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QUANTUM MECHANICS OF TOROIDAL ANYONS

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I. Introduction

Since Wilczek's profound papers /1/ there is an increasing activity in studying the systems containing an electric charge and a magnetic flux. These composites called anyons carry fractional angular momentum and possess unusual statistics. Physically, they are manifested as quasiparticles in the fractional quantum Hall effect /2/ and probably in high-temperature superconductivity /3/. Up to now only two-dimensional anyons were considered (see, e.g., /4/). In the three-dimensional space they can be realized as a cylindrical solenoid with an electric charge attached to it. It is the goal of the present paper to study the system composed of a toroidal solenoid and a charged particle. The plan of our exposition is as follows. In § 2 we write out the Lagrangian and the Schroedinger equation describing an interaction of two anyons without specifying the type of the solenoid used. In § 3 the main facts concerning toroidal solenoid are presented. In § 4 the particular model of a toroidal anyon is proposed. The arguments for existence of multivalued wave functions in the field of impenetrable toroidal solenoid are given in § 5 . Two interacting toroidal anyons are considered in § 6 . It turns out that the statistical properties of their wave function depend on both anyons mutual separation and orientation. The influence of the particular gauge choice on the exchange properties of wave functions is studied in § 7 . In the next section we explain why the arguments denying the existence of fractional statistics in the three-dimensional space do not work in the treated case.

2. Basic equations

A composite consisting of a particle with a charge e and a solenoid with a magnetic flux Φ is called an anyon /1/. The Lagrangian of this system is /5,6/

$$L = \frac{1}{2}m\vec{v}^2 + \frac{1}{2}M\vec{V}^2 + \frac{e}{c}\vec{A}(\vec{r}-\vec{R})\cdot(\vec{v}-\vec{V}). \tag{2.1}$$

The evident notation $m, \vec{v}$ and $M, \vec{V}$ refers to the charged particle and the solenoid; $\vec{A}$ is the vector potential (vp) produced by the solenoid situated at $\vec{R}$ at the position $\vec{r}$ of the charged particle. The Lagrangian (2.1) describes both the Aharonov – Bohm and Aharonov – Casher /5/ effects. The latter was recently /7/ confirmed experimentally. Consider two anyons $(e_1\, \Phi_1)$ and $(e_2\, \Phi_2)$ with masses m_1 and m_2. They are described by the following Lagrangian /6/

$$L = \frac{1}{2}m_1\vec{v}_1^2 + \frac{1}{2}m_2\vec{v}_2^2 + \frac{e}{c}\vec{A}\cdot\vec{v} - V. \tag{2.2}$$

Here we put $e\vec{A} = e_1\vec{A}_{12}(\vec{r}_1-\vec{r}_2) - e_2\vec{A}_{21}(\vec{r}_2-\vec{r}_1)$; $\vec{A}_{12}(\vec{A}_{21})$ is the vp generated by the anyon 2(1) at the position of anyon 1(2); $\vec{v} = \vec{v}_1 - \vec{v}_2$ and V is the electromagnetic interaction between charged particles. It is suggested here that the particular anyon feels only the electromagnetic field of the other. The self-interaction between the magnetic flux and the charge of the same anyon is disregarded. It is the routine operation in the anyon theory. By quantizing the Lagrangian we obtain the following Schroedinger Equation (SE):

$$-\frac{\hbar^2}{2M}\vec{\nabla}_R^2\Psi - \frac{\hbar^2}{2\mu}\left(\vec{\nabla}_r - \frac{ie}{\hbar c}\vec{A}\right)^2\Psi + V\Psi = E\cdot\Psi. \tag{2.3}$$

Here $M = m_1 + m_2$, $\mu = \frac{m_1 m_2}{m_1+m_2}$, $\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2}$, $\vec{r} = \vec{r}_1 - \vec{r}_2$.

If in addition $e_1 = e_2 = e$ and $m_1 = m_2 = m$, then this Eq. reduces to

$$-\frac{\hbar^2}{4m}\vec{\nabla}_R^2\Psi - \frac{\hbar^2}{m}\left(\vec{\nabla}_r - \frac{ie}{\hbar c}\vec{A}\right)^2\Psi + V\Psi = E\cdot\Psi,$$
$$\vec{A} = \vec{A}_{12}(\vec{r}) - \vec{A}_{21}(-\vec{r}) \tag{2.4}$$

Separating the center-of-mass coordinates $(\Psi = \exp(i\vec{K}\vec{R})\cdot\psi)$ we get

$$\left(\vec{\nabla}_r - \frac{ie}{\hbar c}\vec{A}\right)^2\psi + (\epsilon - v)\psi = 0,$$
$$\epsilon = \frac{mE}{\hbar^2} - \frac{K^2}{4}, \quad v = mV/\hbar^2. \tag{2.5}$$

Let the anyon's solenoid be toroidal. Similarly to the term "cyon" used for the cylindrical anyon /8/, the term "toron" will be used for the toroidal anyon. Some facts concerning the toroidal solenoid which are needed for the subsequent discussion will be presented in the next section.

