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NEW FORMULATION OF THE CLASSICAL DYNAMICS OF THE RELATIVISTIC STRING WITH MASSIVE ENDS

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1. INTRODUCTION

A dynamic basis of the hadron model[1] is a relativistic string with point masses at the ends[5]. Until now no general solutions have been derived to the equations describing the dynamics of a relativistic string with massive ends, therefore it seems to be of interest to consider a new mathematical formulation of that problem which would promote the investigation of its dynamics and derivation of new exact solutions.

The action functional for a relativistic string with point masses at the ends[1] results in equations of motion of the string and in boundary conditions that physically represent the equations of motion of two masses interacting through the string. An analogy arises between that system and classical electrodynamics with charges in which the field is described by the Maxwell equations with charges and the dynamics of charges interacting with the field is given by the Lorentz equations. Wheelear and Feynman[2], considering the action to propagate at a distance with a finite velocity, have eliminated the field variables from the equations of motion in electrodynamics, and have formulated the interaction between charges in terms of retarded and advanced propagation functions when there is no absorption and emission of the electromagnetic field.

For a system of a relativistic string with masses at the ends we may also utilize the principle of action at a distance to enable us to find equations of motion in terms of trajectories along which the masses are moving provided the string variables are eliminated. It is clear that owing to the problem being relativistic, it cannot be formulated within the equal-time formalism. In the simplest nonrelativistic limit we arrive at a system of two masses coupled by a linearly growing potential[3,4,11].

In this paper, we derive equations for the curvature $\mathbf{k}_i$ and torsion $\kappa_i(\tau)$ of world trajectories of masses tied by the relativistic string. The curvatures turn out to be constants, connected with the masses and tension of the string, and they together with torsions $\kappa_i(\tau)$ determine the curves[9] along which the masses are moving in the Minkowski space $E_2^1(t,x,y)$. Once these characteristics are known, we can determine the string coordinates $x^\mu(\tau,\sigma)$ as functions of parameters τ and σ up to shifts and rotations in the space E_2^1.

The equations define $\kappa_i(\tau)$ up to an arbitrary functions given in the interval $(0-\pi)$ which allow us to solve the Cauchy problem for that system of equations [6]. For constant torsions κ_{0i} admissible by the equations we obtain well-known helices along which the masses are moving; in this case the world surface of the string is a helicoid [5,6] in the Minkowski space E_2^1. In addition a new solution is also found with periodic torsions $\kappa_i(\tau) = \kappa_i(\tau + 2\pi)$ in II; it describes a more intricate motion than rotation of a stretched string in the (x, y) plane. The coordinates of the string are expressed through elliptic functions with a real-valued period. As shown in ref. [7], to find in classical dynamics corrections to a linearly growing potential between quarks connected via a relativistic string, it is necessary to know the solution to the boundary conditions for that system which contains transverse vibrations of the string. The solution that will be obtained meets this requirement and may be used for determining corrections to a linearly growing potential at the classical level.

Sections 2 and 3 are devoted to the derivation of equations for trajectories of the string massive ends in the space E_2^1.

2. EQUATIONS OF MOTION AND BOUNDARY CONDITIONS

Consider the dynamics of a relativistic string with point masses m_1 and m_2 at the ends. The world surface M_1^1 with coordinates $x^\mu(\tau,\sigma)$, $\mu = 0, 1, \ldots, d-1$ swept out by the strings in the course of motion through the Minkowski space is an extremal of the functional of the action [1,6]:

$$S = -\gamma \int_{\tau_1}^{\tau_2} d\tau \int_{\sigma_1(\tau)}^{\sigma_2(\tau)} d\sigma \sqrt{(\dot{x}x')^2 - \dot{x}^2 x'^2} + \sum_{i=1}^{2} m_i \int_{\tau_1}^{\tau_2} d\tau \sqrt{\left(\frac{dx^\mu(\tau,\sigma)}{d\tau}\right)^2}, \tag{2.1}$$

where the first term is the action of a massless relativistic string; γ is the string tension, $\tau = u^0$ and $\sigma = u^1$ are parameters on the surface M_1^1, and the derivatives are as follows:

$$\dot{x}^{\mu} = \frac{\partial x^{\mu}(\tau, \sigma)}{\partial \tau}, \quad x'^{\mu} = \frac{\partial x^{\mu}(\tau, \sigma)}{\partial \sigma}$$

$$\frac{d x^{\mu}(\tau, \sigma_i(\tau))}{d\tau} = \dot{x}^{\mu}(\tau, \sigma_i) + \dot{\sigma}_i(\tau)\, x'^{\mu}(\tau, \sigma_i).$$

