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ON THE FIRST ORDER DEVIATION EQUATIONS AND SOME QUANTITIES CONNECTED WITH THEM

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INTRODUCTION

In many applications different kinds of approximate deviation equations, viz. the so-called first and high (second, third and so on) order deviation equations[1,6-9] found usage. Most frequently one comes across the first order deviation equations and especially the first order equation of geodesic deviation which with necessary precision are used in the investigation of a number of physical phenomena[1,7-9].

First order deviation equations, or local deviation equations, are called the ordinary differential equations of second order satisfied by the first order deviation vector which practically everywhere is supposed to be an infinitesimal vector. Usually, these equations are derived from the conditions defining the concrete problem under consideration and on their basis the corresponding high order deviation equations[1,2,6-9] by one or another method are derived.

In this work, on the bedrock of the generalized (nonlocal) deviation equation in an arbitrary space (manifold) with affine connection and general linear transport along curves[2-5], we obtain the most general concrete form of the first order deviation equation for the first order deviation vector as well as for some other important physical quantities connected with it.

The paper is organized as follows: In Sec.2 we deal with some approximations to the general linear transports along curves by means of which some approximate vectors are obtained and in particular, the first order deviation vector. Section 3 is devoted to the first order deviation equations satisfied by the first order deviation vectors. In Sec.4 we find the first order relationship between the deviation velocity and the relative velocity of two particles and an ordinary differential equation of the first order satisfied by the relative velocity (the first order deviation equations for the relative velocity). In Sec.5 we derive up to second order terms an ordinary differential equation of first order satisfied by the relative momentum of two point particles (the first order deviation equation for the relative momentum) which has the meaning of the first order equation of

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motion for these particles. In Sec.6, we consider similar to Sec.4 problems but concerning the deviation acceleration and the relative acceleration between two particles. Section 7 contains some concluding remarks.

2. ON THE APPROXIMATE TRANSPORTS ALONG CURVES AND APPROXIMATE DEVIATION VECTORS

In [2,3] we proved that in any local basis $\{\partial_a\}$ (for simplicity we shall use only coordinate bases) every general linear transport (I-transport) of the vectors $I^{\gamma}_{x\to y}: T_x(M) \to T_y(M)$ along the curve $\gamma:[r', r''] \to M$, $x, y \in \gamma([r', r''])$ (M is a differentiable manifold and $T_x(M)$ is the tangent to it space at x) is uniquely defined by the matrix $\|H^{\alpha}_{.\beta}(y, x; \gamma)\|$ (all indices run from 1 to $n = \dim M$ and the usual summation rule is understood), so that for any $A = A^{\alpha}\partial/\partial x^{\alpha} \in T_x(M)$

$$I^{\gamma}_{x\to y} A = (H^{\alpha}_{.\beta}(y, x; \gamma) A^{\beta})\partial/\partial y^{\alpha}, \text{ i.e. } (I^{\gamma}_{x\to y}A)^{\alpha} = H^{\alpha}_{.\beta}(y, x; \gamma) A^{\beta}. \quad (2.1)$$

Let us note that for every x and γ

$$H^{\alpha}_{.\beta}(x, x; \gamma) \equiv \delta^{\alpha}_{\beta}, \quad (2.2)$$

where $\delta^{\alpha}_{\beta} = 1$ for $\alpha = \beta$ and $\delta^{\alpha}_{\beta} = 0$ for $\alpha \neq \beta$. (Other special properties of $\|H^{\alpha}_{.\beta}(y, x; \gamma)\|$ are insignificant for us now).

If the functions $H^{\alpha}_{.\beta}(y, x; \gamma)$ satisfy certain conditions (e.g., if they are of the class C^{N+1} with respect to y), there exists an integer N such that they can be put in one of the following equivalent forms (see (2.2))

$$H^{\alpha}_{.\beta}(y, x; \gamma) = \delta^{\alpha}_{\beta} + \sum_{m=1}^{N} \frac{1}{m!} H^{\alpha}_{.\beta\sigma_1 \dots \sigma_m}(x; y) \times \\ \times (y^{\sigma_1} - x^{\sigma_1}) \dots (y^{\sigma_m} - x^{\sigma_m}) + O((y - x)^{N+1}), \quad (2.3a)$$

$$H^{\alpha}_{.\beta}(\gamma(r_2), \gamma(r_1); \gamma) = \delta^{\alpha}_{\beta} + \sum_{m=1}^{N} \frac{1}{m!}\, {}^{m}H^{\alpha}_{.\beta}(r_1; \gamma)(r_2 - r_1)^{m} + \\ + O((r_2 - r_1)^{N+1}), \quad (2.3b)$$

where $(y - x)^{N+1} := [\max_{\alpha}(y^{\alpha} - x^{\alpha})]^{N+1}$, $x = \gamma(r_1)$, $y = \gamma(r_2)$, $r_1, r_2 \in [r', r'']$ and

$$H^{\alpha}_{.\beta\sigma_1 \dots \sigma_m}(x; \gamma) := \frac{\partial^m (H^{\alpha}_{.\beta}(y, x; \gamma))}{\partial y^{\sigma_m} \dots \partial y^{\sigma_1}}\Big|_{y = x}, \quad (2.4a)$$

$${}^{m}H^{\alpha}_{.\beta}(r_1; \gamma) := \frac{\partial^m (H^{\alpha}_{.\beta}(\gamma(r_2), \gamma(r_1); \gamma))}{\partial r_2^{m}}\Big|_{r_2 = r_1}. \quad (2.4b)$$

