigma_{\mu\nu} = \frac{1}{4}[\gamma_\mu, \gamma_\nu]$, $\alpha = 1,2,3,4$, and $\Sigma_{\mu\nu}$ are the matrices of the finite-dimensional representation of the Lorentz group acting on the index (A) of the functions (1.1). The expressions (1.7) differ from the corresponding ones of paper [1] due to the fact that we have set $k_\mu = 0$, used equality (2.16) of [1] (i.e., the substitution $\frac{\partial}{\partial \rho} \to d$ and $\frac{\partial}{\partial \lambda} \to -iz$) and we have not separated S^+, S^-, T^+ and T^-. Now we can see that the variable θ^-_α enters into the expressions for the generators of the algebra (1.7) always multiplied by $\partial/\partial\theta^-_\alpha$ except for the two first terms of the operator S_α, where θ^-_α enters independently. That is why, in the absence of these two terms, all the generators would keep invariant the space of functions independent of θ^-_α. It is easy to see that this takes place if and only if $\Sigma_{\mu\nu} = 0$ and equation (1.5) holds. Thus we are led to the conclusion, that in the space of functions independent of θ^-_α representations of the conformal superalgebra with $\Sigma_{\mu\nu} \neq 0$ cannot be realized. The same result can be obtained directly from the method of induced representations taking into account that representations of the stability superalgebra (in this case, it is the algebra with the basis $M_{\mu\nu}$, K_μ, S^-_α, T^+_α, T^-_α, D and Π in the notations of paper /1/) for which the following additional relations take place

$$K_\mu = S^-_\alpha = T^+_\alpha = T^-_\alpha = 0, \qquad (1.8)$$
$$M_{\mu\nu} = \Sigma_{\mu\nu} \neq 0, \quad D \neq 0, \quad \Pi \neq 0,$$

cannot exist.

This means, that the scalar representation of the conformal superalgebra of the type (1.6) are on a distinct status among all representations of the algebra and have no analogs of other Lorentz-structure.

If we mention, in addition, the relatively poor content of the ordinary fields in the superfield of the kind (1.6) the necessity to look for other representations of the considered algebra is obvious. These

of a different Lorentz-structure (for instance, scalar and spinor) should act in a space of functions of one and the same type. It is the purpose of this paper to find such representations and to calculate the various two-point functions. Moreover, in the process of solving that problem it will become clear what we mean by the expression "functions of one and the same type".

2. We first consider the scalar case assuming $2d \neq 3z$. The space, where the operators (1.7) act, consists of functions of the type (1.1) which are expanded in powers of θ^-_β. We write the expansion in the form:

$$\Phi_{(d,z)}(x,\theta^+,\theta^-) = F(x,\theta^+) + \chi^\alpha(x,\theta^+)\,\theta^-_\alpha + H(x,\theta^+)\,\theta^-\gamma^0\theta^- \qquad (2.1)$$

Acting on the functions (2.1) by the operators (1.7) and making a subsequent reexpansion in powers of θ^-_α, we obtain the action of the basis elements of our superalgebra (1.7) on the coefficients $F(x,\theta^+)$, $\chi^\alpha(x,\theta^+)$ and $H(x,\theta^+)$. It is readily seen that as a result of the action of the operators (1.7) these coefficients are mixed. Let us introduce the following terminology:

a) We call an operator A diagonal if in the expansion

$$A\Phi_{(d,z)}(x,\theta^+,\theta^-) = F'_A(x,\theta^+) + \chi'^\alpha_A(x,\theta^+)\,\theta^-_\alpha + H'_A(x,\theta^+)\,\theta^-\gamma^0\theta^- \qquad (2.2)$$

the coefficient function $F'_A(x,\theta^+)$ is expressed by itself only; $\chi'^\alpha_A(x,\theta^+)$ by $\chi^\alpha(x,\theta^+)$ only and $H'_A(x,\theta^+)$ by $H(x,\theta^+)$ only. For example the operators $\frac{\partial}{\partial x^\mu}$, $f(\theta^+, \frac{\partial}{\partial\theta^+})$ and $\theta^-_\alpha \frac{\partial}{\partial\theta^-_\beta}$ are diagonal.

