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MINIMAL QUANTIZATION OF LINEARIZED GRAVITY

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1. *Introduction*

The description of fundamental particle interactions with the help of gauge fields introduces superfluous degrees of freedom into the theory. This manifests itself in the singular nature of the corresponding Lagrangians or in the presence of constraints in the equivalent hamiltonian formulation [1]. All constrained systems can be described naturally in terms of symplectic geometry [2]. However, there are two symplectic manifolds associated with each constrained system: the extended and the reduced phase spaces respectively. Classically, there is no formal distinction between working in the extended phase space with allowance for the constraints and solving the constraints, reducing the system and working in the reduced phase space. But these two approaches are not necessarily equivalent on the quantum level and may engender real and significant physical differences in the quantum behaviour of the system. For example, the reduced phase space method [3,4] fails to give the correct results in the cases where the degree of freedom corresponding to the constraint in principle could be quantum mechanically excited [5]. However, this is not the case we meet in gauge theories. On the other hand, there are examples where Dirac method [2,6] leads to negative-energy states due to a quantum tunnelling into a classically forbidden (by the constraints) region [7]. As has been pointed out [8], this is an artifact of the canonical quantization which originates in the choice of polarization incompatible with the constraints.*)

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*) From a geometric-quantization point of view [9] canonical quantization is equivalent to a geometric quantization in the 'vertical' polarization, the leaves of the latter being the fibers of the cotangent bundle projection $T^{*}Q \longrightarrow Q$, where Q is the configuration space of the system.

In the present paper, a new quantization procedure for linearized gravity is proposed without gauge fixing as an initial preposition. The results obtained point out the importance of physically motivated assumptions about the small metric-tensor components to be neglected. The main peculiarity of our approach is that after reduction of the configuration space the theory is formulated and quantized in the space of gauge orbits of the dynamical fields.

2. *Vertical paths, metric and Gauss' law*

Due to the gauge invariance the basic objects of the theory - the gauge potentials - form an overcomplete basis. Gauge fields which are connected by an infinitesimal gauge transformation actually describe the same physical state. Thus the vector potentials are divided into equivalence classes with respect to the gauge group G action. An equivalence class represents an orbit in the gauge-field configuration space. It is rather the space of orbits than the function space of the gauge fields that has to be viewed as a physical space. Transitions along the orbits correspond to pure gauge transformations. These vertical paths are of no physical importance. Physically significant are only horizontal paths, i.e. paths which are perpendicular to the orbits [10]. These paths describe the time evolution of the physical system. Fixing the gauge, one tries to solve the problem of constructing such horizontal paths. In fact, this means that the reduced phase space (on which an unconstrained Hamiltonian can be defined) is identified with the vector potentials and their conjugate momenta in this gauge. It was first Gribov who pointed out that the gauge, regarded as a map from the physical fields to the space of vector potentials, is singular [11], the result being later generalized by Singer [10].

This problem has one more aspect. In the path-integral quantization of gauge fields one starts from an initial configuration at $t=0$ and integrates over all histories, i.e. all paths in the gauge-field configuration space. In such a way the genuine dynamical time evolution is not distinguished from the time evolution generated by gauge transformations. An attempt to circumvent this difficulty consists in imposing a gauge condition [12]; globally, however, this approach fails because garden-variety gauges are only locally unique [13]. Thus, one needs a prescription for choosing the paths so as to eliminate the spurious time development due to gauge transformations.

Using time as a parameter of the paths in the bundle of all spatial potentials, we define the tangent of the path $\vec{A}(t)$

$$\vec{\tau} = \frac{d}{dt}\vec{A}(t)$$

In particular, for vertical paths

$$\vec{A}(t) = g(t)^{-1}\vec{A}(t)g(t) - ig(t)^{-1}\vec{\nabla}g(t)$$

with $g(t)$ an element of G, the tangent vector is

$$\vec{\tau} = \dot{\vec{A}}(t) = \vec{D}\dot{\varepsilon},$$

where $\varepsilon = \{\varepsilon^a\}$ are the parameters of the infinitesimal transformation corresponding to $g(t)$. Thus, vertical paths (i.e. paths along the orbits) have tangent vectors of the form

$$\vec{\tau} = \vec{D}\varphi, \tag{1}$$

with φ an arbitrary Lie-algebra valued function. To eliminate the time development due to gauge transformations, one should restrict the paths in the path integral to those that are purely horizontal. We can define the horizontal vector $\vec{\sigma}$ [10,14] as a vector orthogonal to all vertical vectors $\vec{\tau}$ with respect to the scalar product < , > in the orbit space, i.e.