The functions (2.4b) may easily be expressed through the functions (2.4a), for example,

$${}^{1}H^{\alpha}_{.\beta}(r_1; \gamma) = H^{\alpha}_{.\beta\sigma_1}(x; \gamma)\dot{\gamma}^{\sigma_1}(r_1), \quad (2.5a)$$

$${}^{2}H^{\alpha}_{.\beta}(r_1; \gamma) = H^{\alpha}_{.\beta\sigma_1\sigma_2}(x; \gamma)\dot{\gamma}^{\sigma_1}(r_1)\dot{\gamma}^{\sigma_2}(r_2) + \\ + H^{\alpha}_{.\beta\sigma_1}(x; \gamma)\, d\dot{\gamma}^{\sigma_1}(r)/dr\,|_{r = r_1} \quad (2.5b)$$

and so on, where

$$\dot{\gamma}^{\alpha}(r) := d\gamma^{\alpha}(r)/dr \quad (2.6)$$

are the components of the tangent to γ vector $\dot{\gamma}$ at $r \in [r', r'']$.

Remark. The expansion (2.3a) is more general than (2.3b) because the second one is valid only if γ is a C^{N+1} map and the first one is valid for any map γ.

If in (2.1) we replace $H^{\alpha}_{.\beta}(y, x; \gamma)$ by the N-th approximation

$${}^{(N)}H^{\alpha}_{.\beta}(y, x; \gamma) := \delta^{\alpha}_{\beta} + \sum_{m=1}^{N} \frac{1}{m!}\, {}^{m}H^{\alpha}_{.\beta}(r_1; \gamma)(r_2 - r_1)^{m} = \\ = \delta^{\alpha}_{\beta} + \sum_{m=1}^{N} \frac{1}{m!} H^{\alpha}_{.\beta\sigma_1 \dots \sigma_m}(x; \gamma)(y^{\sigma_1} - x^{\sigma_1}) \dots (y^{\sigma_m} - x^{\sigma_m}), \quad (2.7)$$

we get a map ${}^{(N)}I^{\gamma}_{x\to y}: T_x(M) \to T_y(M)$ which may be called the

N-th order I-transport and it is defined by

$$ {}^{(N)}I^{\gamma}_{x\to y}A := \left({}^{(N)}H^{\alpha}_{\cdot\beta}(y,x;\gamma)A^{\beta}\right)\partial/\partial y^{\beta}. \tag{2.8} $$

Evidently ${}^{(N)}I^{\gamma}_{x\to y}$ describes the I-transport up to the (N+1)-order quantites, i.e., ${}^{(N)}I^{\gamma}_{x\to y}$ is an N-th approximation to $I^{\gamma}_{x\to y}$.

Further, in our paper we shall work only with the zeroth (N = 0) and first (N = 1) order approximations for which due to (2.7) we have

$$ {}^{(0)}H^{\alpha}_{\cdot\beta}(y,x;\gamma) \equiv \delta^{\alpha}_{\beta}, \tag{2.9a} $$

$$ {}^{(1)}H^{\alpha}_{\cdot\beta}(y,x;\gamma) = \delta^{\alpha}_{\beta} + H^{\alpha}_{\cdot\beta\sigma}(x;\gamma)(y^{\sigma}-x^{\sigma}) = \tag{2.9b} $$

$$ = \delta^{\alpha}_{\beta} + {}^{1}H^{\alpha}_{\cdot\beta}(r_1;\gamma)(r_2-r_1) = \delta^{\alpha}_{\beta} + H^{\alpha}_{\cdot\beta\sigma}(x;\gamma)\dot{\gamma}^{\sigma}(r_1)(r_2-r_1), $$

the last two equalities in the chain (2.9b) being valid if γ is a C^1-curve. Note that the functions (2.9a) are simply constants and thus they do not depend on x,y and γ as well as on the I-transport used.

It is important to note and easy to prove that the functions $[-H^{\alpha}_{\cdot\beta\sigma}(x;\gamma)]$ appearing in (2.9b) and defined by (2.4a) for n=1 define an affine connection along γ, i.e., they transform like coefficients of a generally nonsymmetric affine connection. (This statement is a simple corollary from the fact (see (2.1) and [2,3]) that $H^{\alpha}_{\cdot\beta}(y,x;\gamma)$ are components of a two-point tensor from $T_y(M)\otimes T^*_x(M)$.) Let us note without proof that the connection along γ defined by $(-H^{\alpha}_{\cdot\beta\sigma}(x;\gamma))$ is flat, i.e., its curvature tensor is equal to zero.

Let an affine connection with (local) coefficients $\Gamma^{\alpha}_{\cdot\beta\gamma}(x)$ at $x\in M$ on M be defined, i.e., a covariant derivative ∇ on M to be defined so that with respect to the basic vector $\partial/\partial x^{\beta}$ the covariant derivative of $A(x) = A^{\alpha}(x)\partial/\partial x^{\alpha}\in T_x(M)$ to be with components

$$ (\nabla_{\beta}A(x))^{\alpha} := A^{\alpha}_{|\beta}(x) = \partial A^{\alpha}(x)/\partial x^{\beta} + \Gamma^{\alpha}_{\cdot\sigma\beta}(x)A^{\sigma}(x). $$

Then, it is not difficult to prove (cf. [2]) that for the defined by this connection paralled transport

$$ H^{\alpha}_{\cdot\beta\sigma}(x;\gamma;\|) = -\Gamma^{\alpha}_{\cdot\beta\sigma}(x), \tag{2.10} $$

where the additional argument $\|$ indicates that the functions $H^{\alpha}_{\cdot\beta\sigma}(x;\gamma)$ are computed for the given parallel transport.