b) We call an operator A, a reducing operator if $F'_A(x,\theta^+)$ in (2.2) is expressed by $\chi^\alpha(x,\theta^+)$; $\chi'^{\alpha}_A(x,\theta^+)$ by $H(\dot{x},\theta^+)$ and $H'_A(x,\theta^+)=0$. For instance, $\frac{\partial}{\partial\theta^-_\alpha}$ is a reducing operator.

c) We call an operator A a raising operator if in (2.2) $F'_A(x,\theta^+)=0$; $\chi'^{\alpha}_A(x,\theta^+)$ is expressed by $F(x,\theta^+)$ and $H'_A(x,\theta^+)$ by $\chi^\alpha(x,\theta^+)$. For instance the multiplication by θ^-_α is a raising operator.

In accordance with these definitions the generators of the conformal subalgebra (i.e. P_μ, K_μ, D and $M_{\mu\nu}$) contain only diagonal and reducing operators. The same is true also for the generator Π. It is also easy to see that S_α and T_α (only the term containing S) contain raising terms too. Therefore, the subspaces R_0 and R_1, composed of the functions, respectively

$$R_0:\ \Phi^{(0)}_{(d,z)}(x,\theta^+,0) = F(x,\theta^+),$$

$$R_1:\ \Phi^{(1)}_{(d,z)}(x,\theta^+,\theta^-) = F(x,\theta^+) + \chi^\alpha(x,\theta^+)\theta^-_\alpha, \tag{2.3}$$

are invariant under the action of the conformal subalgebra and the Π-operation. The invariance of these subspaces is broken only by the operators S. (T may not be considered since they contain raising operators only through S). Let us calculate the action of S_α on an arbitrary function from R_1 (R_0 can be regarded as a subspace of R_1). After a straightforward calculation we find:

$$S_\alpha\Phi^{(1)}_{(d,z)}(x,\theta^+,\theta^-) = [\,i(\gamma^\circ\frac{\partial}{\partial\theta^+})_\alpha + 8(\gamma^\nu\theta^+)_\alpha\frac{\partial}{\partial x^\nu}]F(x,\theta^+) +$$
$$+\{[\,i(\gamma^\circ\frac{\partial}{\partial\theta^+})_\alpha + 8(\gamma^\nu\theta^+)_\alpha\frac{\partial}{\partial x^\nu}]\chi^\beta(x,\theta^+) + \tag{2.4}$$
$$+4(3z-2d)\delta^\beta_\alpha F(x,\theta^+)\}\theta^-_\beta + (6z-4d-8)(\chi(x,\theta^+)\gamma^\circ)_\alpha\theta^-\gamma^\circ\theta^-.$$

Equality (2.4) shows that terms quadratic in θ^-_α will not appear in the two cases

1. $\chi^\alpha(x,\theta^+)\equiv 0$, (2.5)
2. $d+2=\frac{3}{2}z$. (2.6)

In the first case we will necessarily get the representation already obtained in paper /1/ (which we do not consider here).

The new equality is eq. (2.6). We see that, when it holds, the subspace R_1 is invariant with respect to the action of the superalgebra (1.7).

Let us now consider the spinor superfield. In that case the functions of type (1.1) have the following decomposition

$$\Phi^\alpha_{(d,z)}(x_\mu,\theta^+,\theta^-) = \Phi^{+\alpha}_{(d,z)}(x,\theta^+,\theta^-) + \Phi^{-\alpha}_{(d,z)}(x,\theta^+,\theta^-), \tag{2.7}$$

where $\alpha=1,2,3,4$ is a spinor index, and the sign "$\pm$" has the same meaning as in $\theta^\pm$, that is

$$\Phi^{\pm\alpha}_{(d,z)}(x,\theta^+,\theta^-) = \frac{1}{2}[\,(1\pm i\gamma_5)\Phi_{(d,z)}(x,\theta^+,\theta^-)\,]^\alpha. \tag{2.8}$$