The motion of the string ends in the plane of the parameters τ and σ is described by the functions $\sigma_i(\tau)$, i = 1, 2. As for massless string, the action (2.1) is invariant under nondegenerate changes of variables, $\tau = \bar{\tau}(\tau, \sigma)$ and $\sigma = \bar{\sigma}(\tau, \sigma)$ on the surface M_1^1, which allow us to eliminate two of the three independent components of the metric induced on M_1^1:

$$g_{ij} = \eta_{\mu\nu} \frac{\partial x^{\mu}}{\partial u^i} \cdot \frac{\partial x^{\nu}}{\partial u^j}, \quad i, j = 0, 1. \tag{2.2}$$

It is convenient to introduce isothermal coordinates τ and σ in terms of which the metric (2.2) is diagonal and traceless

$$g_{00} + g_{11} = 0, \quad g_{01} = g_{10} = 0. \tag{2.3}$$

The flat metric $\eta_{\mu\nu}$ of the enveloping d-dimensional space-time is taken with the signature η = (+, -, -,...-).

Variation of the action (2.1) with respect to $x^{\mu}(\tau, \sigma)$ gives equations of motion linear in the gauge (2.3)

$$\ddot{x}^{\mu}(\tau, \sigma) - x''^{\mu}(\tau, \sigma) = 0 \tag{2.4}$$

and nonlinear boundary conditions at the string ends

$$m_1 \frac{d}{d\tau}\left[\frac{\dot{x}^{\mu}(\tau, \sigma_1) + \dot{\sigma}_1(\tau)\, x'^{\mu}(\tau, \sigma_1)}{\sqrt{(1 - \dot{\sigma}_1^2(\tau)) \cdot \dot{x}^2(\tau, \sigma_1)}}\right] = \gamma[x'^{\mu}(\tau, \sigma_1) + \dot{\sigma}_1(\tau)\, \dot{x}^{\mu}(\tau, \sigma_1)] \tag{2.5}$$

$$m_2 \frac{d}{d\tau}\left[\frac{\dot{x}^{\mu}(\tau, \sigma_2) + \dot{\sigma}_2(\tau)\, x'^{\mu}(\tau, \sigma_2)}{\sqrt{(1 - \dot{\sigma}_2^2(\tau))\dot{x}^2(\tau, \sigma_2)}}\right] = -\gamma[x'^{\mu}(\tau, \sigma_2) + \dot{\sigma}_2(\tau)\, \dot{x}^{\mu}(\tau, \sigma_2)].$$

Varying (2.1) with respect to $\sigma_i(\tau)$ we arrive at the same equations (2.5), therefore the functions $\sigma_i(\tau)$ $(i = 1, 2)$ are not dynamical variables[8].

A general solution to the equations of motion (2.4) and gaige conditions (2.3) is of the form

$$x^\mu(\tau, \sigma) = \frac{1}{2}[\psi_+^\mu(u^+) + \psi_-^\mu(u^-)], \quad u^+ = \tau + \sigma, \quad u^- = \tau - \sigma, \tag{2.6}$$

where $\psi_+'^\mu(u^+)$ and $\psi_-'^\mu(u^-)$ are two isotopic vectors,

$$\psi_+'^2(u^+) = 0, \quad \psi_-'^2(u^-) = 0 \tag{2.7}$$

tangent to the string world surface M_1^1. The conditions (2.7) may be satisfied if we represent $\psi_\pm'^\mu$ through the following expansions

$$\psi_+'^\mu(u^+) = \frac{A_+(u^+)}{\sqrt{\sum_{a=2}^{d-1} f_a'^2(u^+)}}[e_0^\mu + e_1^\mu \frac{1}{2}\sum_{a=2}^{d-1} f_a^2(u^+) + \sum_{a=2}^{d-1} e_a^\mu f_a(u^+)]$$

$$\psi_-'^\mu(u^-) = \frac{A_-(u^-)}{\sqrt{\sum_{a=2}^{d-1} g_a'^2(u^-)}}[e_0^\mu + e_1^\mu \frac{1}{2}\sum_{a=2}^{d-1} g_a^2(u^-) + \sum_{a=2}^{d-1} e_a^\mu g_a(u^-)], \tag{2.8}$$

where the constant basis e_0^μ, e_1^μ, e_a^μ is formed from two isotropic vectors e_0^μ, e_1^μ, $e_0^2 = 0$, $e_1^2 = 0$, $(e_0 e_1) = 1$ and $(d-1)$ space-like vectors e_a^μ, $(e_a \cdot e_b) = -\delta_{ab}$, $(e_0 e_a) = (e_1 e_a) = 0$ $(a = 2, 3, \ldots, d-1)$. The representations (2.8) fully determine the world surface of a relativistic string without boundary in a d-dimensional space-time and allow us to construct its basic quadratic forms.