3. The electromagnetic field of the toroidal solenoid

The magnetic field of the toroidal solenoid $(\rho-d)^2 + z^2 = R^2$ equals $\vec{H} = \vec{e}_\varphi \cdot g/\rho$ inside the solenoid and zero outside it. The constant g is expressed through the magnetic flux Φ: $g = \Phi\cdot\left[2\pi\left(d - \sqrt{d^2-R^2}\right)\right]^{-1}$. In the Coulomb gauge $(\operatorname{div}\vec{A} = 0)$ the vp of the toroidal solenoid was obtained in ref. /9/. Later it was used for the description of the electron scattering on the toroidal solenoid /10–13/. Here, we present its components only for the infinitely thin ($R \ll d$) solenoid:

$$A_z = \frac{\Phi d}{2\pi}\frac{1}{(d\rho)^{3/2}}\frac{1}{\operatorname{sh}\nu}\cdot\left[\rho\cdot Q^1_{\frac{1}{2}}(\operatorname{ch}\nu) - d\cdot Q^1_{-\frac{1}{2}}(\operatorname{ch}\nu)\right],$$

$$A_x = \frac{x}{\rho} A_\rho, \quad A_y = \frac{y}{\rho} \cdot A_\rho, \tag{3.1}$$
$$A_\rho = -\frac{\Phi d}{2\pi} \frac{1}{(d\rho)^{3/2}} \frac{1}{\operatorname{sh}\nu} \cdot z \cdot Q^1_{\frac{1}{2}}(\operatorname{ch}\nu).$$

Here $\operatorname{ch}\nu = \frac{r^2+d^2}{2d\rho}$ and Q^1_σ are the Legendre functions of the 2-nd kind. At large distances $\vec{A}$ falls as r^{-3}

$$A_x \approx \frac{3\pi g R^2 d}{4} \frac{xz}{r^5}, \quad A_y \approx \frac{3\pi g R^2 d}{4} \frac{yz}{r^5},$$

$$A_z \approx -\frac{\pi g R^2 d}{4} \frac{r^2 - 3z^2}{r^5}.$$

As outside the solenoid $\vec{H} = \operatorname{rot}\vec{A} = 0$, the v.p. may be presented as a gradient of some function χ : $\vec{A} = \operatorname{grad}\chi$. This function turns out to be multivalued (more accurately: discontinuous) as $\oint \vec{A}\, d\vec{x} = \Phi$ for any contour passing through the solenoid 's hole. To write out this function explicitly we introduce the toroidal coordinates

$$x = a\frac{\operatorname{sh}\mu\cos\varphi}{\operatorname{ch}\mu - \cos\theta}, \quad y = a\frac{\operatorname{sh}\mu\cdot\sin\varphi}{\operatorname{ch}\mu-\cos\theta}, \quad z = a\frac{\sin\theta}{\operatorname{ch}\mu-\cos\theta} \tag{3.2}$$

$$(0 < \mu < \infty, \; -\pi < \theta < \pi, \; 0 < \varphi < 2\pi).$$

Let $\mu = \mu_0$ correspond to the toroidal solenoid S. Then for $\mu > \mu_0$ $(< \mu_0)$ the point $P(x,y,z)$ (where x,y,z are given by (3.2)) lies inside (outside) S. For μ fixed (say, $\mu = \mu_0$) the points $P(x,y,z)$ fill the surface of the torus $(\rho-d)^2 + z^2 = R^2$ with the parameters $d = a\cdot\operatorname{cth}\mu_0$, $R = a/\operatorname{sh}\mu_0$. The value of the angle θ jumps from $-\pi$ to π when one

intersects the circle of the radius d-R lying in the $z=0$ plane. Now we are able to write out the χ function explicitly /9,10/:

$$\chi = \chi_0(\theta) + \frac{4\sqrt{2}\, ga}{\pi}\cdot(\operatorname{ch}\mu - \cos\theta)^{1/2} \cdot \sum_{n=1}^{\infty} \beta_n \cdot P_{n-\frac{1}{2}} \cdot \sin n\theta. \tag{3.3}$$

Here $P_{n-\frac{1}{2}}$ is the Legendre function of the 1-st kind;

$\beta_n = -\sum_{k=n}^{\infty} Q_{k-\frac{1}{2}}(0)\cdot Q_{k+\frac{1}{2}}(0)$; χ_0 is given by

$$\chi_0 = \frac{1}{4\pi}\cdot\Phi\cdot\left\{-2\theta + \sum_{n=1}^{\infty}\frac{1}{n}\sin n\theta\cdot\left[P_{-\frac{1}{2}}\left(Q_{n+\frac{1}{2}} + Q_{n-\frac{3}{2}}\right) - 2P_{\frac{1}{2}}\cdot Q_{n-\frac{1}{2}}\right]\right\}.$$