In order to obtain the conformal superalgebra generators acting in the space of functions (2.7) it is necessary to substitute in (1.7)

$$\Sigma_{\mu\nu} = i\sigma_{\mu\nu} \tag{2.9}$$

and to have in mind that this matrix acts on the spinor indices of the functions (2.7)

First of all we notice that the decomposition (2.7) is invariant under the action of the conformal superalgebra generators obtained after the substitution (2.9). There fore both terms in (2.7) can be considered independently. We decompose as before the field $\Phi^{+\alpha}_{(d,z)}(x,\theta^+,\theta^-)$ in powers of θ^-_α

$$\Phi^{+\alpha}_{(d,z)}(x,\theta^+,\theta^-) = \Psi^{+\alpha}(x,\theta^+) + G^\mu(x,\theta^+)(\gamma_\mu\theta^-)^\alpha + B^{+\alpha}(x,\theta^+)\,\theta^-\gamma^\circ\theta^- . \quad (2.10)$$

The same arguments as in the scalar case lead here to the conclusion that the subspace M^+_1 formed out of the functions of the type

$$\Phi^{+\alpha}_{1(d,z)}(x,\theta^+,\theta^-) = \Psi^{+\alpha}(x,\theta^+) + G^\mu(x,\theta^+)(\gamma_\mu\theta^-)^\alpha , \quad (2.11)$$

is invariant with respect to the conformal subalgebra and the operation Π. Acting on the functions (2.11) by the operators S_α we find

$$(S_\alpha)^\beta{}_\gamma\Phi^\gamma_{(1)(d,z)}(x,\theta^+,\theta^-) =$$

$$= (2d-3z+4)\,G^\mu(x,\theta^+)(\gamma_\mu\gamma^\circ(1-i\gamma_5))^\beta_\alpha\,\theta^-\gamma^\circ\theta^- + \quad (2.12)$$

$$+ (4d-6z)(\gamma^\mu\Psi^+)_\alpha(\gamma_\mu\theta^-)^\beta +$$

$$+ [8(\gamma^\nu\theta^+)_\alpha\frac{\partial}{\partial x^\nu} + i(\gamma^\circ\frac{\partial}{\partial\theta^+})_\alpha]\Phi^\beta_{(1)(d,z)}(x,\theta^+,\theta^-) .$$

The latter relation shows that the space M^+_1 is invariant with respect to the spinor representation of the conformal superalgebra (1.7).

Similar calculations show, that the subspaces composed of functions of the type $\Phi^{-\alpha}_{(d,z)}(x,\theta^+,\theta^-)$ do not contain any simpler subspaces. Thus the results of this section can be summarized in the following theorem:

Theorem: Given eq. (2.6), the scalar and the spinor representations of the algebra (1.7) obtained by the substitution $\Sigma_{\mu\nu}=0$ and $\Sigma_{\mu\nu}=i\sigma_{\mu\nu}$, respectively, are reducible in the sense that the spaces of the functions where the operators (1.7) are acting contain invariant subspaces. In the scalar case it is the subspace of functions R_1 defined by formula (3.2), in the spinor case the space of functions M^+_1 defined by formula (2.11).

It is clear that R_1 and M^+_1 are formed out of functions of one and the same type, i.e., functions linear in θ^-_α.