In the space E_2^1 with $d = 3$ and with $f_2(u^+) = f(u^+)$ and $g_2(u^-) = g(u^-)$ in (2.8) we obtain for the only nonzero component of the metric tensor g_{00} (2.3)

$$g_{00} = \dot{x}^2(u^+, u^-) = \frac{1}{2}(\psi_+'(u^+)\, \psi_-'(u^-)) = \frac{A_+(u^+) A_-(u^-)}{4f'(u^+) g'(u^-)}[f(u^+) - g(u^-)]^2. \tag{2.9}$$

As is known[1], in a three-dimensional space E_2^1 the Gauss equation for the world surface M_1^1 of a relativistic string reduces to the Liouville equation for $g_{00} = \dot{x}^2(u^+, u^-)$

$$\frac{\partial^2 \ln \dot{x}^2(u^+, u^-)}{\partial u^+ \partial u^-} = \frac{A_+(u^+)\, A_-(u^-)}{2\dot{x}^2(u^+, u^-)} \tag{2.10}$$

and (2.9) is the general solution to this equation.

Computation of the coefficients of the second quadratic form

$$b_{a|ij} = (n_a \cdot \frac{\partial^2 x}{\partial u^i \partial u^j}) \qquad i, j = 0, 1; \quad a = 2, 3, \ldots, d-1 \tag{2.11}$$

requires a special choice of the orthonormalized system of unit normals $n_a^\mu(u^+, u^-)$

$$(n_a \cdot \frac{\partial x}{\partial u^i}) = 0, \quad (n_a \cdot n_b) = -\delta_{ab} \tag{2.12}$$

to the surface M_1^1, which together with tangent vectors $\dot{x}^\mu$ and x'^μ constitute a moving frame of reference. This can most easily be done for d = 3 when the field of normals (2.12) contains only one vector $n^\mu(u^+, u^-)$ that may be constructed in terms of the vectors $\dot{x}^\mu$ and x'^μ as follows:

$$n^\mu(u^+, u^-) = \frac{[\dot{x} \cdot x']}{\dot{x}^2(u^+, u^-)}, \tag{2.13}$$

where $[\dot{x} \cdot x'] = \epsilon^{\mu\nu\rho} \dot{x}_\nu x'_\rho$, and $\epsilon^{\mu\nu\rho}$ is a totally antisymmetric unit tensor. Inserting the relations (2.8) with d = 3 into (2.13) and considering that $[e_0 \cdot e_1] = e_2$, $[e_1 \cdot e_2] = -e_1$, $[e_0 \cdot e_2] = e_0$ we arrive at the expansion of the normal n^μ over the isotropic basis $e_0^\mu, e_1^\mu, e_2^\mu$:

$$n^\mu(u^+, u^-) = \frac{2e_0^\mu + f(u^+)\, g(u^-)\, e_1^\mu + [\, f(u^+) + g(u^-)\,]\, e_2^\mu}{f(u^+) - g(u^-)}. \tag{2.14}$$

Using the expansions (2.8) with d = 3 and (2.14) for coefficients of the second quadratic form $b_{2|ij} = b_{ij}$ of the string world surface M_1^1, according to (2.11) we obtain

$$b_{00} = b_{11} = \frac{A_+(u^+) - A_-(u^-)}{2}; \quad b_{01} = b_{10} = \frac{A_+(u^+) + A_-(u^-)}{2} \tag{2.15}$$

The first equality of (2.15) shows that the string world surface is minimal, i.e., its mean curvature is zero [9],

$$H = \frac{1}{2} b_{ij} g^{ij} = \frac{b_{00} - b_{11}}{2 g_{00}} = 0; \qquad (2.16)$$

it is assumed that for any point of the surface M_1^1 there holds the condition $g_{00} = \dot{x}^2 > 0$ or $f'(u^+) g'(u^-) > 0$ and $f(u^+) \neq g(u^-)$ as follows from (2.9).

For any arbitrary dimensionality d of the enveloping space the condition of minimality (2.16) in the coordinate system (2.3) should be replaced by the relations

$$b_{a|00} = b_{a|11}, \quad a = 2, 3, \ldots, d-1. \qquad (2.17)$$

For a relativistic string with massive ends the coordinates of the minimal surface M_1^1 obey the nonlinear boundary conditions (2.5). Substituting (2.6) into (2.5) for the isotropic vectors (2.7) and functions $\sigma_i(\tau)$ we get

$$(-1)^{i+1} m_i \frac{d}{d\tau} \left\{ \frac{\psi_+'^{\mu}(u_i^+)\, \dot{u}_i^+ + \psi_-'^{\mu}(u_i^-)\, \dot{u}_i^-}{\sqrt{\frac{1}{2} \psi_+'(u^+) \psi_-'(u^-) \dot{u}_i^+ \dot{u}_i^-}} \right\} =$$

$$= \gamma [\psi_+'^{\mu}(u^+)\, \dot{u}_i^+ - \psi_-'^{\mu}(u^-)\, \dot{u}_i^-], \quad u_i^+ = \tau + \sigma_i(\tau), \quad u_i^- = \tau - \sigma_i(\tau) \qquad (2.18)$$

