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SYMPLECTIC STRUCTURE AND QUANTIZATION OF GAUGE THEORIES

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1. *Introduction*

The description of fundamental particle interactions with the help of the gauge fields introduces superfluous degrees of freedom into the theory. This manifests itself in the singular nature of the corresponding Lagrangian or in the presence of constraints in the equivalent hamiltonian formulation /1/. The phase space in this case is larger than the physical one which is a hypersurface determined by these constraints. On the physical subspace a usual hamiltonian system can be defined without any gauge symmetries and constraints.

Due to the gauge invariance the basic objects of the theory - the gauge potentials- form an overcomplete basis. Gauge fields which are connected by an infinitesimal gauge transformation actually describe the same physical state. Thus the vector potentials are divided into equivalence classes with respect to the gauge group $\mathcal{G}$ action. An equivalence class represents an orbit in the gauge-field configuration space. It is rather the space of orbits than the function space of the gauge fields that has to be viewed as the physical space. Transitions along the orbits correspond to pure gauge transformations. These vertical paths are of no physical importance. Physically significant are only horizontal paths, i.e. paths which are perpendicular to the orbits /2/. These paths describe the time evolution of the physical system. Fixing the gauge, one tries to solve the problem of constructing such horizontal paths. In fact, this means that the reduced phase space (on which an unconstrained Hamiltonian can be defined) is identified with the vector potentials and their conjugate momenta in this gauge. However, Gribov pointed out that the standard Coulomb gauge fails to specify a unique vector potential for each physical field and that the gauge, regarded as a map from the physical fields to the space of vector potentials, is singular /3/. This result was generalized by Singer /2/ who showed that there is no complete gauge for non-Abelian Yang-Mills fields on a 3-sphere and thus no complete gauge on R^3 for which the vector potentials are sufficiently regular at infinity.

This problem has one more aspect. In the path-integral quantization of gauge fields one starts from an initial configuration at $t = 0$ and integrates over all histories, i.e. all paths in the gauge-field configuration space. In such a way the genuine dynamical time evolution is not distinguished from the time evolution generated by gauge transformations. An attempt to circumvent this difficulty consists in imposing a gauge condition /4/; globally, however, this approach fails because garden - variety gauges are only locally unique /5/. Thus, one needs a prescription for choosing the paths so as to eliminate the spurious time development due to gauge transformations.

Using time as a parameter of the paths in the bundle of all spatial potentials, we define the tangent of the path $\vec{A}(t)$

$$\vec{\tau} = \frac{d}{dt} \vec{A}(t).$$

In particular, for vertical paths

$$\vec{A}(t) = g^{-1}(t)\vec{A}(t)g(t) - ig^{-1}(t)\vec{\nabla}g(t)$$

with $g(t)$ an element of $\mathcal{G}$, the tangent vector is

$$\vec{\tau} = \dot{\vec{A}}(t) = \vec{D}\dot{\varepsilon},$$

where $\varepsilon = \{ \varepsilon^{a} \}$ are the parameters of the infinitesimal transformation corresponding to $g(t)$. Thus, vertical paths (i.e. paths along the orbits) have tangent vectors of the form

$$\vec{\tau} = \vec{D}\phi, \tag{1}$$

with ϕ an arbitrary Lie-algebra valued function. To eliminate the time development due to gauge transformations, one should restrict the paths in the path integral to those that are purely horizontal. We can define the horizontal vector $\vec{\sigma}$ /2,6/ as a vector orthogonal to all vertical vectors $\vec{\tau}$ with respect to the scalar product $< \, , >$ in the orbit space, i.e.

$$\langle\vec{\sigma},\vec{\tau}\rangle = 0 \quad \text{for all vertical } \vec{\tau}\text{'s}.$$

Using expression (1), we find /7/

$$0 = \mathrm{Tr} \int d^3x \; \sigma_i(x) \; P_{ij} \; D_j \; \phi(x) =$$

$$= - \mathrm{Tr} \int d^3x \; [D_i \; P_{ij} \; \sigma_j(x)] \; \phi(x),$$

which implies

$$D_i \, P_{ij} \, \sigma_j = 0, \tag{2}$$

(where P_{ij} is the metric in the orbit space) because of the arbitrariness of $\phi(x)$. A path $A_i(t)$ is horizontal if its tangent is everywhere horizontal and condition (2) leads to the following definition of a horizontal path in a space with a metric P_{ij}

$$D_i \, P_{ij} \, \dot{A}_j = 0 . \tag{3}$$

Thus, the correct definition of the metric in the orbit space comes out to be very important for singling out the horizontal paths.