$$\langle\vec{\sigma},\vec{\tau}\rangle = 0 \quad for\ all\ vertical\ \vec{\tau}'s.$$

Using expression (1), we find [13]

$$0 = Tr\int d^3x\ \sigma_i(x)\ \eta_{ij}\ D_j\ \varphi(x) = -\ Tr\int d^3x\ [D_i\ \eta_{ij}\ \sigma_j(x)]\ \varphi(x),$$

which implies

$$D_i\ \eta_{ij}\ \sigma_j = 0, \tag{2}$$

(where η_{ij} is the metric in the orbit space) because of the arbitrariness of $\varphi(x)$. A path $A_i(t)$ is horizontal if its tangent is everywhere horizontal and condition (2) leads to the following definition of a horizontal path in a space with metric η_{ij}

$$D_i\ \eta_{ij}\ \partial_o A_j = 0. \tag{3}$$

Thus, the correct definition of the metric in the orbit space comes out to be very important for singling out the horizontal paths.

The dynamical-field gauge-orbit space is equipped with a natural projective metric, which provides introduction of a symplectic structure therein and construction of Poincare-group representation with a nonstandard action on the gauge fields [16]. The canonical energy-momentum tensor obtained is symmetric and gauge invariant but differs from the Belinfante one, being its reduction on the equations of motion and thus the minimal symmetric energy-momentum tensor for the theory under consideration. A transition to independent physical variables is then possible that allows further canonical quantization of the theory with commutation relations for the new fields, which coincide with their classical Poisson-bracket ones.

From the explicit expression for this metric and condition (3) it follows that it is precisely Gauss' law which provides a natural definition of horizontal paths. Integration only over this class of paths means that each physical path (i.e. path in which all gauge-equivalent potentials are identified) gives rise to a unique, everywhere horizontal path in the orbit space. This is the best that can be achieved in the absence of a global gauge. An analogous statement has been proved in ref./13/ for the special case of the temporal gauge. We do not fix the gauge but instead solve explicitly the constraint equation for A_o and concentrate on the structure of the orbit space for thus reduced configuration space and especially on its nonstandard metric.

The explicit solution of the constraint equation together with the importance of the Belinfante tensor have been postulated in the minimal quantization method [18-21]. As we have seen, these steps have not only physical but also deep geometrical motivations.

3. *Gravitational waves in the linearized gravity*

Quantization of the linearized gravity gives an evidence about the importance of physically motivated assumptions for the small metric-tensor components to be neglected, which concerns the existence of gravitational waves in the conventional understanding of this problem.

Gravitational waves in Einstein theory are considered as quantum excitations of weak classical fields. In this context, construction of a gravity quantization scheme which is adequate to the problem of elementary excitations is important. From such a point of view the minimal quantization method with an explicit solution of the constraint equations [18-21] is distinguished among the large variety of gravitational field quantization approaches [22].

Consider Einstein theory

$$S = \int R \sqrt{-g}\; d^4x \qquad (4)$$

in a week field approximation

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad h_{\mu\nu} \ll 1. \qquad (5)$$

The Lagrangian then takes the form (up to $O(h^3)$-terms)

$$\mathcal{L}^{(2)} = \tfrac{1}{2}\,(h_{\mu\mu} \Box h_{\nu\nu} - h_{\mu\nu} \Box h_{\mu\nu}) + h_{\mu\nu}\partial_\nu\partial_\sigma h_{\mu\sigma} - h_{\mu\mu}\partial_\nu\partial_\sigma h_{\nu\sigma}$$

$$\mathcal{L}^{(2)}(h_{oo}) = h_{oo}(\partial_k\partial_l h_{kl} - \partial_k{}^2 h_{jj}) \qquad (6)$$

$$\mathcal{L}^{(2)}(h_{ol}) = -2\, \partial_o h_{ol}(\partial_k h_{kl} - \partial_l h_{jj}) - \partial_k h_{ol}(\partial_l h_{ok} - \partial_k h_{ol}).$$