3. As was already mentioned it is unlikely that the representations in the space of functions (1.6) though being quite simple, will be of a wide practical use due to their great "individuality". At the same time the representations obtained in the previous section on the one hand are not very complicated, and on the other hand (which is more important) they can possess a more copious Lorentz structure. In this section we will discuss scalar and spinor superfields, transforming according to the representations obtained and calculate the various two-point functions of these fields. We write the scalar superfield in the form:

$$F_{(d)}(x,\theta^+,\theta^-) = \phi_{(d)}(x,\theta^+) + i\psi^\beta_{(d)}(x,\theta^+)\theta^-_\beta \quad (3.1)$$

and the spinor one in the form:

$$\Phi_{(d)\alpha}(x,\theta^+,\theta^-) = \eta^+_{(d)\alpha}(x,\theta^+) + iG_{(d)\mu}(x,\theta^+)(\gamma^\mu\theta^-)_\alpha \quad (3.2)$$

The subscript (d) denotes the corresponding irreducible representation $z=\frac{2}{3}(d+2)$. We have separated an "i" in order to get the correct conjugation properties of the two-point functions. It is more simply to write the transformation laws of the fields (3.1) and (3.2) in an infinitesimal form. For the components of the scalar superfield we have:

$$[S_\alpha, \phi_{(d)}(x,\theta^+)] = [i(\gamma^\circ\frac{\partial}{\partial\theta^+})_\alpha + 8(\gamma^\nu\theta^+)_\alpha\frac{\partial}{\partial x^\nu}]\phi_{(d)}(x,\theta^+),$$

$$\{S_\alpha,\psi_{(d)\beta}(x,\theta^+)\} = [i(\gamma^\circ\frac{\partial}{\partial\theta^+})_\alpha + 8(\gamma^\nu\theta^+)_\alpha\frac{\partial}{\partial x^\nu}]\psi_{(d)\beta}(x,\theta^+) -$$

$$-8i(1-i\gamma_5)_{\alpha\beta}\phi_{(d)}(x,\theta^+), \quad (3.3)$$

$$[T_\alpha,\phi_{(d)}(x,\theta^+)] = [ix_\rho(\gamma^\rho\gamma^\circ\frac{\partial}{\partial\theta^+})_\alpha + 8(\gamma^\rho\gamma^\sigma\theta^+)_\alpha x_\rho\frac{\partial}{\partial x^\sigma} -$$

$$-8(\theta^+\gamma^\circ\theta^+)(\gamma^\circ\frac{\partial}{\partial\theta^+})_\alpha + 16(d+1)\theta^+_\alpha]\phi_{(d)}(x,\theta^+) + (\gamma^\circ\psi_{(d)}(x,\theta^+))_\alpha$$

$$\{T_\alpha,\psi_{(d)\beta}(x,\theta^+)\} = [ix_\rho(\gamma^\rho\gamma^\circ\frac{\partial}{\partial\theta^+})_\alpha + 8(\gamma^\rho\gamma^\sigma\theta^+)_\alpha x_\rho\frac{\partial}{\partial x^\sigma} - \quad (3.4)$$

$$-8(\theta^+\gamma^\circ\theta^+)(\gamma^\circ\frac{\partial}{\partial\theta^+})_\alpha + 16d\theta^+_\alpha]\psi_{(d)\beta}(x,\theta^+) -$$

$$-8ix_\mu(\gamma^\mu(1-i\gamma_5))_{\alpha\beta}\phi_{(d)}(x,\theta^+).$$

For the spinor superfield we have, respectively:

$$\{S_\alpha,\eta^+_{(d)\beta}(x,\theta^+)\} = [i(\gamma^\circ\frac{\partial}{\partial\theta^+})_\alpha + 8(\gamma^\nu\theta^+)_\alpha\frac{\partial}{\partial x^\nu}]\eta^+_{(d)\beta}(x,\theta^+)$$

$$[S_\alpha, G^\mu_{(d)}(x,\theta^+)] = [i(\gamma^\circ\frac{\partial}{\partial\theta^+})_\alpha + 8(\gamma^\nu\theta^+)_\alpha\frac{\partial}{\partial x^\nu}]G^\mu_{(d)}(x,\theta^+) + \quad (3.5)$$

$$+ 8i(\gamma^\mu\eta^+_{(d)}(x,\theta^+))_\alpha,$$

$$\{T_\alpha,\eta^+_{(d)\beta}(x,\theta^+)\} = [ix_\rho(\gamma^\rho\gamma^\circ\frac{\partial}{\partial\theta^+})_\alpha + 8(\gamma^\rho\gamma^\sigma\theta^+)_\alpha x_\rho\frac{\partial}{\partial x^\sigma} -$$