For each of value $i = 1,2$, only $d - 1$ of the d equations (2.18) are independent of each other since the projections of the system (2.18) onto the vectors tangent to the surface M_1^1

$$\dot{x}^{\mu}(u^+, u^-) = \frac{\psi_+'^{\mu}(u^+) + \psi_-'^{\mu}(u^-)}{2}; \quad x'(u^+, u^-) = \frac{\psi_+'^{\mu}(u^+) - \psi_-'^{\mu}(u^-)}{2}$$

coincide. Thus, $2(d-1)$ equations (2.18) contain, as $2d$ unknown quantities, two functions $\sigma_i(\tau)$ and $2(d-1)$ independent components of the isotropic vectors $\psi_{\pm}'^{\mu}$ expressed, according to (2.8), through $A_{\pm}$, f_a, g_a which are, as we see from the boundary conditions (2.18), functionals of $\sigma_i(\tau)$. The functions $\sigma_i(\tau)$ may be fixed from the invariance of equations (2.7) and (2.18) under conformal transformations of the parameters $\bar{u}^{\pm} = \bar{u}^{\pm}(u^{\pm})$, where $\bar{u}^{\pm}$ are two arbitrary functions of one variable. So, the definition of system (2.18) may be supplemented by imposing two auxiliary conditions fixing the parameters u^+, u^- or τ, σ on the surface M_1^1. Taking the equalities

$$A_+(u^+) = A_-(u^-) = A = \text{const}. \tag{2.19}$$

to be gauge conditions, we fix the functions $\sigma_i(\tau)$ in eqs. (2.18). Indeed, consider projections (2.18) onto normals n_a^μ, (a = 2, 3,..., d - 1) and taking account of (2.6) and (2.11) we obtain 2(d - 1) equalities

$$[1 + \dot{\sigma}_i^2(\tau)]\, b_{a|00} + 2\dot{\sigma}_i(\tau)\, b_{a|01} = 0. \tag{2.20}$$

At d = 3 $n_2^\mu = n^\mu$, $b_{2|ij} = b_{ij}$ and from (2.15) it follows that gauge (2.19) fixes the asymptotic coordinates τ and σ on the world surface of a relativistic string

$$b_{00} = (n\ddot{x}) = 0, \quad b_{01} = (n\dot{x}') = A. \tag{2.21}$$

From (2.20) and (2.21) it follows that

$$\sigma_i'(\tau) = 0, \quad i = 1, 2. \tag{2.22}$$

Consequently, σ_i are constants and we put $\sigma_1 = 0$ and $\sigma_2 = \pi$. For d = 4 from (2.20) we may also derive eqs. (2.22) using the arbitrariness in choice of the field of normals n_a^μ corresponding to the group of transformations SO(d - 2).

Indeed, utilizing the expansions (2.8) for the vectors in the gauge (2.19) we get

$$\ddot{x}^2 + \dot{x}'^2 = -A^2; \; (\ddot{x}\dot{x}') = 0. \tag{2.23}$$

Therefore, when $d \geq 4$, we may, without loss of generality, direct the normals n_2^μ and n_3^μ along two mutually orthogonal spacelike vectors $x^\mu_{;01}$ and $x^\mu_{;00}$, respectively: $n_3^\mu = x^\mu_{;00}$ and $n_2^\mu = x^\mu_{;01}$. As a result, the coefficients of the second quadratic form (2.11) become equal:

$$\begin{aligned} & b_{2|00} = 0; \quad b_{2|01} = -\sqrt{-x^2_{;01}} \\ & b_{3|00} = -\sqrt{-x^2_{;00}}; \quad b_{3|01} = 0 \\ & b_{a|ij} = 0; \quad a = 4, 5, \dots, d-1. \end{aligned} \tag{2.24}$$

Here semicolon stands for the covariant differentiation with respect to the metric tensor (2.2).
With the latter equalities, eqs. (2.20) for a = 4,,5,...,d - 1

are identically satisfied, and for a = 2,3 take the form

$$[1+\dot{\sigma}_i^2(\tau)]\, x^2_{;00}(u_i^+, u_i^-) = 0; \quad 4\dot{\sigma}_i^2(\tau)\, x^2_{;01}(u_i^+, u_i^-) = 0.$$