This action contains constraints which introduce a transverse structure

$$\frac{\delta \mathcal{L}^{(2)}}{\delta h_{oo}} = 0 \;\Rightarrow\; \partial_k A^T_k = 0, \quad A^T_k = \partial_i h_{ki} - \partial_k h_{jj} \qquad (7)$$

with the corresponding equation of motion

$$\frac{\delta \mathcal{L}^{(2)}}{\delta h_{ol}} = 0 \;\Rightarrow\; \partial_o A^T_k = \partial_i(\partial_i h_{ok} - \partial_k h_{oi}). \qquad (8)$$

On the solutions of constraints (7),(8) Lagrangian (6) reads

$$\mathcal{L}^{(2)} = \tfrac{1}{2}\, \partial^\mu h_{lk}\, \mathfrak{P}(lk|lm)\, \partial_\mu h_{lm}\,,$$

$$\mathfrak{P}(lk|lm) = \delta_{lk}\delta_{lm} + \delta_{ll}\delta_{km} - \frac{\partial_l\partial_l}{\Delta}\,\delta_{km} - \frac{\partial_k\partial_m}{\Delta}\,\delta_{ll}\,, \qquad (9)$$

where the projection operator $\mathfrak{P}(lk|lm)$ can be considered as defining the distance in the space of dynamical field h_{lk} orbits with respect to infinitesimal gauge transformations

$$h_{lk} \longrightarrow h_{lk} + \partial_l\lambda_k + \partial_k\lambda_l\,.$$

From Lagrangian (9) canonical momenta are obtained

$$p_{rs}(x) = \mathfrak{P}(rs|lm)\, \partial_o h_{lm}(x),$$

which obey the following Poisson-bracket relations

$$\{\, h_{lm}(x),\, p_{rs}(y)\, \} = \mathfrak{P}(lm|rs)(x)\delta(x-y).$$

The energy-momentum tensor is obtained to be symmetric and gauge invariant

$$T^{(c)}_{\mu\nu} = \partial_\mu h_{lk}\, \mathfrak{P}(lk|lm)\, \partial_\nu h_{lm} - \tfrac{1}{2}\, g_{\mu\nu}\, \partial^\sigma h_{lk}\, \mathfrak{P}(lk|lm)\, \partial_\sigma h_{lm}\,.$$

It does not represent a full derivative and gives rise to a set of Poincare-group generators in which boost generators induce an additional gauge transformation of the dynamical fields

$$i[M_{lo}, h_{rs}(x)] = (x_o\partial_l - x_l\partial_o)\mathfrak{P}(rs|lm)h_{lm}(x) + (\partial_r \frac{1}{\Delta} p_{ls} + \partial_s \frac{1}{\Delta} p_{rl}).$$

This additional gauge transformation leads to a time-axis rotation that ensures the relativistic covariance of this manifestly non-covariant quantization procedure.

The basis in this space can be defined as

$$\mathfrak{P}(lk|lm) = \varepsilon^a_{lk}\,\varepsilon^a_{lm} \qquad (10)$$

$$\varepsilon^a_{lk}\,\mathfrak{P}(lk|lm)\,\varepsilon^b_{lm} = \delta^{ab}$$

$$\varepsilon^a_{lk}\,P_{lk} = 0,$$

where P_{lk} is the three-dimensional projection operator

$$P_{lk} = \delta_{lk} - \frac{\partial_l\partial_k}{\Delta}, \quad P_{lk} = e_l^{\alpha}e_k^{\alpha}, \quad e_l^{\alpha}P_{lk}e_k^{\beta} = \delta^{\alpha\beta}, \quad \alpha,\beta = 1,2.$$

Thus, the relevant polarizations are found to be

$$\varepsilon^1_{lk} = e_l^{\,1}e_k^{\,1} - e_l^{\,2}e_k^{\,2} \qquad \text{and} \qquad \varepsilon^2_{lk} = e_l^{\,1}e_k^{\,2}$$

and for the independent physical variables

$$h^a = \varepsilon^a_{lk}\,\mathfrak{P}(lk|lm)\,h_{lm} \qquad (11)$$

the free two-component scalar field Lagrangian is obtained

$$\mathcal{L} = \frac{1}{2}\,\partial^{\mu}h^a\,\partial_{\mu}h^a\,; \qquad (12)$$

hence, plane waves are present in the excitation spectrum of the linearized Einstein theory.