An important property of the functions (2.10) is their independence of the curve γ.

As has been pointed out in [5], part V, on the bašis of the expansions (2.3) one can derive the corresponding approximate displacement vectors, deviation vectors and deviation equations. Now we shall get the expression for the first order deviation vector and in Sec.3 we shall consider the corresponding to it deviation equation.

Let there be given curves $x_a:[s'_a,s''_a]\to M$, a = 1,2 and $x:[s',s'']\to M$ and one-to-one maps $\tau_a:[s',s'']\to[s'_a,s''_a]$ which map the parameter $s\in[s',s'']$ onto the parameters $s_a = \tau_a(s)\in[s'_a,s''_a]$, a = 1,2. Let also be given two one-parameter families of curves $\gamma_s:[r'_s,r''_s]\to M$ and $\eta_s:[\rho'_s,\rho''_s]\to M$, such that $\gamma_s(r'_s) := x_1(\tau_1(s))$ $:= \eta_s(\rho'_s)$, $\gamma_s(r''_s) := x_2(\tau_2(s))$ and $\eta_s(\rho''_s) := x(s)$, $s\in[s',s'']$. Thus, defining $\dot{\gamma}^{\alpha}_s(r) := \partial\gamma^{\alpha}_s(r)/\partial r$, $r\in[r',r'']$ we see that the deviation vector of x_2 with respect to x_1 relatively to x at x(s) is [2,3]

$$ h = h(s;x) = I^{\eta_s}_{x_1(s_1)\to x(s)} \int_{r'_s}^{r''_s} I^{\gamma_s}_{\gamma_s(r)\to x_1(s_1)}\dot{\gamma}_s(r)\,dr \in T_{x(s)}(M). \tag{2.11} $$

Using (2.3) for N = 0 and (2.1) from (2.11), we get

$$ h^{\alpha} = x^{\alpha}_2(s_2) - x^{\alpha}_1(s_1) + O((x(s)-x_1(s_1))(x_2(s_2)-x_1(s_1))) + $$

$$ + O((x(s)-x_1(s_1))(x_2(s_2)-x_1(s_1))^2) + O((x_2(s_2)-x_1(s_1))^2) = \tag{2.12} $$

$$ = \dot{\gamma}^{\alpha}_s(r'_s)(r''_s-r'_s) + O((\rho''_s-\rho'_s)(r''_s-r'_s)) + $$

$$ + O((\rho''_s-\rho'_s)(r''_s-r'_s)^2) + O((r''_s-r'_s)^2). $$

From here we see that $x^{\alpha}_2(s_2) - x^{\alpha}_1(s_1)$ or equivalently $\dot{\gamma}^{\alpha}_s(r'_s)(r''_s-r'_s)$ (with the same precision) is the lowest approximate expression for the deviation vector which will be called the first order deviation vector and will be denoted by $\zeta = \zeta(s)$.

Further, for brevity and simplicity we shall consider only the cases $\tau_a = \mathrm{id}$ (i.e. $\tau_a(s) = s_a = s$), a = 1,2 (this is not an essential restriction: it simply means to denote $x_a\circ\tau_a$ by x_a and to parametrize it by s) and $x = x_1$; one usually finds these assumptions in the literature [1,6-9]. Besides, we shall use the independent parameters $s\in[s',s'']$ and $r\in[r',r'']$ such that $r_s = f_s(r)$, where $f_s:[r',r'']\to[r'_s,r''_s]$

is one-to-one C^2-function and $f_s(r') = r'_s$ and $f_s(r'') = r''_s$. Thus, (2.11) and (2.12) take respectively the form

$$h = \int_{r'}^{r''} I^{\gamma_s}_{\gamma(s,r)\to x_1(s)} \dot\gamma(s,r)\,dr, \qquad (2.11')$$

$$h^a = x_2^a(s) - x_1^a(s) + O((x_2(s) - x_1(s))^2) =$$
$$= \dot\gamma^a(s,r')(r''-r') + O((r''-r')^2) = \zeta^a + O((r''-r')^2), \qquad (2.12')$$

where

$$\gamma(s,r) := \gamma_s(f_s(r)) = (\gamma_s \circ f_s)(r), \quad \dot\gamma^a(s,r) := \partial\gamma^a(s,r)/\partial r$$

and in this case the first order deviation vector is

$$\zeta = \zeta(s) = \dot\gamma(s,r')(r''-r') \subset T_{x_1(s)}(M). \qquad (2.13)$$

The feature of (2.13) is its independence of γ up to terms of an order of $O((r''-r')^2)$.

If $r''-r'$ is an infinitesimal constant, this vector is sometimes called the infinitesimal deviation vector [2,6,8].

At the end of this section we want to present one useful approximate formula.