$$-8(\theta^+\gamma^\circ\theta^+)(\gamma^\circ\frac{\partial}{\partial\theta^+})_\alpha + 16(d+1)]\eta^+_{(d)\beta}(x,\theta^+) +$$

$$+8(\sigma_{\rho\sigma}\theta^+)_\alpha(\sigma^{\rho\sigma}\eta^+_{(d)}(x,\theta^+))_\beta + \frac{1}{2}(\gamma_\mu\gamma^\circ(1-i\gamma_5))_{\alpha\beta}G^\mu_{(d)}(x,\theta^+), \quad (3.6)$$

$$[T_\alpha, G^\mu_{(d)}(x,\theta^+)] = [ix_\rho(\gamma^\rho\gamma^\circ\frac{\partial}{\partial\theta^+})_\alpha + 8(\gamma^\rho\gamma^\sigma\theta^+)_\alpha x_\rho\frac{\partial}{\partial x^\sigma} -$$

$$-8(\theta^+\gamma^\circ\theta^+)(\gamma^\circ\frac{\partial}{\partial\theta^+})_\alpha + 16d\theta^+_\alpha]G^\mu_{(d)}(x,\theta^+) +$$

$$+16(\sigma^{\mu\sigma}\theta^+)_\alpha G_{(d)\sigma}(x,\theta^+) - 8x_\rho(\gamma^\rho\gamma^\mu\eta^+_{(d)}(x,\theta^+))_\alpha.$$

We do not write the rest of the commutation relations because the generators S_α and T_α can reproduce the whole superalgebra. To calculate the two-point functions we need also the conjugated fields:

$$\tilde{\tilde{F}}_{(d)}(x,\xi^-,\xi^+) = \tilde{\tilde{\phi}}_{(d)}(x,\xi^-) + i\tilde{\tilde{\psi}}^{\beta}_{(d)}(x,\xi^-)\xi^+_{\beta}.$$

$$\Phi_{(d)\ \alpha}(x,\xi^-,\xi^+) = \tilde{\tilde{\bar{\eta}}}_{(d)\alpha}(x,\xi^-) - i\tilde{\tilde{G}}_{(d)\mu}(x,\xi^-)(\gamma^{\mu}\xi^+)_{\alpha} \tag{3.7}$$

The transformation laws of these fields are easily obtained from eq. (3.3) - (3.6) by the substitution $\theta^+_{\alpha} \to \xi^-_{\alpha}$, $\theta^-_{\alpha} \to \xi^+_{\alpha}$.

Remark: We use the terminology "conjugated fields" or "conjugated representation" in a conventional sense. The field transformation laws (3.7) are such that the invariant two-point functions of these fields with the fields (3.1) and (3.2) do not vanish (as it will be seen in the following). The representations in the space (3.7) can be found by the induced representation methc if another stability algebra is used, obtained from the old one by the substitution S^- and $T^+ \to S^+$ and T^-, respectively. We write the two-point function of the scalar superfields in the form:

$$\Delta^{++}_{(d_1 d_2)}(x-y;\theta^+_1,\theta^-_1;\theta^+_2,\theta^-_2) =$$

$$= \langle 0|F_{(d_1)}(x,\theta^+_1,\theta^-_1)\, F_{(d_2)}(y,\theta^+_2,\theta^-_2)|0\rangle,$$

$$\Delta^{+-}_{(d_1 d_2)}(x-y;\theta^+,\theta^-;\xi^-,\xi^+) = \tag{3.8}$$

$$= \langle 0|F_{(d_1)}(x,\theta^+,\theta^-)\, \tilde{\tilde{F}}_{(d_2)}(y,\xi^-,\xi^+)|0\rangle .$$