Hence, owing to (23) we obtain eqs. (2.22) and, setting $\sigma_i = (0, \pi)$ the conditions

$$x^2_{;00}(\tau, 0) = x^2_{;00}(\tau, \pi) = 0. \tag{2.25}$$

The $2(d-1)$ functions $f_a(u^+)$ and $g_a(u^-)$, $a = 2,3,\ldots, d-1$ remaining upon gauge (2.19) will obey two conditions (2.25) and $2(d-4)$ relations (2.24) when $d \geq 4$, and also two projections of the boundary conditions (2.18) on the vectors $\dot{x}^\mu$ and x'^μ tangent to the surface. For projecting it is convenient to employ the conditions (2.5) that with the use of (2.22) may be written in the form

$$\ddot{x}^\mu(\tau, 0) - \frac{(\dot{x}\ddot{x})}{\dot{x}^2}\dot{x}^\mu(\tau, 0) = \frac{\gamma}{m_1}\sqrt{\dot{x}^2}\, x'^\mu(\tau, 0), \; \sigma = 0$$

$$\ddot{x}^\mu(\tau, \pi) - \frac{(\dot{x}\ddot{x})}{\dot{x}^2}\dot{x}^\mu(\tau, \pi) = -\frac{\gamma}{m_2}\sqrt{\dot{x}^2}\, x'^\mu(\tau, \pi), \; \sigma = \pi. \tag{2.26}$$

Taking advantage of the conformal gauge (2.23) and equations of motion (2.4) it is easy to show that the projections (2.26) onto $\dot{x}^\mu(\tau, \sigma_i)$, $i = 1,2$ vanish, and projections onto $x'^\mu(\tau, \sigma_i)$ give the equations

$$\frac{\partial}{\partial\sigma}\left(\frac{1}{\sqrt{\dot{x}^2(\tau, \sigma)}}\right)\Big|_{\sigma=\sigma_i} = (-1)^i \frac{\gamma}{m_i}, \quad i = 1, 2. \tag{2.27}$$

For d = 3, eqs.(2.27) results in two equations for the functions $f(u^+)$ and $g(u^-)$, i.e. we arrive at the boundary value problem for the Liouville equation (2.10).

For d = 4 eqs.(2.27) are to be supplemented with conditions (2.25) for four unknown functions $f_a(u^+)$, $g_a(u^-)$ (a = 2,3).

3. EQUATIONS FOR TRAJECTORIES OF STRING MASSIVE ENDS IN A THREE-DIMENSIONAL SPACE-TIME

As has been shown above, in a 3-dimensional space coordinates (2.6) of the minimal surface of a relativistic string with massive ends in the representation (2.8) and gauge (2.19) are defined by two functions $f(u^+)$ and $g(u^-)$ that obey the boundary conditions (2.27). Inserting the general solution (2.9) of the Liouville equation (2.10) into (2.27) we obtain the system of two ordinary differential equations with deviating arguments for the functions $f(\tau)$ and $g(\tau)$:

$$\frac{d}{d\tau} \ln \frac{g'(\tau)}{f'(\tau)} + 2\,\frac{f'(\tau) + g'(\tau)}{f(\tau) - g(\tau)} = \frac{\gamma}{m_1}\,\frac{|f(\tau) - g(\tau)|}{\sqrt{f'(\tau)\,g'(\tau)}}\,|A| \tag{3.1}$$

$$\frac{d}{d\tau} \ln \frac{g'(-)}{f'(+)} + 2\,\frac{f'(+) + g'(-)}{f(+) - g(-)} = -\frac{\gamma}{m_2}\,\frac{|f(+) - g(-)|}{\sqrt{f'(+)\,g'(-)}}\,|A|, \tag{3.2}$$

where $g(-) = g(\tau - \pi)$, $f(+) = f(\tau + \pi)$. Shifting in (3.2) τ to $\tau - \pi$ we get

$$\frac{d}{d\tau} \ln \frac{g'(\tau - 2\pi)}{f'(\tau)} + 2\,\frac{f'(\tau) + g'(\tau - 2\pi)}{f(\tau) - g(\tau - 2\pi)} = -\frac{\gamma}{m_2}\,\frac{|f(\tau) - g(\tau - 2\pi)|}{\sqrt{f'(\tau)\,g'(\tau - 2\pi)}}\,|A|.$$

Here will be used the notation $g(\tau - 2\pi) = g(\cdot)$.

For $m_1 = m_2 = 0$ the system (3.1), (3.2) has periodic solutions $g(\tau) = f(\tau)$, $f(\tau) = f(\tau + 2\pi)$ that according to (2.9) violate the minimality condition (2.16) at the points $\sigma = \sigma_i$ ($i = 1,2$), and conversely, if one of the functions, either $f(\tau)$ or $g(\tau)$, is periodic, the other is also periodic and $m_1 = m_2 = 0$. Therefore, periodic solutions to eqs.(3.1) and (3.2) can exist only for a massless string /8/.