Let an affine connection be defined on M, B be a C^1 vector field defined on $\{\gamma(s,r)\}$ and

$$\Delta B_{21} := \Delta B(s; x_1; I^{\gamma_s}) :=$$
$$= I^{\gamma_s}_{x_2(s)\to x_1(s)} B(x_2(s)) - B(x_1(s)) \in T_{x_1(s)}(M). \qquad (2.14)$$

Applying (2.1) and (2.3b) for N=1 to the first term of this definition and then taking into account (2.5a), (2.13) and that $x_1(s) = \gamma(s,r')$ and $x_2(s) = \gamma(s,r'')$ we get after some simple calculations

$$\Delta B^a_{21} = \frac{DB^a(\gamma(s,r))}{\partial r}\Big|_{r=r'} (r''-r') +$$
$$+ \Delta^a_{.\beta\sigma}(x_1(s))\, B^\beta(x_1(s))\, \zeta^\sigma(s) + O((r''-r')^2), \qquad (2.15)$$

where $D/\partial r := \dot\gamma^\sigma(s,r)\nabla_{\partial/\partial\gamma^\sigma(s,r)}$ is the covariant derivative along $\gamma(s,r)$ for fixed s $(DB^a/\partial r = \partial B/\partial r + \Gamma^a_{.\beta\sigma} B^\beta \dot\gamma^\sigma(s,r))$ and Δ is a tensor field of the type (1,2) defined on $\{\gamma(s,r)\}$ with the components

$$\Delta^a_{.\beta\sigma}(\gamma(s,r)) := \Gamma^a_{.\beta\sigma}(\gamma(s,r)) + H^a_{.\beta\sigma}(\gamma(s,r);\gamma_s). \qquad (2.16)$$

Note that if I^{γ_s} is a parallel transport along γ_s then due to (2.10) the tensor field Δ is identically equal to zero.

3. FIRST ORDER DEVIATION EQUATION FOR THE DEVIATION VECTOR

Here and hereafter in our text we suppose to be defined an affine connection on M and a general linear transport (see Sec.2) having a needed number of derivatives (see below).

For any C^2-vector field $\xi = \xi(s)$ defined along any C^2-curve $y:[s', s''] \to M$ the following identity [2,6] is valid.

$$\frac{D^2\xi^a}{ds^2} = R^a_{.\beta\gamma\delta} u^\beta u^\gamma \xi^\delta + \xi^a_{.|\beta} F^\beta + u^\beta \frac{D}{ds}(T^a_{.\beta\gamma}\xi^\gamma) +$$
$$+ u^\beta u^\gamma \underset{\xi}{\pounds} \Gamma^a_{.\beta\gamma}, \qquad (3.1)$$

where all quantities are defined at $y(s)$, $s \in [s', s'']$ and $D/ds := u^\beta \nabla_\beta$, $u^\beta := dy^\beta(s)/ds$, $F^a := Du^a/ds$, $T^a_{.\beta\gamma} := -\Gamma^a_{.\beta\gamma} + \Gamma^a_{.\gamma\beta}$ is the torsion tensor,

$$R^a_{.\beta\gamma\delta} = -\partial\Gamma^a_{.\beta\gamma}/\partial x^\delta + \partial\Gamma^a_{.\beta\delta}/\partial x^\gamma - \Gamma^\epsilon_{.\beta\gamma}\Gamma^a_{.\epsilon\delta} + \Gamma^\epsilon_{.\beta\delta}\Gamma^a_{.\epsilon\gamma}$$

is the curvature tensor and $\underset{\xi}{\pounds}$ is the Lie derivative along ξ.

For $y = x$ and $\xi = h$ (the deviation vector - see Sec.2) the identity (3.1) with some additional conditions which must satisfy h is called the (generalized) deviation equation [2,6]. If the expansion (2.12') is substituted into this equation, then one can get the first order deviation equation which must satisfy the first order deviation vector (2.13). But the latter equation may be obtained (as an exact equation) directly from (3.1) in the following way [6]:

At first, we have to put in (3.1) $y = x = x_1$, so $u^a = V_1^a := dx_1^a(x)/ds$ and $\xi = \zeta$ and then to use as an additional condition the equality $\partial^2\gamma^a(s,r)/\partial s\partial r = \partial^2\gamma^a(s,r)/\partial r\partial s$ (γ is a C^2 map) which for $r = r'$ is equivalent to $\underset{\zeta}{\pounds} V_1 = 0$. Hence, using the identities (for any V_1 and ζ; see [6])

$$\underset{\zeta}{\pounds} F^a = F^a_{\cdot|\beta}\zeta^\beta - \zeta^a_{|\beta}F^\beta + T^a_{\cdot\beta\gamma}F^\beta\zeta^\gamma$$

and

$$V_1^\beta V_1^\gamma \underset{\zeta}{\pounds}\Gamma^a_{\cdot\beta\gamma} = \underset{\zeta}{\pounds} F^a - D(\underset{\zeta}{\pounds} V_1^a)/ds - V^a_{1|\beta}\underset{\zeta}{\pounds} V_1^\beta,$$

we get after some simple calculations:

$$\frac{D^2\zeta^a}{ds^2} = R^a_{\cdot\beta\gamma\delta}V_1^\beta V_1^\gamma\zeta^\delta + T^a_{\cdot\beta\gamma}F^\beta(s,r')\zeta^\gamma +$$

$$+ V_1^\beta\frac{D(T^a_{\cdot\beta\gamma}\zeta^\gamma)}{ds} + \frac{DF^a(s,r)}{\partial r}\Big|_{r=r'}(r''-r') = \tag{3.2}$$

$$= R^a_{\cdot\beta\gamma\delta}V_1^\beta V_1^\gamma\zeta^\delta + \frac{D}{ds}(T^a_{\cdot\beta\gamma}V_1^\beta\zeta^\gamma) + \frac{DF^a(s,r)}{\partial r}\Big|_{r=r'}(r''-r'),$$

where ζ and V_1 have an argument s, $R^a_{\cdot\beta\gamma\delta}$, $T^a_{\cdot\beta\gamma}$ have an argument $x_1(s)$, and

$$F^a(s,r) := \frac{D}{\partial s}\left(\frac{\partial\gamma^a(s,r)}{\partial s}\right). \tag{3.3}$$

Eq. (3.2) is the first order deviation equation. It was obtained (in the general form) by another method in [4]. Also it was derived in a number of special cases on the bedrock of other principles, e.g., in [1,7-9].