In this paper we show that the physical (orbit) space is equipped with a natural projective metric, which provides introduction of a symplectic structure therein and construction of Poincare-group representation with a nonstandard action on the gauge fields. A transition to physical variables is defined which transforms the initial theory into a manifestly gauge invariant and Lorentz covariant theory of a two-component scalar field. Quantization, then, is straightforward and quantum commutation relations coincide with the classical Poisson-bracket ones.

2. *The metric of the orbit space*

The notion of a metric is connected with the definition of the measure for a distance between the points in the space. As has been mentioned above, one should consider as points in the physical space the gauge orbits as a whole rather than the vector potentials. We shall show that the consequences of this almost obvious observation are far from being trivial.

Let us consider the simple example of free electrodynamics

$$\mathcal{L} = -\frac{1}{4} \, F_{\mu\nu} \, F^{\mu\nu} \tag{4}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \, .$$

From Lagrangian (4) the following equations of motion are obtained

$$\partial^\mu \, F_{\mu\nu} = 0,$$

the one with $\nu = 0$ being, in fact, a constraint. Its explicit solution with respect to the time component of the gauge field A

$$A_0 = \frac{1}{\Delta}\partial_i\partial_0 A_i, \qquad (5)$$

$$\frac{1}{\Delta} f(x) = \int d^3y\, G(x-y)\, f(y),$$

where $G(x-y)$ is the Green function of the Laplace operator, represents A_0 as a functional of the space components A_i thus reducing the gauge-field configuration space

$$\Gamma = \{ A^{\mu} \} \longrightarrow \mathfrak{M} = \{ A^{i} \}.$$

On solution (5) the kinetic term of Lagrangian (4) takes the form

$$\mathcal{L} = \frac{1}{2}\partial_0 A_i\, P_{ij}\, \partial_0 A_j,$$

where

$$P_{ij} = \delta_{ij} - \partial_i \frac{1}{\Delta}\partial_j . \qquad (6)$$

Thus, by analogy with the free-particle action, the natural way to define the infinitesimal squared distance in the space $\mathfrak{M}$ is /8/

$$\delta s^2 = \delta A_i\, P_{ij}\, \delta A_j . \qquad (7)$$

So the space $\mathfrak{M}$ is endowed with a natural metric P.

The operator P is a projection operator since $P^2 = P$. As can easily be shown by partial integration

$$u \cdot P \cdot v = (P \cdot u) \cdot v,$$

so it is self-adjoint. Also the following relations take place:

$$\partial \cdot P = P \cdot \partial = 0,$$

$$P \cdot \delta A = 0, \quad \delta A_i = \partial_i \varepsilon,$$

where ε are the infinitesimal gauge-transformation parameters.

Thus, the distance between any two points on an orbit vanishes and P, in fact, may be considered as a metric in the orbit space where points are the orbits as a whole. This metric separates the invariant (horizontal) characteristic of the orbit. In the case of free electrodynamics, this means that all gauge fields with one and the same transverse part belong to the same equivalence class or represent the same physical situation.

From the explicit expression for P_{ij} it follows that condition (3), which defines the horizontal trajectories in the orbit space, is precisely Gauss law. So Gauss law provides a natural definition of horizontal paths. Integration only over this class of paths means that each physical path (i.e. path in which all gauge-equivalent potentials are identified) gives rize to a unique, everywhere horizontal path in the orbit space. This is the best that can be achieved in the absence of a global gauge. An analogous statement has been proved in ref.[5] for the special case of the temporal gauge. We do not fix the gauge but instead solve explicitly the constraint equation for A_o and concentrate on the structure of the orbit space for thus reduced configuration space and especially on its non-standard metric. The explicit solution of the constraint equation has been postulated in the minimal quantization method [9-12]. As we have seen, this step has not only physical but also deep geometrical motivations.

On solution (5) Lagrangian (4) takes the form

$$\mathcal{L} = \frac{1}{2}\,\partial^{\mu}A_i\;P_{ij}\;\partial_{\mu}A_j \tag{8}$$

and is Lorentz and gauge invariant. From (8) the canonical momenta are obtained

$$\Pi_i = P_{ij}\;\partial_o A_j \tag{9}$$

and the Hamiltonian is found to be

$$\mathcal{H} = \frac{1}{2}\,\Pi_i\;P_{ij}\;\Pi_j + \frac{1}{2}\,\partial_k A_i\;P_{ij}\;\partial_k A_j\,. \tag{10}$$

The coordinates A_i and their conjugated momenta Π_j (9) form the phase space of the theory (8). In the usual way a symplectic structure can be introduced in it with a symplectic unit

$$E = \begin{pmatrix} 0 & P \\ -P & 0 \end{pmatrix}$$

and Poisson-bracket relations

$$\begin{aligned} \{\, A_i(x),\ \Pi_j(y)\ \} &= P_{ij}\;\delta(x-y) \\ \{\, A_i(x),\ A_j(y)\ \} &= \{\, \Pi_i(x),\ \Pi_j(y)\ \} = 0\,. \end{aligned} \tag{11}$$

The question one must answer now is whether a representation of the Poincare group can be constructed in this space and if so, which are the transformation properties of the fields A_i with respect to it. The next paragraph is devoted to these problems.