Further we may prove that the system (3.1), (3.2′) does not change under the Moebius transformation /12/ of the functions $f(\tau)$ and $g(\tau)$:

$$f(u^+) \Rightarrow \frac{af(u^+) + b}{cf(u^+) + d}, \quad g(u^-) \Rightarrow \frac{ag(u^-) + b}{cg(u^-) + d}, \quad ad - bc = 1, \tag{3.3}$$

which is a consequence of the relativistic invariance of the theory since under the Lorentz transformations of vectors $\psi_\pm'^{\mu}$ and also, according to (2.8), the vectors of the isotropic basis e_0^μ, e_1^μ, e_2^μ, the functions $f(u^+)$ and $g(u^-)$ undergo the transformations (3.3). Therefore relativistically invariant expressions, for instance (2.9), in terms of the functions $f(\tau)$ and $g(\tau)$ should be invariants with respect to (3.3).

Now let us demonstrate that the minimal surface M_1^1 is fully determined by the world trajectories $x^\mu(\tau, \sigma_i)$ of the massive ends. To this end, for $d = 3$ we shall describe the trajectories in terms of geometric invariants, curvatures k_i and torsions κ_i .As is well-known[9], these invariants define a curve in a three-dimensional space up to its position. In general, the curvature of a space-like curve $x^\mu(\tau)$ is given by the following expression[9]

$$k(\tau) = \frac{1}{\dot{x}^2(\tau)} \sqrt{\frac{(\dot{x}\ddot{x})^2}{\dot{x}^2} - \ddot{x}^2}.$$

Substituting the l.h.s. of eqs.(2.26) for $x^\mu(\tau, \sigma_i)$ $i = 1, 2$, into this formula and using the conditions (2.3) we obtain

$$k_i = \frac{\gamma}{m_i} \quad i = 1, 2. \tag{3.4}$$

Torsion of an arbitrary space-like curve $x^\mu(\tau)$ is defined by the formula[9]

$$\kappa(\tau) = \frac{\epsilon_{\mu\nu\sigma} \dot{x}^\mu \ddot{x}^\nu \dddot{x}^\sigma}{(\dot{x}\ddot{x})^2 - \dot{x}^2 \cdot \ddot{x}^2}.$$

Differentiating eqs.(2.26) with respect to τ and inserting the expressions for $\ddot{x}^\mu(\tau, \sigma_i)$ and $\dddot{x}^\mu(\tau, \sigma_i)$ $(i = 1, 2)$, we arrive at the torsions of the trajectories

$$\kappa_i(\tau) = \frac{\epsilon_{\mu\nu\sigma} \dot{x}^\mu \dot{x}'^\nu \dot{x}'^\sigma}{(\dot{x}^2(\tau, \sigma_i))^2}$$

which, owing to the definitions (2.11), (2.13) and condition (2.21), are reduced to the form

$$\kappa_i(\tau) = \frac{A}{\dot{x}^2(\tau, \sigma_i)}, \quad i = 1, 2. \tag{3.5}$$

Substituting $\dot{x}^2(\tau, \sigma_i)$ from (2.9) into (3.5) we obtin the expressions for torsions

$$\kappa_1(\tau) = \frac{4f'(\tau)\, g'(\tau)}{A[\, f(\tau) - g(\tau)]^2} \tag{3.6}$$

$$\kappa_2(\tau) = \frac{4f'(+)\, g'(-)}{A[\, f(+) - g(-)]^2} \quad \text{or} \quad \kappa_2(\tau - \pi) = \frac{4f'(\tau)\, g'(\cdot)}{A[f(\tau) - g(\cdot)]^2} \tag{3.7}$$

invariant under the transformations (3.3).

Formulae (3.6), (3.7) together with eqs.(3.1), (3.2) allow us to express the functions $f(\tau)$ and $g(\tau)$ in terms of the torsions $\kappa_i(\tau)$ as follows. Calculating from (3.6), (3.7) the differences of the functions

$$\frac{1}{|f(\tau)-g(\tau)|}=\frac{\sqrt{A\kappa_1(\tau)}}{2\sqrt{f'(\tau)\,g'(\tau)}}\;;\;\frac{1}{|f(\tau)-g(\cdot)|}=\frac{\sqrt{A\kappa_2(\tau-\pi)}}{2\sqrt{f'(\tau)\,g'(\cdot)}} \tag{3.8}$$

and then inserting them into the boundary conditions (3.1), (3.2) with allowance made for (3.4), we get

$$\begin{aligned}&\frac{d}{d\tau}\ln\frac{g'(\tau)}{f'(\tau)}+\epsilon_1\sqrt{A\kappa_1(\tau)}\left(\sqrt{\frac{f'(\tau)}{g'(\tau)}}+\sqrt{\frac{g'(\tau)}{f'(\tau)}}\right)=2k_1\sqrt{\frac{A}{\kappa_1(\tau)}}\,,\\&\frac{d}{d\tau}\ln\frac{g'(\cdot)}{f'(\tau)}+\epsilon_2\sqrt{A\kappa_2(-)}\left(\sqrt{\frac{f'(\tau)}{g'(\cdot)}}+\sqrt{\frac{g'(\cdot)}{f'(\tau)}}\right)=-2k_2\sqrt{\frac{A}{\kappa_2(-)}}\,,\end{aligned} \tag{3.9}$$