Let us emphasize two features of eq.(3.2). Firstly, it is independent of the concrete choice of the used I-transport and up to terms of an order of $O((r''-r')^2)$ of the family of curves $\gamma_s(r_s) = \gamma(s,r)$. Secondly, it is an exact equation, i.e., it does not contain any correcting higher order terms like $O((r''-r')^2)$ (if one defines (noncovariantly) the first order deviation vector instead of (2.13) by the equation $\zeta^a = x_2^a(s) - x_1^a(s)$, as it is done, e.g., in [6,7], then in the right-hand side of (3.2) there must be added $O((x_2(s) - x_1(s))^2)$).

In the general form, the physical meaning of the deviation equations was discussed, e.g., in [2,6,9]. This interpretation is valid and for the first order deviation equation (3.2) because it is a special case of the generalized deviation equation [2]. From this view-point, in Sec.6 we shall pay special attention to the last term in eq.(3.2).

4. FIRST ORDER DEVIATION EQUATION FOR THE RELATIVE VELOCITY

Let along the curves x_1 and x_2 be moving two particles 1 and 2, respectively, i.e., x_1 and x_2 to be world lines (trajectories) of these particles. Then, their velocities V_1 and V_2 have components

$$V_a^a = V_a^a(s) := \frac{dx_a^a(s)}{ds}, \quad a = 1, 2 \tag{4.1}$$

and the relative velocity of the second particle with respect to the first one is [5]

$$\Delta V_{21} := I^{\gamma_s}_{x_2(s)\to x_1(s)} V_2 - V_1. \tag{4.2}$$

First of all, we want to find the connection betweeen ΔV_{21} and the first order deviation velocity $D\zeta/ds$ which due to (2.12') is connected with the deviation velocity Dh/ds [5] by

$$Dh/ds = D\zeta/ds + O((r''-r')^2). \tag{4.3}$$

Taking into account that $x_1(s) = \gamma(s,r')$ and $x_2(s) = \gamma(s,r'')$, from (2.13) and (4.1), we find:

$$\zeta^a(s) = x_2^a(s) - x_1^a(s) + O((r''-r')^2), \tag{4.4}$$

$$V_2^a(s) - V_1^a(s) = \frac{\partial}{\partial r}\,\frac{\partial\gamma^a(s,r)}{\partial s}\Big|_{r=r'}(r''-r') + O((r''-r')^2). \tag{4.5}$$

Differentiating (4.4) covariantly along $x_1(s)$ and using (4.1) and (4.5), we get

$$\frac{D\zeta^{\alpha}(s)}{ds} = \frac{D}{dr}\left.\frac{\partial\gamma^{\alpha}(s,r)}{\partial s}\right|_{r=r'}(r''-r') + $$
$$+ T^{\alpha}_{\cdot\beta\sigma}(x_1(s))V_1^{\beta}(s)\zeta^{\sigma}(s) + O((r''-r')^2). \quad (4.6)$$

On the other hand, from (2.15) for $B^{\alpha}(\gamma(s,r)) = \partial\gamma^{\alpha}(s,r)/\partial s$, (2.14), (4.1) and (4.2), we derive

$$\Delta V^{\alpha}_{21} = \frac{D}{dr}\left.\frac{\partial\gamma^{\alpha}(s,r)}{\partial r}\right|_{r=r'}(r''-r') + $$
$$+ \Delta^{\alpha}_{\cdot\beta\sigma}(x_1(s))V_1^{\beta}(s)\zeta^{\sigma}(s) + O((r''-r')^2). \quad (4.7)$$

Comparing (4.7) with (4.6) we find the following relationship between the relative velocity ΔV_{21} and the first order deviation velocity $D\zeta/ds$, both being the first order (in $r''-r'$) quantities:

$$\frac{D\zeta^{\alpha}(s)}{ds} = \Delta V^{\alpha}_{21} + [T^{\alpha}_{\cdot\beta\sigma}(x_1(s)) - \Delta^{\alpha}_{\cdot\beta\sigma}(x_1(s))]V_1^{\beta}(s)\zeta^{\sigma}(s) + $$
$$+ O((r''-r')^2). \quad (4.8)$$

So up to second order terms we can make the conclusion that the deviation velocity describes the "general relative velocity" of the particle 2 with respect to the particle 1 and it is caused by the (nongravitational) interaction of the particles as well as by all the properties (curvature, torsion, I-transport) of the space M, but the relative velocity (4.2) is called forth only by the (nongravitational) interaction of particles and by the used I-transport.