For the spinor superfields we have, respectively

$$S^{++}_{(d_1 d_2)\alpha\beta}(x-y;\theta^+_1,\theta^-_1;\theta^+_2,\theta^-_2) =$$

$$= \langle 0|\Phi_{(d_1)\alpha}(x,\theta^+_1,\theta^-_1)\, \Phi_{(d_2)\beta}(y,\theta^+_2,\theta^-_2)|0\rangle, \tag{3.9}$$

$$S^{+-}_{(d_1 d_2)\alpha\beta}(x-y;\theta^+,\theta^-;\xi^-,\xi^+) =$$

$$= \langle 0|\Phi_{(d_1)\alpha}(x,\theta^+,\theta^-)\, \tilde{\tilde{\Phi}}_{(d_2)\beta}(y,\xi^-,\xi^+)|0\rangle .$$

Performing an infinitesimal transformation of the fields in the right hand side of eq. (3.8) and (3.9) using the formulae (3.3)-(3.6) and the invariance of the vacuum state:

$$S_{\alpha}|0\rangle = T_{\alpha}|0\rangle = 0$$

for each function (3.8)-(3.9) we get a system of partial differential equations. These equations are rather cumbersome and we do not write them here. But it is not, in principle, difficult to solve them. (The equations seem uncommon due to the presence of Grassman variables and derivatives with respect to them. However, to find the primitives of Grassman variables is a rather trivial operation). Therefore we give only the final results:

$$\Delta^{++}_{(d_1 d_2)}(x-y,\theta^+_1,\theta^-_1;\theta^+_2,\theta^-_2) = 0$$

for any d_1 and d_2.

$$\Delta^{+-}_{(d_1 d_2)}(x-y,\theta^+,\theta^-;\xi^-,\xi^+) = 0 \quad \text{if } d_1 \neq d_2 \tag{3.10}$$

and

$$\Delta^{+-}_{(d\,d)}(x-y,\theta^+,\theta^-;\xi^-,\xi^+) = \mathrm{const}\{1-16i[\theta^-\gamma^\circ\xi^- + \xi^+\gamma^\circ\theta^+ -$$

$$-(\theta^-\gamma^\circ\gamma^\mu\xi^+)(x-y)_\mu]\}e^{8i(\theta^+\gamma^\circ\gamma^\nu\xi^-)\frac{\partial}{\partial x^\nu}}[(x-y)^2]^{-d},$$

$$S^{+-}_{(d_1 d_2)}(x-y,\theta^+_1,\theta^-_1,\theta^-_2,\theta^+_2) = 0,$$

for any d_1 and d_2.

$$S^{+-}_{(d_1 d_2)}(x-y,\theta^+,\theta^-;\xi^-,\xi^+) = 0 \quad \text{if } d_1 \neq d_2$$

$$S^{+-}_{(d,d)}(x-y,\theta^+,\theta^-;\xi^-,\xi^+) = \mathrm{const}\{1-16i[\theta^-\gamma^\circ\xi^- + \xi^+\gamma^\circ\theta^+ - \tag{3.11}$$

$$-(\theta^-\gamma^\circ\gamma^\mu\xi^+)(x-y)_\mu]\}e^{8i(\theta^+\gamma^\circ\gamma^\nu\xi^-)\frac{\partial}{\partial x^\nu}}(\gamma^\mu\gamma^\circ(1+i\gamma_5))_{\alpha\beta}(x-y)_\mu[(x-y)^2]^{-d-\frac{1}{2}}$$

The presence of nonvanishing two-point functions of the transforming according to the representations (3.3)-(3.6) superfields and their conjugated leads to a possibility of writing invariant bilinear forms out of these superfields. Such bilinear forms are evidently the following:

$$\int \Delta^{+-}_{(d\,d)}(x-y,\theta^+,\theta^-;\xi^-,\xi^+)F^{(1)}_{(d)}(x,\theta^+,\theta^-)\tilde{\tilde{F}}^{(2)}_{(d)}(y,\xi^-,\xi^+) \times \tag{3.12}$$
$$\times\, d\mu(x,y,\theta^+,\theta^-;\xi^-,\xi^+)$$