From now we do not indicate the argument of the Legendre function if it equals $\operatorname{ch}\mu$; Further, $Q_\nu(0) \equiv Q_\nu(\operatorname{ch}\mu_0)$, $P_\nu(0) \equiv P_\nu(\operatorname{ch}\mu_0)$ Clearly, χ transforms into χ_0 for the infinitely thin ($R \ll d$ or $\mu_0 \gg 1$) solenoid. We see that χ suffers a jump from $-\frac{1}{2}\Phi$ to $\frac{1}{2}\Phi$ when one intersects the circle of the radius d-R lying in the $z=0$ plane. At large distances χ falls as r^{-2} (this follows from (3.3))

$$\chi \approx -\frac{\Phi d R^2}{8(d - \sqrt{d^2 - R^2})} \frac{\cos\theta_s}{r^2}.$$

(r and θ_s are the usual spherical coordinates). The unitary transformation $\psi = \psi' \cdot \exp(ie\chi/\hbar c)$ may be used to eliminate v.p. outside the solenoid. The transformed wave function (w.f.) is a multivalued (mv) one if the initial w.f. ψ is a single-valued (sv) one:

$$\psi'(\rho \le d-R, z=0-) = \psi'(\rho \le d-R, z=0+)\cdot\exp(-i\gamma) \tag{3.4}$$

Here $\gamma = e\Phi/\hbar c$. The reverse is also true.

For the arbitrary orientation of the solenoid the v p at the point $\vec{r}$ is given by

$$\tilde{A}_i(\vec{r}) = \sum R_{ik}(\varphi, \theta, \psi) A_k(\vec{r}).$$

Here A_K are given by Eqs. (3.1), R is the usual rotation matrix and φ, θ, ψ are the angles defining the orientation of the solenoid fixed frame wrt the laboratory one.

4. The particular realization of a toron

Usually, treating the anyon problems one does not specify the way in which the charge is attached to the solenoid. One of the possible ways to do this is to charge the surface ($\mu = \mu_0$) of the toroidal solenoid. To exclude the appearance of the currents at the solenoid surface the latter should be at the constant electrostatic potential Φ_0. The elementary calculations show /14/ that the electric charge should be distributed over the solenoid surface with the density

$$\rho(\theta) = \delta(\mu - \mu_0) \frac{(ch\mu_0 - \cos\theta)^{5/2}}{\sqrt{2}\,\pi^2 a^2} \frac{\Phi_0}{sh\mu_0} \sum_{n=0}^{\infty} \frac{\cos n\theta}{1 + \delta_{n0}} \cdot \left[P_{n-\frac{1}{2}}(0)\right]^{-1}.$$

From now we suggest that the summation, if it is not specified, extends from $n = 0$ to $n = \infty$. The electrostatic potential generated by this density equals Φ_0

inside the toroidal solenoid ($\mu > \mu_0$) and

$$\Phi = \frac{2\sqrt{2}}{\pi} \Phi_0 \cdot (ch\mu - \cos\theta)^{1/2} \cdot \sum P_{n-\frac{1}{2}} \frac{Q_{n-\frac{1}{2}}(0)}{P_{n-\frac{1}{2}}(0)} \frac{\cos n\theta}{1 + \delta_{n0}}$$

outside it. The total surface charge e may be expressed through Φ_0:

$$e = \frac{4a\Phi_0}{\pi} \sum \frac{1}{1 + \delta_{n0}} \frac{Q_{n-\frac{1}{2}}(0)}{P_{n-1/2}(0)}.$$

The subsequent consideration does not depend on this particular realization.

5. The possibility of the multivalued wave functions in the field of toroidal solenoid

We present here the arguments for the mv wf existence in the magnetic field of an impenetrable toroidal solenoid. But at first, we repeat the similar arguments /15-17/ for the well-known case of a cylindrical solenoid. Consider two identical charged particles 1 and 2 in the field of an infinite cylindrical solenoid (fig.1). Now we exchange particles 1 and 2. This procedure is path dependent if the mv wf are used. The wf remains the same if there is no net magnetic flux inside the closed contour composed of exchange paths 1 and 2: $\Psi(2,1) = \Psi(1,2)$. On the other hand, the wf changes when the finite magnetic flux Φ is presented inside the above closed contour $\Psi(2,1) = \Psi(1,2)\cdot \exp(i\gamma)$, $\gamma = e\Phi/\hbar c$. If $\gamma = 2\pi n$ then $\Psi(2,1) = \Psi(1,2)$ i.e. the presence of the magnetic flux does not affect the exchange properties of wf. When $\gamma = \pi\cdot(2n+1)$ one has $\Psi(2,1) = -\Psi(1,2)$, that is the particles behave as fermions. For the arbitrary γ one has the intermediate (between bosons and fermions) statistics. The impenetrability of the solenoid

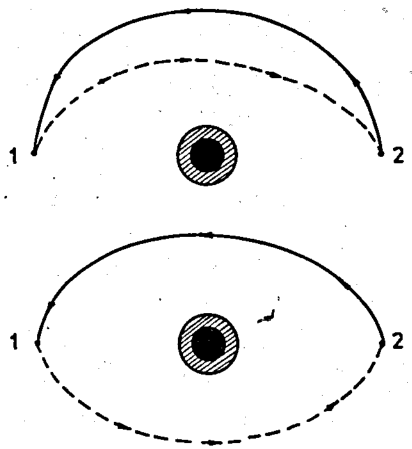

Fig.1. The trivial (upper part) and nontrivial (lower part) exchange paths in the field of cylindrical solenoid. The space region where $H \neq 0$ is darkened. The inaccessible region is hatched.