Therefore, minimal quantization of weak gravitational fields reproduces the radiational-gauge results together with the corresponding additional conditions which are in fact generated by the equations of motion for the nondynamical fields h_{oo} and h_{ol}.

For the original theory (4) without any additional assumptions about the fields minimal quantization consists in excluding nonphysical degrees of freedom through the *exact* solutions of their equations of motion (constraints). However, the coincidence of the linearized expansion of the action obtained with the one considered above (in the naive linearization scheme) is by no means obvious, the reason being the distinguished role of the Newton component g_{oo}. Thus, assuming that condition (5) concerns only dynamical fields h_{lk}, we are forced to consider also components h_{lo} as small variables because of the constraints, but no restrictions are imposed on the

Newton component h_{oo}. In the minimal quantization method this component is considered as a classical one and assumption $h_{oo} \ll 1$ is by no means motivated because we don't know the strength of the Newton potential the gravitational wave is interacting with.

To better realize the difference between these two cases, let us consider a simplified problem, namely constructing the effective Lagrangian $\mathcal{L}_{eff}(h_{lk})$ by explicitly solving the constraint equations for components h_{ok} and h_{lk} in Lagrangian (4) already expanded over these small fields but in an arbitrary Newton field g_{oo}.

To this end, we shall make use of the following expansion of the metric tensor $g_{\mu\nu}$ [23]

$$g = \exp(\hat{\varphi}/2)\,\exp(\hat{h})\,\exp(\hat{\varphi}/2),$$

where

$$\hat{\varphi} = \begin{pmatrix} h_{oo} & 0 \\ 0 & 0 \end{pmatrix}, \qquad \hat{h} = \begin{pmatrix} 0 & \eta_k \\ \eta_k & h_{lj} \end{pmatrix}.$$

As a result, up to second order in η_k and h_{lj} terms, the metric tensor components take the form

$$g_{oo} = e^{h_{oo}}\left(1 - \frac{1}{2}\eta_k^{\,2}\right) + \cdots \;; \qquad g^{oo} = e^{-h_{oo}}\left(1 - \frac{1}{2}\eta_k^{\,2}\right) + \cdots$$

$$g_{ol} = e^{\frac{h_{oo}}{2}}\,\eta_k\,(\delta_{lk} - \frac{1}{2}h_{lk}) + \cdots \;; \qquad g^{ol} = e^{-\frac{h_{oo}}{2}}\,\eta_k\,(\delta_{lk} + \frac{1}{2}h_{lk}) + \cdots$$

$$g_{lj} = -\delta_{lj} + h_{lj} - \frac{1}{2}(h^2)_{lj} + \frac{1}{2}\eta_l\eta_j + \cdots$$

$$g^{lj} = -\delta_{lj} - h_{lj} - \frac{1}{2}(h^2)_{lj} + \frac{1}{2}\eta_l\eta_j + \cdots$$

With this metric taken into account the Dirac Lagrangian [22]

$$\mathcal{L} = -\frac{1}{4}\sqrt{-g}\,(\partial_{\rho}g_{\mu\nu})\,(\partial_{\sigma}g_{\alpha\beta})\,\{(g^{\mu\alpha}g^{\nu\beta} - g^{\mu\nu}g^{\alpha\beta})g^{\rho\sigma} + 2(g^{\mu\rho}g^{\alpha\beta} - g^{\mu\alpha}g^{\rho\beta})g^{\varphi\sigma})\}$$

can be presented in the following form

$$\mathcal{L} = e^{-h_{oo}/2}A - e^{h_{oo}/2}(B + \partial_k C_k),$$

where A, B and C are complicated expressions depending on time and space derivatives of the dynamical fields h_{lj} (A and B) and generating constraint (7) (C_k). Therefore, consideration of the simplified model Lagrangian

$$\mathcal{L}^{M} = e^{-h_{oo}/2} (\partial_o h^a)^2 - e^{h_{oo}/2} (\partial_k h^a)^2 \qquad (13)$$

provides an idea about the spectrum of this theory. Lagrangian (13) coincides with Lagrangian (12) in the limit $h_{oo} \ll 1$ but with an arbitrary h_{oo} it leads to the equations

$$(\partial_k h^b)^2\, \partial_o^2 h^a + (\partial_o h^b)^2\, \partial_k^2 h^a = 0$$

with solutions $p = 0$ or $E = 0$, which is equivalent to rotation of the light cone by $\pi/4$ and does not represent a plane wave in the initial frame.