3. *Poincare group representation in the space* $\mathfrak{M}$

The canonical energy-momentum tensor for a theory with Lagrangian $\mathcal{L}$ is defined as

$$T_{\mu\nu} = \frac{\delta\ \mathcal{L}}{\delta\ \partial^{\mu}A^{i}}\ \partial_{\nu}A_{i} - g_{\mu\nu}\ \mathcal{L}\ .$$

For the theory (8) with the metric (6) taken into account we find

$$T_{\mu\nu} = \partial_{\mu}A_{i}\ P_{ij}\ \partial_{\nu}A_{j} - \frac{1}{2}\, g_{\mu\nu}\ \partial^{\sigma}A_{i}\ P_{ij}\ \partial_{\sigma}A_{j}\ . \qquad (12)$$

We would like to emphasize that in such a way a symmetric and gauge invariant canonical energy-momentum tensor is obtained. It differs from the Belinfante one, being its reduction on the equations of motion. Thus, tensor (12) appears to be the minimal symmetric energy-momentum tensor for the theory under consideration.

The generators of the Poincare group are

$$\begin{aligned} H &= P_{o} = \int d^{3}x\ T_{oo}(x) \\ P_{k} &= \int d^{3}x\ T_{ko}(x) \\ M_{ij} &= -\int d^{3}x\ [x_{i}T_{jo}(x) - x_{j}T_{io}(x)] \\ M_{ko} &= x_{o}P_{k} - \int d^{3}x\ x_{k}T_{oo}(x)\ . \end{aligned} \qquad (13)$$

It is not difficult to get convinced in the validity of the Poincare algebra for generators (13)

$$[\ P_{i},\ P_{j}\ ] = [\ P_{i},\ H\ ] = [\ M_{ij},\ H\ ] = 0$$

$$[\ P_{i},\ M_{jk}\ ] = i(\delta_{ik}\ P_{j} - \delta_{ij}\ P_{k})$$

$$[\ H,\ M_{oi}\ ] = \ i\ P_{i}$$

$$[\ P_{i},\ M_{ok}\ ] = -i\ \delta_{ik}\ H$$

$$[\ M_{ij},\ M_{mn}\ ] = -i(\delta_{im}\ M_{nj} + \delta_{jm}\ M_{in} - \delta_{in}\ M_{mj} - \delta_{jn}\ M_{im})$$

$$[\ M_{ok},\ M_{ij}\ ] = i(\delta_{kj}\ M_{oi} - \delta_{ki}\ M_{oj})$$

$$[\ M_{oi},\ M_{oj}\ ] = -i\ M_{ij}\ .$$

Now the transformation properties of the field A_{i} under the action of generators (13) can be obtained

$$i[\, P_\mu ,\ A_i\ ] = \ \partial_\mu P_{ik}\ A_k$$
$$i[\, M_{ij},\ A_k(x)\ ] = -\ (x_i\partial_j - x_j\partial_i)\ P_{kn}\ A_n \qquad (14)$$
$$i\ \varepsilon_i[\ M_{oi},\ A_j(x)\ ] = \delta_L\ P_{jk}\ A_k + \partial_j\Lambda,$$
where

$$\delta_L = -\varepsilon_i[x_o\partial_i - x_i\partial_o], \qquad \Lambda = \varepsilon_i\ \frac{1}{\Delta}\ \Pi_i = \varepsilon_i\ \frac{1}{\Delta}\ P_{ij}\ \partial_o A_j\ .$$

Relations (14) show that translations and space rotations act on the field A_i in the usual way with the metric tensor P_{ij} present in the transformation laws. Contrary to this, Lorentz rotations act in a more specific manner: in addition to the ordinary Lorentz transformation a gauge transformation occurs with a gauge parameter which depends on the metric.

To summarize, in the space of gauge orbits $\mathfrak{M}$ with metric (6) and symplectic structure (Poisson-bracket relations) (11) a representation of the Poincare group can be constructed in which boost generators induce gauge transformations of the field A_i, or equivalently, vertical translations in the spatial-potential configuration space.