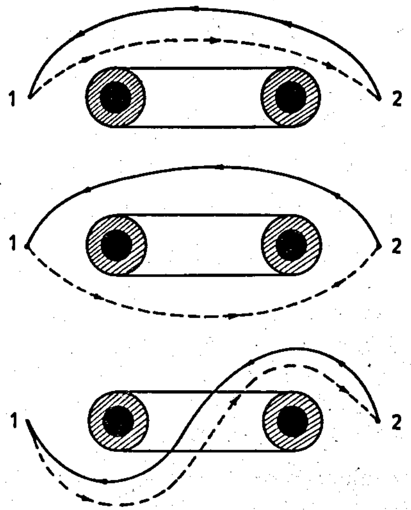

Fig.2. The trivial exchange paths in the field of toroidal solenoid.

guarantees that exchange paths shown at the lower part of fig.1 cannot be continuously deformed (or shrinked to a point) into that presented at the upper part of the same figure.

Now we turn to the behaviour of wf in the magnetic field of impenetrable toroidal solenoid. In fig.2 there are shown exchange paths which do not embrace the magnetic flux Φ. Each of them can be contracted to a point without intersecting the impenetrable torus. Thus, $\Psi(2,1)=\Psi(1,2)$ for them. Some of the topologically nontrivial exchange paths are shown in fig.3. All of them cannot be either shrinked to a point or deformed into each other without intersecting the impenetrable torus (and the flux Φ). If the mv wf are used one has $\Psi(2,1) = \Psi(1,2)\cdot\exp(\mp i\gamma)$ for the upper and middle parts on fig.3, resp., while $\Psi(2,1) = \Psi(1,2)\cdot\exp(-2i\gamma)$ for its lower part (as before $\gamma = e\Phi/\hbar c$ and Φ is the magnetic flux inside the toroidal solenoid). This situation strongly resembles that of the cylindrical solenoid, but there are more topologically nonequivalent possibilities. As for the cylindrical solenoid the double exchange of particles does not in general lead to the initial wf. For example, under the double exchange composed of the single exchanges (each in the clock-wise direction), presented in the upper part of fig.3, the wf acquires the factor $\exp(-2i\gamma)$.

6. The interacting torons

Now we return to the interacting torons. Consider first the simplified case when the symmetry axes of toroidal solenoids 1 and 2 are parallel to the z axis. In addition, the solenoids are assumed to be thin ($R_1 \ll d_1$, $R_2 \ll d_2$). Then, it follows from (3.1) that outside the solenoids one has

$$A_{21z} = \frac{1}{2\pi}\frac{\varphi_1 d_1}{(d_1\rho)^{3/2}}\frac{1}{sh\nu_1}\cdot\left[\rho\cdot Q^{1}_{\frac12}(1) - d_1\cdot Q^{1}_{-\frac12}(1)\right],$$

$$A_{21x} = \frac{x}{\rho}A_{21\rho}, \quad A_{21y} = \frac{y}{\rho}A_{21\rho}, \qquad (6.1)$$

$$A_{21\rho} = -\frac{1}{2\pi}\varphi_1 d_1 \frac{z}{(d_1\rho)^{3/2}}\frac{1}{sh\nu_1}\cdot Q^{1}_{\frac12}(1).$$

Here $Q^{\sigma}_{\lambda}(1) \equiv Q^{\sigma}_{\lambda}(ch\nu_1)$, $ch\nu_1 = \frac{r^2+d_1^2}{2d_1\rho}$, $x = x_1 - x_2$ etc. $\rho = \sqrt{x^2+y^2}$, $r = (x^2+y^2+z^2)^{1/2}$. The vp $\vec{A}_{12}$ is obtained from $\vec{A}_{21}$ by the interchanging of particle indices 1 and 2. The vp thus obtained are symmetrical wrt interchange of particle coordinates:

$$\vec{A}_{12}(\vec{r}_1,\vec{r}_2) = \vec{A}_{12}(\vec{r}_2,\vec{r}_1), \quad A_{21}(\vec{r}_1,\vec{r}_2) = A_{21}(\vec{r}_2,\vec{r}_1). \qquad (6.2)$$

The vp $\vec{A}_{12}$ and $\vec{A}_{21}$ may be expressed as gradients of mv functions $\chi_{12}(\vec{r})$ and $\chi_{21}(\vec{r})$:

$$\vec{A}_{21} = \mathrm{grad}_{\vec{r}}\,\chi_{21}, \quad A_{12} = \mathrm{grad}_{\vec{r}}\,\chi_{12}.$$