Substituting (4.8) into the left-hand side of (3.2) and performing some evident calculations we find the first order deviation equation for the relative velocity ΔV_{21} in the form:

$$\frac{D\Delta V^{\alpha}_{21}}{ds} = R^{\alpha}_{\cdot\beta\gamma\delta}(x_1(s))V_1^{\beta}(s)V_1^{\gamma}(s)\zeta^{\delta}(s) + $$
$$+ \frac{D}{ds}[\Delta^{\alpha}_{\cdot\beta\sigma}(x_1(s))V_1^{\beta}(s)\zeta^{\sigma}(s)] + \quad (4.9)$$
$$+ \left.\frac{DF^{\alpha}(s,r)}{\partial r}\right|_{r=r'}(r''-r') + O((r''-r')^2).$$

This equation up to second order terms describes the change of the relative velocity of the second particle with respect to the first one along the world line of the first particle.

5. FIRST ORDER DEVIATION EQUATION FOR THE RELATIVE MOMENTUM

The momenta of the considered in sec.4 particles are [5,8]

$$p_a = p_a(s) := \mu_a(s)V_a(s), \quad a = 1, 2, \quad (5.1)$$

where $\mu_1(s) \neq 0$ and $\mu_2(s) \neq 0$ are some scalar functions of s (if, e.g., the first particle has a nonzero (rest) mass $m_1(s) \neq 0$, then $\mu_1(s) = m_1(s)$; see [5,8]) and the relative momentum of the second particle with respect to the first one is [5]

$$\Delta p_{21} = I^{\gamma_s}_{x_2(s)\to x_1(s)}p_2 - p_1. \quad (5.2)$$

Using (5.1) and (4.2), we derive (cf. [5] , eq,(2.4))

$$\Delta p_{21} = \mu_2(s)\Delta V_{21} + (\mu_2(s)/\mu_1(s) - 1)p_1(s). \quad (5.3)$$

(The same result may be obtained as an approximate expression from (4.7) and (2.15) for $B^{\alpha}(\gamma(s,r)) = \mu(s,r)\partial\gamma^{\alpha}(s,r)/\partial s$, , where $\mu(s,r)$ is a C^1-function of r , $\mu(s,r') = \mu_1(s)$ and $\mu(s,r'') = \mu_2(s)$ (see also below (7.1)).

Differentiating (5.3) along x_1, we get

$$\frac{D\Delta p_{21}}{ds} = \mu_2(s)\frac{D\Delta V_{21}}{ds} + \frac{d\mu_2(s)}{ds}\Delta V_{21} + \frac{D}{ds}\left[\left(\frac{\mu_2(s)}{\mu_1(s)} - 1\right)p_1(s)\right] \quad (5.4)$$

and substituting here (4.9) we find:

$$\frac{D\Delta p^{\alpha}_{21}}{ds} = \frac{\mu_2(s)}{(\mu_1(s))^2}R^{\alpha}_{\cdot\beta\gamma\delta}(x_1(s))p_1^{\beta}(s)p_1^{\gamma}\zeta^{\delta}(s) + $$
$$+ \mu_2(s)\frac{D}{ds}\left[\frac{1}{\mu_1(s)}\Delta^{\alpha}_{\cdot\beta\sigma}(x_1(s))p_1^{\beta}(s)\zeta^{\sigma}(s)\right] + \frac{d\mu_2(s)}{ds}\Delta V^{\alpha}_{21} + $$

$$+\frac{D}{ds}\left[\left(\frac{\mu_2(s)}{\mu_1(s)}-1\right)p_1^{\alpha}(s)\right]+\mu_2(s)\,\frac{DF^{\alpha}(s,r)}{\partial r}\Big|_{r=r'}\ (r''-r')+ \tag{5.5}$$

$$+O((r''-r')^2),$$

where if it is needed, one can substitute ΔV_{21} from (4.7) or it may be obviously obtained from (5.3).

Equation (5.5) is the first order deviation equation for the relative momentum Δp_{21} of the second particle with respect to the first one and it describes up to second order terms the evolution of Δp_{21} along the world line of the first particle. The physical interpretation of this equation will be considered in sec.7.

6. FIRST ORDER DEVIATION EQUATION FOR THE RELATIVE ACCELERATION

The accelerations A_1 and A_2 of the considered above particles are with components

$$A_a^{\alpha}=A_a^{\alpha}(s):=\frac{dV_a^{\alpha}(s)}{ds}=\frac{D}{ds}\left(\frac{dx_a^{\alpha}(s)}{ds}\right),\quad a=1,2 \tag{6.1}$$

and the relative acceleration of the second particles with respect to the first one is[5]

$$\Delta A_{21}=I^{\gamma_s}_{x_2(s)\to x_1(s)}A_2-A_1. \tag{6.2}$$

From (6.1) and (3.3) we see that

$$A_1=F(s,r'),\ A_2=F(s,r''). \tag{6.3}$$

So putting $B(\gamma(s,r))=F(s,r)$ in (2.14) and (2.15), we obtain

$$\Delta A_{21}^{\alpha}=\Delta F_{21}^{\alpha}=\frac{DF^{\alpha}(s,r)}{\partial r}\Big|_{r=r'}\ (r''-r')+$$
$$+\Delta^{\alpha}_{\cdot\beta\sigma}(x_1(s))\,A_1^{\alpha}(s)\,\zeta^{\sigma}(s)+O((r''-r')^2). \tag{6.4}$$

From (3.3) and physical interpretation of the deviation equation (3.2) (see [2,4,6,9]) it is clear that $F(s,r)$ has a meaning of a (nongravitational) force per unit mass acting on a particle situated at the point $\gamma(s,r)$, i.e., $F(s,r)$ is the acceleration of this particle (cf.[4]). Thus, equality (6.4) shows that up to second order terms the expression $(DF^{\alpha}(s,r)/\partial r)|_{r=r'}\,(r''-r')$ which appears above in (3.2), (4.9) and (5.5), is equal to the relative acceleration between the considered particles (computed by means of an I-transport) or, all the same, it is equal to the (covariant) difference of the (nongravitational) forces per unit mass, $F(s,r'')$ and $F(s,r')$, acting on these particle which is defined by means of the given I-transport along $\gamma(s,r)$ for fixed s (cf.[9]).