However, the fields A_i are still not independent. Transition to independent physical variables can be performed by the introduction of an appropriate basis $e_i^{\ \alpha}$,

$$\sum_{\alpha=1}^{2} e_i^{\ \alpha}\ e_j^{\ \alpha} = P_{ij}, \qquad e_i^{\ \alpha}\ P_{ij}\ e_j^{\ \beta} = \delta^{\ \alpha\beta}\ . \qquad (15)$$

Then, the independent physical variables are defined as

$$\mathcal{A}^\alpha = e_i^{\ \alpha}\ P_{ij}\ A_j, \qquad e_i^{\ \alpha}\ \mathcal{A}^\alpha = P_{ij}\ A_j\ . \qquad (16)$$

In terms of independent variables (16) Lagrangian (8) and thus the initial Lagrangian (4) can be rewritten as

$$\mathcal{L} = \frac{1}{2}\ \partial_\mu\mathcal{A}^\alpha\ \partial^\mu\mathcal{A}^\alpha\ . \qquad (17)$$

Starting from Lagrangian (17), we can repeat the whole procedure described above. Now the canonical momenta and the Hamiltonian read

$$p_\alpha = \partial_o\mathcal{A}_\alpha$$

$$\mathcal{H} = \frac{1}{2}\ p_\alpha\ p_\alpha + \frac{1}{2}\ \partial_i\mathcal{A}_\alpha\ \partial_i\mathcal{A}_\alpha$$

and the energy-momentum tensor is

$$T_{\mu\nu} = \partial_\mu \mathcal{A}_\alpha \, \partial_\nu \mathcal{A}_\alpha - \frac{1}{2} g_{\mu\nu} \, \partial^\sigma \mathcal{A}_\alpha \, \partial_\sigma \mathcal{A}_\alpha .$$

This energy-momentum tensor gives rise to a set of generators

$$H, \; p_i, \; m_{ij}, \; m_{ok},$$

for which the Poincare algebra is closed. So, a representation of the Poincare group is defined which transforms the field $\mathcal{A}^\alpha$ as a usual two-component Lorentz scalar

$$\begin{aligned}
&i[\, H, \mathcal{A}^\alpha \,] = \partial_o \mathcal{A}^\alpha \\
&i[\, p_k, \mathcal{A}^\alpha \,] = \partial_k \mathcal{A}^\alpha \\
&i[\, m_{ok}, \mathcal{A}^\alpha \,] = -(x_o \partial_k - x_k \partial_o)\, \mathcal{A}^\alpha \\
&i[\, m_{ij}, \mathcal{A}^\alpha \,] = -(x_i \partial_j - x_j \partial_i)\, \mathcal{A}^\alpha .
\end{aligned} \tag{18}$$

Relations (18) convince us in the relativistic invariance of Lagrangian (17). Its invariance under gauge transformations of the initial fields A_μ follows from definition (16) of the independent physical variables $\mathcal{A}^\alpha$. It is free of constraints and contains only physical degrees of freedom. Quantization of this Lagrangian is straightforward: classical Poisson-bracket relations

$$\{\, \mathcal{A}_\alpha(x), \, p_\beta(y) \,\} = \delta_{\alpha\beta} \, \delta(x-y)$$

$$\{\, \mathcal{A}_\alpha(x), \, \mathcal{A}_\beta(y) \,\} = \{\, p_\alpha(x), \, p_\beta(y) \,\} = 0$$

become commutation relations in quantum theory. Note, that classical and quantized fields are now transformed in a uniform way under the action of the Poincare group. Thus, Lagrangian (17) provides a consistent description of the electromagnetic field in an explicitly Lorentz covariant and gauge invariant form.

4. *Concluding remarks*

In the previous sections we have considered the free electromagnetic field as an illustration of the main ideas of our approach. The same procedure can be performed for the free Yang - Mills field as well, though with some peculiarities. In this case, the metric has the form

$$P_{ab} = \delta_{ab} - D_{ac} \, D^{-2}_{cd} \, D_{db} \, , \tag{19}$$

where D is the covariant derivative. Contrary to (6), metric (19) depends on the gauge field through the derivative D which means that the orbit space of the free Yang - Mills field is not flat but has an intrinsic curvature. However, the same is true for the abelian gauge field coupled to scalar matter fields /8/. Consideration of the singular points of the orbit space with an infinite value of the curvature is connected with a more detailed analysis of the zero modes of the inverse operator in the metric. In the latter, the effects of the nontrivial topology of the orbit space are manifested. In some simple models this provides an interesting topological confinement mechanism /9/ thus suggesting a possible solution of the open-colour problem in the more realistic theories.

With the help of the genuine symplectic structure of the physical (orbit) space the theory of the free gauge field is formulated in a manifestly relativistic-covariant form providing its straightforward quantization with the same transformation properties of the quantized fields with respect to the Lorentz-group action as in the classical theory. The Lagrangian obtained describes an unconstrained hamiltonian system. Thus, in the path-integral construction one should not encounter difficulties connected with the singular nature of the original gauge-field Lagrangian such as the necessity of additional conditions and, consequently, the problem of the equivalence of different gauges, gauge ambiguities, ghosts and so on. These problems will be considered in detail in a separate paper.

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