where ϵ_i $i = 1,2$ are the signs of the products $f'(\tau)[f(\tau)-g(\tau)]$ and $f'(\tau)[f(\tau)-g(\cdot)]$, respectively. Taking the logarithm and differentiating with respect to τ, formulas (3.6), (3.7) with the use of (3.8) are transformed to

$$\begin{aligned}&\frac{d}{d\tau}\ln[f'(\tau)\,g'(\tau)]-\epsilon_1\sqrt{A\kappa_1(\tau)}\left(\sqrt{\frac{f'(\tau)}{g'(\tau)}}-\sqrt{\frac{g'(\tau)}{f'(\tau)}}\right)=\frac{\dot\kappa_1(\tau)}{\kappa_1(\tau)}\\&\frac{d}{d\tau}\ln[f'(\tau)\,g'(\cdot)]-\epsilon_2\sqrt{A\kappa_2(-)}\left(\sqrt{\frac{f'(\tau)}{g'(\cdot)}}-\sqrt{\frac{g'(\cdot)}{f'(\tau)}}\right)=\frac{\dot\kappa_2(\tau-\pi)}{\kappa_2(\tau-\pi)}\,.\end{aligned} \tag{3.10}$$

The sum and difference of (3.9) and (3.10) give the following system of equations for the first boundary

$$\begin{aligned}&2\frac{d}{d\tau}\left(\frac{1}{\sqrt{f'(\tau)}}\right)=\left[\sqrt{\kappa_1(\tau)}\,\frac{d}{d\tau}\left(\frac{1}{\sqrt{\kappa_1(\tau)}}\right)+\kappa_1\sqrt{\frac{A}{\kappa_1(\tau)}}\right]\frac{1}{\sqrt{f'(\tau)}}-\epsilon_1\sqrt{\frac{A\kappa_1(\tau)}{g'(\tau)}}\\&2\frac{d}{d\tau}\left(\frac{1}{\sqrt{g'(\tau)}}\right)=\left[\sqrt{\kappa_1(\tau)}\,\frac{d}{d\tau}\left(\frac{1}{\sqrt{\kappa_1(\tau)}}\right)-\kappa_1\sqrt{\frac{A}{\kappa_1(\tau)}}\right]\frac{1}{\sqrt{g'(\tau)}}+\epsilon_1\sqrt{\frac{A\kappa_1(\tau)}{f'(\tau)}}\end{aligned} \tag{3.11}$$

and for the second boundary:

$$2\frac{d}{d\tau}\left(\frac{1}{\sqrt{f'(\tau)}}\right)=\left[\sqrt{\kappa_2(-)}\,\frac{d}{d\tau}\left(\frac{1}{\sqrt{\kappa_2(-)}}\right)-k_2\sqrt{\frac{A}{\kappa(-)}}\right]\frac{1}{\sqrt{f'(\tau)}}-\epsilon_2\sqrt{\frac{A\kappa_2(-)}{g'(\cdot)}}$$

$$2\frac{d}{d\tau}\left(\frac{1}{\sqrt{g'(\cdot)}}\right)=\left[\sqrt{\kappa_2(-)}\,\frac{d}{d\tau}\left(\frac{1}{\sqrt{\kappa_2(-)}}\right)+k_2\sqrt{\frac{A}{\kappa_2(-)}}\right]\frac{1}{\sqrt{g'(\cdot)}}+\epsilon_2\sqrt{\frac{A\kappa_2(-)}{f'(\tau)}}. \qquad (3.12)$$

And finally, eliminating $1/\sqrt{f'(\tau)}$ and then $1/\sqrt{g'(\tau)}$ from (3.11) we arrive at the equations which connect $f(\tau)$ and $g(\tau)$ with the torsion $\kappa_1(\tau)$:

$$D(f(\tau))=D\left(\int^{\tau}\sqrt{A\kappa_1(\eta)}\,d\eta\right)+\frac{\kappa_1(\tau)}{2}\left(1-\frac{k_1^2}{\kappa_1^2(\tau)}\right)-2k_1\frac{d}{d\tau}\sqrt{\frac{A}{\kappa_1(\tau)}}$$

$$D(g(\tau))=D\left(\int^{\tau}\sqrt{A\kappa_1(\eta)}\,d\eta\right)+\frac{\kappa_1(\tau)}{2}\left(1-\frac{k_1^2}{\kappa_1^2(\tau)}\right)+2k_1\frac{d}{d\tau}\sqrt{\frac{A}{\kappa_1(\tau)}} \qquad (3.13)$$