The functions χ_{21} and χ_{12} are obtained from Eq. (3.3) by making the substitution: $\varphi \to \Phi_1$, $\mu_0 \to \mu_1$, $ga \to g_1 a_1 = \frac{\Phi_1}{2\pi}(cth\mu_1 - 1)^{-1}$ for χ_{21} and $\Phi \to \Phi_2$, $\mu_0 \to \mu_2$, $ga \to g_2 a_2 = \frac{\Phi_2}{2\pi}(cth\mu_2 - 1)^{-1}$ for χ_{12}. The linear combination $e\vec{A} \equiv e_1\vec{A}_{12}(\vec{r}) - e_2\vec{A}_{21}(\vec{r})$ entering into (2.3) may be presented in the form

$$e\vec{A} = \mathrm{grad}_{\vec{r}}\, e\chi, \quad e\chi \equiv e_1\chi_{12}(\vec{r}) - e_2\cdot\chi_{21}(\vec{r}).$$

Now we can write the classical Lagrangian (2.2) in the gauge invariant form

$$L = \frac{1}{2}m_1\vec{v}_1^{\,2} + \frac{1}{2}m_2\vec{v}_2^{\,2} + \frac{1}{c}\frac{d}{dt}\left[e_1\chi_{12}(\vec{r}) - e_2\chi_{21}(\vec{r})\right].$$

The unitary transformation

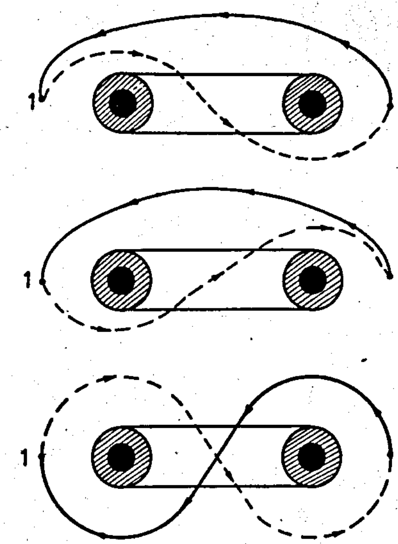

Fig.3. The nontrivial exchange paths in the field of toroidal solenoid.

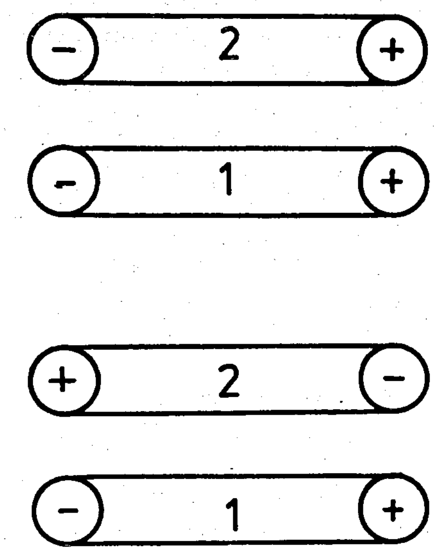

Fig.4. Two toroidal solenoids with equal (upper part) and opposite (lower part) magnetic fluxes.

$$\Psi = \Psi' \cdot \exp\left(\frac{ie\chi}{\hbar c}\right) \tag{6.2}$$

eliminates vp from Eq . (2.3). Consider the particular cases. Let the toron parameters be the same ($m_1=m_2=m, \ell_1=\ell_2=\ell, d_1=d_2=d, R_1=R_2=R$) except for the magnetic fluxes Φ_1 and Φ_2. Then,

$$\chi_{12} = \Phi_2 \cdot \chi(\vec{r}) \quad , \quad \chi_{21} = \Phi_1 \cdot \chi(\vec{r}) ,$$

where χ is obtained from Eq. (3.3) by dropping the overall factor Φ . Thus

$$e_1 \chi_{12} - e_2 \cdot \chi_{21} = e(\Phi_2 - \Phi_1)\chi$$

and

$$\Psi = \Psi' \exp\left[\frac{ie(\Phi_2-\Phi_1)}{\hbar c}\chi\right] \tag{6.3}$$

Let in addition $\Phi_1 = \Phi_2$. Then $\vec{A}_{21} = \vec{A}_{12}$ and $\Psi = \Psi'$. From Eqs. (2.4) or (2.5) it follows that vp drops out from the SE. This means that the presence of vp outside the torons with $\Phi_1 = \Phi_2$ changes neither the dynamics of torons nor their exchange properties. If $\Phi_1 = -\Phi_2 = \Phi$, then Eq. (6.3) gives

$$\Psi = \Psi' \exp(-2i\gamma\chi) \quad , \quad \gamma = e\Phi/\hbar c . \tag{6.4}$$

If Ψ is chosen to be sv, then Ψ' suffers the discontinuity,

$$\Psi'(\rho \le d-R, z=0-) = \exp(2i\gamma)\cdot\Psi'(\rho \le d-R, z=0+) .$$

It should be noted that the equality $\Phi_1 = -\Phi_2$ does not mean that anyons are different. To see this we turn to fig.4. At the upper part of it we see two identical toroidal solenoids with $\Phi_1 = \Phi_2$. The signs + and − mean that the magnetic field ($\vec{H} = \vec{e}_\varphi \cdot g/\rho$, $g = \Phi[2\pi(d-\sqrt{d^2-R^2})]^{-1}$) is directed from and towards the observer, resp. It has the same direction in both the solenoids.