Expressing $(DF^{\alpha}(s,r)/\partial r|_{r=r'})\,(r''-r')$ from (6.4) and substituting the so-obtained result into (3.2), we find the following relation between the first order deviation acceleration $D^2\zeta(s)/ds^2$ which due to (2.12') is connected with the deviation acceleration D^2h/ds^2 by

$$D^2h/ds^2=D^2\zeta(s)/ds^2+O((r''-r')^2), \tag{6.5}$$

and the relative acceleration ΔA_{21}:

$$\frac{D^2\zeta^{\alpha}(s)}{ds^2}=\Delta A_{21}^{\alpha}+R^{\alpha}_{\cdot\beta\gamma\delta}(x_1(s))\,V_1^{\beta}(s)\,V_1^{\gamma}(s)\,\zeta^{\delta}(s)+$$
$$+T^{\alpha}_{\cdot\beta\gamma}(x_1(s))\,A^{\beta}(s)\,\zeta^{\gamma}(s)-\Delta^{\alpha}_{\cdot\beta\sigma}(x_1(s))\,A_1^{\beta}(s)\,\zeta^{\sigma}(s)+ \tag{6.6}$$
$$+V_1^{\beta}(s)\,\frac{D}{ds}[T^{\alpha}_{\cdot\beta\gamma}(x_1(s))\,\zeta^{\gamma}(s)]+O((r''-r')^2),$$

where for some purposes one may substitute the first order deviation velocity $D\zeta^{\alpha}(s)/ds$ from (4.8).

We should like to mention that due to (2.13) and (6.4) $D^2\zeta^{\alpha}(s)/ds^2$ and ΔA_{21} are first order quantities in $(r''-r')$.

From (6.6) we can conclude that up to second order terms the deviation acceleration is a consequence of the (nongravitational) interaction of the particles as well as of all the properties of the space M and the causes of the relative acceleration are only the (nongravitational) interaction of particles and the used I-transport.

If we express $(DF^{\alpha}(s,r)/\partial r|_{r=r'})\,(r''-r')$ from (4.9) and substitute the obtained result into (6.4), we get the following relation between the relative velocity ΔV_{21} and the relative acceleration ΔA_{21}:

$$\Delta A^{\alpha}_{21} = \frac{D\Delta V^{\alpha}_{21}}{ds} - R^{\alpha}_{\cdot\beta\gamma\delta}(x_1(s))\, V^{\beta}_1(s)\, V^{\gamma}_1(s)\, \zeta^{\delta}(s) - V^{\beta}_1(s)\, \frac{D}{ds}[\Delta^{\alpha}_{\cdot\beta\sigma}(x_1(s))\, \zeta^{\sigma}(s)] + O((r''-r')^2), \tag{6.7}$$

which due to the physical interpretation of the quantities in it may be called a deviation equation for the relative acceleration of the second particle with respect to the first one.

7. CONCLUSION

In this paper we have derived from a unified viewpoint the first order deviation equations for the first order diviation vector, relative velocity, relative momentum and relative acceleration. For the last quantity this equation turns out to be an algebraic one and for other quantities it is an ordinary differential equation which is of a second order for the first and of a first order for the second and third ones.

Besides we have found important relations connecting the deviation velocity and the relative velocity as well as the deviation acceleration and the relative acceleration.

Now we want to pay special attention to the deviation equation (5.5) for the relative momentum. On the basis of the above analysis (see especially Sec.6) we can infer that equation (5.5) is a direct analogue (up to second order terms) of the second Newton law, i.e., of the low of motion (cf. /4/), in the case. To show this, we shall put eq. (5.5) into a slightly different form.

Let $\mu(s,r)$ be a smooth function of r, $\mu(s,r') := \mu_1(s)$, $\mu(s,r'') := \mu_2(s)$ and $K(s,r) := \mu(s,r)\, F(s,r)$ (see (3.3)). The quantity $K(s,r)$ has a meaning of a (nongravitational) force acting on a particle with momentum $\mu(s,r)\,\partial\gamma^{\alpha}(s,r)/\partial s$ at the point $\gamma(s,r)$. Putting $B(\gamma(s,r)) = K(s,r)$ in (2.15) and taking into account (6.3) and

$$\mu_2(s) - \mu_1(s) = \mu(s,r'') - \mu(s,r') = (\partial\mu(s,r)/\partial r|_{r=r'})\,(r''-r') + O((r''-r')^2)$$

we obtain

$$\begin{aligned}\Delta K^{\alpha}_{21} &= (\mu_2(s) - \mu_1(s))\, A^{\alpha}_1(s) + \mu_1(s)\, \frac{\partial F^{\alpha}(s,r)}{\partial r}\Big|_{r=r'} (r''-r') + \mu_1(s)\, \Delta^{\alpha}_{\cdot\beta\sigma}(x_1(s))\, A^{\beta}_1(s)\, \zeta^{\sigma}(s) + O((r''-r')^2) = \\ &= (\mu_2(s) - \mu_1(s))\, A^{\alpha}_1(s) + \mu_2(s)\, \frac{\partial F^{\alpha}(s,r)}{\partial r}\Big|_{r=r'} (r''-r') + \mu_2(s)\, \Delta^{\alpha}_{\cdot\beta\sigma}(x_1(s))\, A^{\beta}_1(s)\, \zeta^{\sigma}(s) + O((r''-r')^2).\end{aligned} \tag{7.1}$$

Physically ΔK_{21} is the (covariant) difference of the forces acting on the considered particles.