for the scalar superfields and

$$\sum_{\alpha\beta}\int S^{+-}_{(d\,d)\,\alpha\beta}(x-y,\theta^+,\theta^-,\xi^-,\xi^+)\Phi^1_{(d)\,\alpha}(x,\theta^+,\theta^-)\Phi^2_{(d)\,\beta}(y,\xi^-,\xi^+)$$
$$\times\, d\mu(x,y,\theta^+,\theta^-,\xi^-,\xi^+), \tag{3.13}$$

for the spinor superfield, where $d\mu(x,y,\theta^+,\theta^-,\xi^-,\xi^+)$ is the invariant measure. It is, in principle, well-known [5] how to find it (see also Appendix) and after tedious calculations we obtain:

$$d\mu(x,y,\theta^+,\theta^-,\xi^-,\xi^+) = \mu(x-y;\theta^+,\theta^-;\xi^-,\xi^+)\times$$
$$\times\, d^4x\,d^4y\,(d\theta^+\gamma^\circ d\theta^+)(d\theta^-\gamma^\circ d\theta^-)(d\xi^-\gamma^\circ d\xi^-)(d\xi^+\gamma^\circ d\xi^+), \tag{3.14}$$

and

$$\mu(x-y,\theta^+,\theta^-,\xi^-,\xi^+) = \mu_0 e^{8i(\theta^+\gamma^\circ\gamma^\nu\xi^-)\frac{\partial}{\partial x^\nu}}[(x-y)^2]^{-4}\times$$

$$\times\{\frac{1}{[1-16i\theta^+\gamma^\circ\xi^+][1-16i\theta^-\gamma^\circ\xi^-]} + 16\cdot 8(x-y)^2(\xi^+\gamma^\circ\xi^+)(\theta^-\gamma^\circ\theta^-) - \tag{3.15}$$

$$-16i[(\xi^+\gamma^\circ\gamma^\rho\theta^-) + 16i(\xi^-\gamma^\circ\gamma^\rho\xi^+)(\theta^-\gamma^\circ\theta^-) -$$

$$-16i(\theta^+\gamma^\circ\gamma^\rho\theta^-)(\xi^+\gamma^\circ\xi^+) -$$

$$-16^2(\theta^+\gamma^\circ\gamma^\rho\xi^-)(\xi^+\gamma^\circ\xi^+)(\theta^-\gamma^\circ\theta^-)](x-y)_\rho\}.$$

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APPENDIX

There exists a general theory for constructing an invariant measure in transitive spaces of Lie groups. In paper [5] this theory is extended to the case of supergroups. Nevertheless, in that paper the conditions are not pointed out when this invariant measure exists (in our case, for example, such an invariant measure formed out of d^4x, $d\theta^+\gamma^\circ d\theta^+$ and $d\theta^-\gamma^\circ d\theta^-$ does not exist). It is therefore rather doubtful if one may apply straightforward the results of this paper. To find

$$d\mu(x,y,\theta,\xi) = \mu(x-y;\theta^+,\xi^-,\theta^-,\xi^+)\,d\tau \qquad \textbf{(A1)}$$

we used the following method. Let us denote by O_A any of the generators S_a, T_a and assume the following commutation relations are known

$$[O_A, d\tau] = M_A(x,y;\theta^+,\theta^-;\xi^-,\xi^+)\,d\tau\,. \qquad \textbf{(A2)}$$

Then, obviously, the function $\mu(x-y,\theta^+,\theta^-,\xi^-,\xi^+)$ should satisfy the commutation relations

$$[O_A,\mu] = -M_A(x,y,\theta^+,\theta^-,\xi^-,\xi^+)\,\mu(x-y,\theta^+,\theta^-,\xi^-,\xi^+)\,. \qquad \textbf{(A3)}$$