Now we begin to rotate the second solenoid around the axis normal to the plane of figure. After the rotation at the angle π it is performed, we obtain the situation shown at the lower part of the same figure. We observe that the direction of magnetic field in the second solenoid has been changed to the opposite. This means that for an external observer the magnetic flux of the second toron has changed its sign and that torons with $\Phi_1 = -\Phi_2$ are indeed the same. Now we try to exchange torons with $\Phi_1 = -\Phi_2$. But at first we must know how the relative toroidal coordinates μ , θ and φ entering into χ_{12} and χ_{21} behave under the particle exchange. It follows from (3.2) that to $\vec{r}_1 \leftrightarrow \vec{r}_2$ there corresponds: $\mu \to \mu$, $\theta \to -\theta + 2\pi n$, $\varphi \to \varphi + (2m+1)\pi$. This leads to the following changing of χ :

$$\chi(-\vec{r}) = -\chi(\vec{r}) - n .$$

If the wf Ψ is chosen to be symmetrical, then this results in the following behaviour of Ψ' under the particle exchange

$$\Psi'(2,1) = \Psi'(1,2)\cdot\exp[-4i\gamma\chi(\vec{r}) - 2i\gamma n] . \tag{6.5}$$

We see that exchange properties of torons depend essentially on their relative positions and orientations. In particular cases we obtain the situation similar to that of cyons. Let torons 1 and 2 be in such a relative position that $\chi(1,2) = 0$. From the explicit expression for χ (see (3.3)) it follows that this occurs, e.g., for $\theta = 0$. This in turn happens either when the equatorial planes of the solenoids lie in the same plane (this corresponds to $z = z_1 - z_2 = 0$ in (3.2)), or when the torons are enough separated. The latter is due to the fact that toroidal angle θ decreases at large distances ($\theta \approx \frac{2\sqrt{d^2-R^2}}{r}\cos\theta_s$ for $r \to \infty$). In this case Eq. (6.5) reduces to

$$\psi'(2,1)=\psi'(1,2)\cdot\exp(-2i\gamma h).$$

From this we obtain Bose, Fermi or intermediate statistics depending on the value of $\gamma\ (\equiv e\Phi/\hbar c)$

The fact that statistical properties of anyons can depend on their mutual separation is not new. For two-dimensional anyons this has recently been proved in a very interesting ref, /18/ (in the framework of the Chern-Simons gauge theory). When the torons dimensions tend to zero we obtain the magnetic toroidal moment /19/ with electric charge attached to it. The unusual statistics is obtained for anyons with the opposite toroidal moments. There is an intuitive explanation of such a different wf behaviour for $\Phi_1=\Phi_2$ and $\Phi_1=-\Phi_2$. When torons with $\Phi_1=\Phi_2$ pass through each other (the upper part of fig.4) the net change of the wf phase equals zero. In fact, the charged particle of toron 2 passing through the hole of toron 1 contributes the value γ to the phase while particle of 1 passing through the hole of 2 contributes $-\gamma$. When $\Phi_1=-\Phi_2$ (the lower part of the same figure) both particles contribute the same phase γ.

When the symmetry axes of the torons have arbitrary orientation (i.e. they are neither parallel nor antiparallel) one should use in Eqs. (2.3) the vp defined as

$$\tilde{A}_{21i}=\sum R_{ik}(\varphi_1,\theta_1,\psi_1)\,A_{21k},\qquad (6.6)$$
$$\tilde{A}_{12i}=\sum R_{ik}(\varphi_2,\theta_2,\psi_2)\,A_{12k}.$$

The angles φ,θ,ψ define the orientation of the particular toron wrt fixed laboratory frame. The v.p. $\vec{A}_{21}$ and $\vec{A}_{12}$ in the r.h.s. of (6.6) are defined by Eqs. (6.1)

7. Ambiguities arising from various choices of gauge

At first we demonstrate arising uncertainties using two interacting cyons as an example. The v.p. of the cylindrical solenoid in a Coulomb gauge is equal to: $\vec{A}=\vec{e}_\varphi\Phi\rho/2\pi R^2$ inside the solenoid $(\rho<R)$ and $\vec{A}=\vec{e}_\varphi\Phi/2\pi\rho$ outside it $(\rho>R)$. It falls as ρ^{-1} at large distances from the axis. The magnetic field equals $\vec{H}=\vec{e}_z\Phi/\pi R^2$ inside the solenoid and zero outside it. On the other hand, one may equally use the following VP /10,11,20/ : $A'_x=0$ everywhere, $A'_y=\Phi\cdot(x+\sqrt{R^2-y^2})/\pi R^2$ inside the solenoid. Outside it A'_y differs from zero inside the hatched strip $(-R<y<R,\ x>\sqrt{R^2-y^2})$ shown in fig.5. It equals there $2\Phi\cdot\sqrt{R^2-y^2}/\pi R^2$. This v.p. generates the same magnetic field as $\vec{A}$. For the infinitely thin solenoid it reduces to: $A'_x=0$, $A'_y=\Phi\cdot\theta(x)\cdot\delta(y)$. Now we return to interacting cyons. Inserting $\vec{A}'$ into Eq.(2.3) we obtain the following net cyon vp entering there: $A_x=0$, $A_y=2\Phi x/\pi R^2$ for $\rho<R$ and