Expressing from (7.1) the term

$$\mu_2(s)\,(\partial F^{\alpha}(s,r)/\partial r|_{r=r'})\,(r''-r')$$

and substituting this result into (5.5) we find (see also (5.1)):

$$\begin{aligned}\frac{D\Delta p^{\alpha}_{21}}{ds} &= \frac{\mu_2(s)}{(\mu_1(s))^2}\, R^{\alpha}_{\cdot\beta\gamma\delta}(x_1(s))\, p^{\beta}_1(s)\, p^{\gamma}_1(s)\, \zeta^{\delta}(s) + \frac{\mu_2(s)}{\mu_1(s)}\, p^{\beta}_1(s)\, \frac{D}{ds}[\Delta^{\alpha}_{\cdot\beta\sigma}(x_1(s))\, \zeta^{\sigma}(s)] + \frac{d\mu_2(s)}{ds}\, \Delta V^{\alpha}_{21} + \\ &+ \frac{1}{\mu_1(s)}\, \frac{d(\mu_2(s) - \mu_1(s))}{ds}\, p^{\alpha}_1(s) + \Delta K_{21} + O((r''-r')^2).\end{aligned} \tag{7.2}$$

This is just another form of the first order deviation equation for the relative momentum. Its left-hand side defines the change of the relative momentum of the second particle with respect to the first one along the world line of the first particle. The terms in its right-hand side describe this change as follows: the first is due to curvature of the space, the source of the second is a tensor Δ with components (2.16), the third and the fourth are caused by the speed of change of the functions $\mu_1(s)$ and $\mu_2(s)$ (the rest masses of the particles if they are nonzero), the fifth is a result of the (nongravitational) interaction of the particles and $O((r''-r')^2)$ contains all high order (at least second order) corrections in $(r''-r')$ (all of the preceding terms are of a first order in $(r''-r')$).

At the end, we shall derive an equation which by analogy with the above results may be called a first order deviation equation for the relative energy.

Let M be endowed also with a metric, i.e., a scalar product of the vectors which we shall denote by a dot: $A \cdot B = A^{\alpha} B^{\beta} g_{\alpha\beta}(y)$, where A and B are vectors at one and the same point $y \in M$ with the components A^{α} and B^{β}, respectively, and $g_{\alpha\beta}(y)$ are the covariant components of the metric at y.

The relative energy of the second particle with respect to the first one is (see (5.2) and [5], eqs. (3.1) and (3.6))

$$E_{21} = \Theta((V_1(s))^2)\, V_1(s) \cdot I^{\gamma_s}_{x_2(s)\to x_1(s)} p_2(s) = \Theta((V_1(s))^2)\,(\Delta p_{21} \cdot V_1(s) + p_1(s) \cdot V_1(s)), \tag{7.3}$$

where $\Theta(\lambda) := +1$ for $\lambda \geq 0$ and $\Theta(\lambda) := -1$ for $\lambda < 0$.

Differentiating (7.3) with respect to s along x_1 (note that $Df/ds = df/ds$ for a scalar f) and substituting in this expression eq. (7.2), we get

$$\begin{aligned}
\frac{dE_{21}}{ds} = {} & \Theta((V_1(s))^2)\Big[\frac{\mu_2(s)}{(\mu_1(s))^3} g_{\alpha\sigma}(x_1(s))\, R^{\alpha}_{\cdot\beta\gamma\delta}(x_1(s)) \times \\
& \times p_1^{\sigma}(s)\, p_1^{\beta}(s)\, p_1^{\gamma}(s)\, \zeta^{\delta}(s) + \\
& + \frac{\mu_2(s)}{(\mu_1(s))^2} p_1^{\beta}(s)\, p_1^{\gamma}(s)\, g_{\alpha\gamma}(x_1(s)) \frac{D}{ds}(\Delta^{\alpha}_{\cdot\beta\sigma}(x_1(s))\, \zeta^{\sigma}(s)) + \\
& + \frac{d\mu_2(s)}{ds} \Delta V_{21} \cdot V(s) + \frac{1}{(\mu_1(s))^2} \frac{d(\mu_2(s) - \mu_1(s))}{ds} p_1(s) \cdot p_1(s) + \\
& + \Delta K_{21} \cdot V_1(s) + \Delta p_{21} \cdot A_1(s) + \Delta p^{\alpha}_{21} V_1^{\beta}(s)\, D g_{\alpha\beta}(x_1(s))/ds + \\
& + \frac{d\mu_1(s)}{ds} V_1(s) \cdot V_1(s) + \mu_1(s)\,(2 V_1(s) \cdot A_1(s) + \\
& + V_1^{\alpha}(s)\, V_2^{\beta}(s)\, D g_{\alpha\beta}(x_1(s))/ds) \Big] + O(r''-r')^2).
\end{aligned} \tag{7.4}$$

This is the first order equation of the relative energy balance which may be regarded as a generalization of the usual equation of the energy conservation.

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