4. *Concluding remarks*

With the help of the genuine symplectic structure of the physical (orbit) space the theory of linearized gravitational field is formulated in a manifestly relativistic-covariant form providing its straightforward quantization with the same transformation properties of the quantized fields with respect to the Lorentz-group action as in the classical theory. The Lagrangian obtained describes an unconstrained hamiltonian system. Thus, in the path-integral construction one should not encounter difficulties connected with the singular nature of the original gauge-field Lagrangian such as the necessity of additional conditions and, consequently, the problem of equivalence of different gauges, gauge ambiguities, ghosts and so on.

The results obtained also point out the necessity of physically motivated assumptions about the small metric-tensor components to be neglected. Thus, with an arbitrary value of the Newton component g_{oo} the existence of plane waves in the excitation spectrum becomes problematic.

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References

1. L.D.Faddeev, Teor.Mat.Fiz., **1** (1969) 3.
2. P.A.M.Dirac, *Lectures in Quantum Mechanics*, Academic, New York, 1964
3. K.Sundermeyer, *Constrained Dynamics*, Lecture Notes in Physics, vol.169, Springer, Berlin, 1982.
4. J.E.Marsden, A.Weinstein, Rep.Math.Phys., **5** (1974) 121.
5. M.Puta, Lett.Math.Phys., **8** (1984) 189.
   M.J.Gotay, J.Math.Phys., **27** (1986) 2051.
6. J.Romano, R.Tate, Class.Quantum Grav., **6** (1989) 1487.
7. A.Ashtekar, G.T.Horowitz, Phys.Rev.D, **26** (1982) 3342.
8. M.J.Gotay, Class.Quantum Grav., **3** (1986) 487.
9. N.M.J.Woodhouse, *Geometric Quantization*, Clarendon, Oxford, 1980.
   V.Guillemin and S.Sternberg, Invent.Math., **67** (1982) 515.
10. I.Singer, Commun.Math.Phys., **60** (1978) 7.
11. V.Gribov, Nucl.Phys.B, **139** (1978) 1.
12. L.D.Faddeev and V.N.Popov, Phys.Lett.B, **25** (1967) 30.
13. J.L.Friedman and N.J.Papastamatiou, Nucl.Phys.B, **219** (1983) 125.
14. O.Babelon and C.M.Viallet, Commun.Math.Phys., **81** (1981) 515.
15. Indices $\mu,\nu,\rho,\sigma,\ldots$ vary from 0 to 3, indices $i,j,k,\ldots$ - from 1 to 3 and indices $\alpha,\beta,\gamma,\ldots$ take values 1 and 2.
16. N.Ilieva, L.Litov, in:"Selected Topics in QFT and Mathematical Physics", World Scientific, Singapore, 1990, p.239.
17. M.Grabiak, B.Muller and W.Greiner, Ann.Phys.(N.Y.), **172** (1986) 213.
18. N.Ilieva, V.N.Pervushin, Proc.8th Int.Sem.on the HEP Problems, Dubna,1986; JINR D1,2-86-668, Dubna, 1987, vol.1, p.69.
   V.N.Pervushin, Riv.Nuovo.Cim., **10**, no.8 (1985) 1.
19. N.Ilieva, Nguyen S.H., V.N.Pervushin, Sov.J.Nucl.Phys., **45** (1987) 1169.
20. Nguyen S.H., V.N.Pervushin, Fortschr.Phys, **37** (1989) 27.
21. N.Ilieva, Int.J.Mod.Phys.A, **4** (1989) 4567.
22. P.A.M.Dirac, Proc.Roy.Soc.A, **246** (1958) 333.
   R.Arnowitt, S.Deser, C.Misner, Phys.Rev., **117** (1960) 1596.
23. V.N.Pervushin, Teor.Mat.Fiz., **27** (1976) 16.

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