$$A_y=\frac{2\Phi}{\pi R^2}\sqrt{R^2-y^2}\cdot\left[\theta(x-\sqrt{R^2-y^2})-\theta(-x-\sqrt{R^2-y^2})\right]\quad \text{for } \rho>R$$

(Here x,y and ρ are relative coordinates). This means that inside the total strip composed of that shown in fig.5 and that symmetrical to it: $A_y=\pm 2\Phi\cdot\sqrt{R^2-y^2}/\pi R^2$ for right and left halfstrips, resp. For the infinitely thin solenoid $A_y=\Phi\cdot\delta(y)\cdot[\theta(x)-\theta(-x)]$. From this it follows that anyons interact only if their relative coordinates lie in the hatched strip. Such a distinct behaviour of $\vec{A}$ and $\vec{A}'$ means that one should not pay too much attention to the particular realization of vp. According Wu and Yang /21/ only the phase factor $\exp\left(\frac{ie}{\hbar c}\oint A_\mu dx_\mu\right)$ is physically meaningful and measurable. In fact, it is the same fo $\vec{A}$ and $\vec{A}'$. The vp $\vec{A}$ and $\vec{A}'$ are connected by the

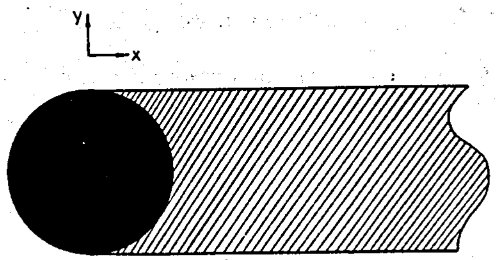

Fig.5. The vector potential of the cylindrical solenoids in a non-standard gauge. Outside the solenoid it differs from zero in the hatched region only.

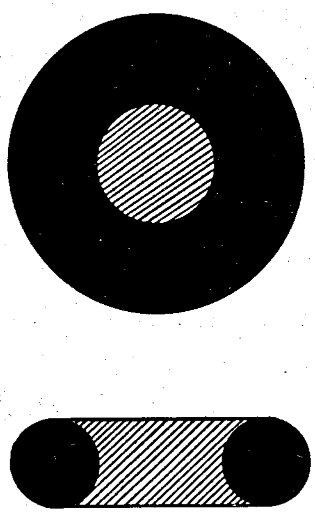

Fig.6. The same as in the previous figure but for the toroidal solenoid.

gauge transformation. The corresponding w.f. Ψ and Ψ' are connected by the unitary transformation. This means that all observables are the same for Ψ and Ψ'.

Going over to interacting torons we observe that in addition to the vp $\vec{A}$ of the toroidal solenoid discussed in §3 there exists $\vec{A}'$ /10,11/ the single nonvanishing component of which (A'_z) differs from zero in the nearest vicinity of the toroidal solenoid. It equals $g\cdot\ln(d+\sqrt{R^2-z^2})/\rho$ inside the solenoid and $g\cdot\ln(d+\sqrt{R^2-z^2})/(d-\sqrt{R^2-z^2})$ outside it, in the hatched region (see fig.6). It is zero in other space regions. For the infinitely thin solenoid ($R \ll d$) it reduces to $A'_z=\Phi\cdot\delta(z)\cdot\theta(d-\rho)$ /22/. The total net vp for two identical interacting torons $(\vec{A}'=\vec{A}'_{12}(\vec{r})-\vec{A}'_{21}(-\vec{r}),\ \vec{r}=\vec{r}_1-\vec{r}_2)$ equals zero if $\Phi_1=\Phi_2$ and twice the value of A'_z if $\Phi_1=-\Phi_2$. This means that in such a gauge the torons interact only if their relative coordinates lie in the hatched region. It is easy to check that both $\vec{A}$ and $\vec{A}'$ satisfy the condition that $\oint A_\ell dx_\ell=\Phi$ for closed contours passing through the solenoid's hole and zero otherwise. The vp $\vec{A}$ and $\vec{A}'$ are connected by the gauge transformation $\vec{A}=\vec{A}'+\mathrm{grad}\,\frac{\partial\alpha}{\partial z}$. For the infinitely thin ($R \ll d$) torons the function α equals /11,23/ $\alpha=\frac{1}{2\pi}\cdot\Phi\iint\frac{dx_1\,dy_1}{|\vec{r}-\vec{r}_1|}$. Here, integration is performed over the circle of the radius d lying in the $z=0$ plane. The double integral may be expressed through the linear one /23/: $\iint|\vec{r}-\vec{r}_1|^{-1}dx_1\,dy_1=2\pi(\sqrt{z^2+d^2}-|z|)-2\sqrt{d}\int_0^\rho dx\cdot x^{-1/2}\cdot Q_{1/2}\left(\frac{r^2+x^2+d^2}{2dx}\right)$. The corresponding w.f. are connected by the singular (for the infinitely thin torons) although unitary transformation: $\Psi=\Psi'\cdot\exp\left(\frac{ie}{\hbar c}\,\frac{\partial\alpha}{\partial z}\right)$