Equations (A3) reduce to a system of partial differential equations if we notice that

$$[O_A,\mu] = \sum_\nu \Big\{\frac{\partial\mu}{\partial x^\nu}[O_A,x^\nu] + \frac{\partial\mu}{\partial y^\nu}[O_A,y^\nu]\Big\} + \sum_\beta\Big[\frac{\partial\mu}{\partial\theta^+_\beta}\{O_A,\theta^+_\beta\} + \frac{\partial\mu}{\partial\theta^-_\beta}\{O_A,\theta^-_\beta\} + \frac{\partial\mu}{\partial\xi^-_\beta}\{O_A,\xi^-_\beta\} + \frac{\partial\mu}{\partial\xi^+_\beta}\{O_A,\xi^+_\beta\}\Big], \qquad \textbf{(A4)}$$

where each commutator (or anticommutator) in (A4) can be found with the help of the operators (1.7). Thus, the problem of obtaining

the function μ reduces to determine the right-hand side of eq. (A2), i.e., to evaluate the function $M_A(x,y,\theta^+,\theta^-,\xi^-,\xi^+)$. As is easily seen, in our case

$$M_A(x,y,\theta^+,\theta^-,\xi^-,\xi^+) = R_A(x,\theta^+,\theta^-) + \tilde{\tilde{R}}_A(y,\xi^-,\xi^+)\,,$$

and

$$[O_A, d^4x\,d\theta^+\gamma^\circ d\theta^+ d\theta^-\gamma^\circ d\theta^-] = R_A(x,\theta^+,\theta^-)\,d^4x\,(d\theta^+\gamma^\circ d\theta^+)(d\theta^-\gamma^\circ d\theta^-)\,, \qquad \text{(A5)}$$

$\tilde{\tilde{R}}_A$ is an analogous functions for the conjugated representation.

Let us assume that we know the global transformations of the variables x_μ, θ^+_a, θ^-_a i.e.,

$$x'_\mu = F_\mu(x,\theta)\,, \qquad \theta^{\pm\prime}_a = G^\pm_a(x,\theta)\,. \qquad \text{(A6)}$$

Using the rules for substitution of variables in Grassman integrals [5,6] we obtain:

$$d^4x'\,d\theta'^+\gamma^\circ d\theta'^+ d\theta'^-\gamma^\circ d\theta'^- = \Delta(x,\theta)\,\frac{d^4x\,d\theta^+\gamma^\circ d\theta^+ d\theta^-\gamma^\circ d\theta^-}{D^2(x,\theta)}\,, \qquad \text{(A7)}$$

where $\Delta(x,\theta)$ is the Jacobian of the transformation (A6) and $\Delta(x,\theta)$ the determinant of the matrix

$$|D_{ij}| = \begin{vmatrix} \dfrac{\partial G^+_a}{\partial\theta^+_\beta} & \dfrac{\partial G^+_a}{\partial\theta^-_\beta} \\[2ex] \dfrac{\partial G^-_a}{\partial\theta^+_\beta} & \dfrac{\partial G^-_a}{\partial\theta^+_\beta} \end{vmatrix}\,.$$

It is easy to show that eq. (A7) is identical with the analogous one of paper [7]. For the purpose we notice that any block matrix given in the form

$$\Lambda = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$$

satisfies the following identity

$$\left(\begin{array}{c|c} A - BD^{-1}C & BD^{-1} \\ \hline 0 & 1 \end{array}\right) \left(\begin{array}{c|c} 1 & 0 \\ \hline C & D \end{array}\right) = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right),$$

and therefore

$$\det (A - BD^{-1}C) \det D = \det \Lambda .$$

In order to obtain eq. (A5) from (A7) it is necessary to introduce the infinitesimal transformations in (A7). As a result, we find:

$$R_{S_\alpha}(x, \theta^+, \theta^-) = -16\, \theta^-_\alpha$$

$$R_{T_\alpha}(x, \theta^+, \theta^-) = -16 (\gamma^\rho \theta^-)_\alpha\, x_\rho .$$

After inserting these expressions for R (and $\tilde{\tilde{R}}$ respectively) into (A3) and solving the so obtained differential equations we find formula (3.14).

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