Hence it follows that ψ and ψ' behave differently under the particle exchange (in spite of the fact that they correspond to VP with the same circulation). The main conclusion of this section is that exchange properties of the w.f. depend on the particular gauge choice. Thus, some caution is needed in their interpretation.

8. Discussion

Here we analyse the frequently used assertion (/15-17/) that there are no nontrivial exchange paths in the three-dimensional space. Usually, one starts with the consideration of a plane with a singular isolated point P in it. It turns out that for the charged particle the mv wf are allowable if this point carries magnetic flux Φ. In fact, closed contour embracing P cannot be shrinked into a point without intersecting P. Going over to the three-dimensional space one encounters the following alternative. First, one may continue to treat P as an isolated singular point. In this case the above contour may be shrinked into a point without intersecting P (for this one at first rotates the half of the contour around the axis lying in the initial plane and passing through P). Therefore, the mv wf are not allowable. On the other hand, one may treat P as a trace of an infinite singular line $\mathcal{L}$ piercing the plane at P. The contour encircling $\mathcal{L}$ cannot be contracted without intersecting it. The mv wf are allowable if $\mathcal{L}$ carries the magnetic flux Φ, thus coinciding with an infinitely thin cylindrical solenoid. Let us have on the plane two singular points with $\Phi_1 = -\Phi_2$. They can be viewed as traces of two parallel singular lines which pierce the plane at those points. For the charged particle one easily recovers

topologically trivial (which embrace either both the solenoids or none of them) and nontrivial (which embrace one of the solenoids) exchange paths /24/. Physically these singular lines can be realized as two cylindrical solenoids with $\Phi_1 = -\Phi_2$. The charge particle scattering on them was studied in refs. /11,13,25/. The singular line may also have a form of the circular filament which carries the magnetic flux Φ and which may be viewed as an infinitely thin toroidal solenoid. For the charged particle mv wf are allowable as the closed contours (passing the solenoid's hole) exist which cannot be shrinked into a point. The above singular lines may be considered as the limiting cases of the finite impenetrable cylindrical and toroidal solenoids shown in figs. 1-3. So far we have considered the behaviour of charged particles in the field of cylindrical and toroidal solenoids. Now we turn again to the toroidal anyons. We have seen in § 6 that they exhibit fractional statistics w.r.t. their exchange. This contradicts the frequently occurring assertion (see, e.g., review /26/ and refs. therein) that exotic statistics does not exist if the number of spatial dimensions is greater than 2. The proof grounds essentially on the fact that after the removal of the points corresponding to the coinciding particle coordinates the remaining portion of space is multiconnected for d=2 and simply-connected for $d \geq 3$ /27,28/. In the treated case the part of space occupied by the coinciding identical torons is isomorphic to the torus. The remaining portion of space (lying outside that torus) is the multiconnected one. The dimensions of the toron may be arbitrary small, yet there remains a finite possibility for one particular toron to penetrate through the hole of the other. This is just the reason for the appearance of nonstandard statistics in the three-dimensional case. The reasoning of the cited referenoes fails as the particles there were minded as point-like structurless objects. The nonstandard statistics disappears when the hole of a toron is closed. The smallness of the toron is not essential as the mutual penetration of torons does not depend on their dimensions. Only their nontrivial topology is important. The question arises how to choose the mutual orientation of the interacting torons? The angles describing this orientation enter into the Hamiltonian as parameters. The reason for this is that kinetic energy of particles is taken to be coinciding with that for the point partioles. For the toron (whatever small its dimensions are) the kinetic energy should depend on the orientation angles and corresponding momenta (like for the quantum symmetrical top). At the present stage of investigation the mutual orientation angles may be chosen from the minimum energy considerations. We do not intend to elaborate further these questions here.

It would be appropriate to mention that there are three-dimensionsl objects exhibiting fractional statistics. We mean the so-called dyons which are composites of monopole and charged particle /6,8/ . Finally, there exists an excellent three-dimensional description /29/ of the quantum Hall effect which does not appeal to its two-dimensional nature.

9. Conclusion

The author being nonspecialist in the solid state physics cannot appreciate the practical meaning of the results obtained. Probably, they have some relation to the recently observed fractional quantum Hall effect in the three-dimensional structures /30/ .

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