The same procedure applied to eqs.(3.12) allows us to express $f(\tau)$ and $g(\tau-2\pi) = g(\cdot)$ in terms of $\kappa_2(\tau-\pi) = \kappa_2(-)$

$$D(f(\tau))=D\left(\int^{\tau}\sqrt{A\kappa_2(-)}\,d\eta\right)+\frac{\kappa_2(-)}{2}\left(1-\frac{k_2^2}{\kappa_2^2(-)}\right)+2k_2\frac{d}{d\tau}\sqrt{\frac{A}{\kappa_2(-)}}$$

$$D(g(\tau-2\pi))=D\left(\int^{\tau}\sqrt{A\kappa_2(-)}\,d\eta\right)+\frac{\kappa_2(-)}{2}\left(1-\frac{k_2^2}{\kappa_2^2(-)}\right)-2k_2\frac{d}{d\tau}\sqrt{\frac{A}{\kappa_2(-)}}\,. \qquad (3.14)$$

In formulae (3.13) and (3.14) we made use of the Schwarz derivative invariant with respect to the transformations (3.3)

$$D(f(\tau))=\frac{f'''(\tau)}{f'(\tau)}-\frac{3}{2}\left(\frac{f''(\tau)}{f'(\tau)}\right)^2=-2\sqrt{f'(\tau)}\,\frac{d^2}{d\tau^2}\left(\frac{1}{\sqrt{f'(\tau)}}\right). \qquad (3.15)$$

Thus, the functions $f(\tau)$, $g(\tau)$ and therefore according to (2.6) and (2.8) coordinates of the minimal surface $x^{\mu}(\tau,\sigma)$ are defined by the torsions $\kappa_i(\tau)$ of the world trajectories of a string with massive ends.

Eliminating $D(f)$ and $D(g)$ from the four eqs.(3.13), (3.15) we obtain for the torsions $\kappa_i(\tau)$ $(i = 1, 2)$ the following two differential equations of second order with shifted arguments:

$$D(f(\tau)) = D\left(\int^{\tau}\sqrt{A\kappa_1(\eta)}\,d\eta\right) + \frac{\kappa_1(\tau)}{2}\left(1-\frac{k_1^2}{\kappa_1^2(\tau)}\right) - 2k_1\frac{d}{d\tau}\sqrt{\frac{A}{\kappa_1(\tau)}} =$$

$$= D\left(\int^{\tau}\sqrt{A\kappa_2(-)}\,d\eta\right) + \frac{\kappa_2(-)}{2}\left(1-\frac{k_2^2}{\kappa_2^2(-)}\right) + 2k_2\frac{d}{d\tau}\sqrt{\frac{A}{\kappa_2(-)}}. \tag{3.16}$$

$$D(g(\tau)) = D\left(\int^{\tau}\sqrt{A\kappa_1(\eta)}\,d\eta\right) + \frac{\kappa_1(\tau)}{2}\left(1-\frac{k_1^2}{\kappa_1^2(\tau)}\right) + 2k_1\frac{d}{d\tau}\sqrt{\frac{A}{\kappa_1(\tau)}} =$$

$$= D\left(\int^{\tau}\sqrt{A\kappa_2(+)}\,d\eta\right) + \frac{\kappa_2(+)}{2}\left(1-\frac{k_2^2}{\kappa_2^2(+)}\right) - 2k_2\frac{d}{d\tau}\sqrt{\frac{A}{\kappa_2(+)}}. \tag{3.17}$$

The system (3.16)-(3.17) is of fundamental importance in studying the world surfaces of a relativistic string with massive ends in the space E_2^1. From this system it follows, for instance, that in the range $0 < \tau < \pi$ the torsions $\kappa_i(\tau)$ are arbitrary functions and are defined only by initial conditions[6] (by the initial position $x^{\mu}(0,\sigma)$ and initial velocity $\dot{x}^{\mu}(0,\sigma)$ of the string $0 < \sigma < \pi$). Continuation of these functions beyond the interval $0 < \tau < \pi$ is made by the integrals of eqs. (3.16) and (3.17), and two conditions of smoothness at the ends $\sigma_i = 0, \pi$ for the continued functions $\kappa_i(\tau)$ may always be fulfilled with the four arbitrary constants.

The simplest solution to eqs.(3.16)-(3.17) are constant torsions $\kappa_i(\tau) = \kappa_{i0}$ when the ends of the string are moving along helices obeying the following conditions[6]:

$$\kappa_{10}\left(1-\frac{k_1^2}{\kappa_{10}^2}\right) = \kappa_{20}\left(1-\frac{k_2^2}{\kappa_{20}^2}\right). \tag{3.18}$$

In this case we obtain from eqs.(3.13) and (3.14) the equalities

$$D(g(\tau)) = D(f(\tau)) = D(g(\tau-2\pi)), \tag{3.19}$$

which, in accordance with the property of the Schwarz derivative[10] , imply that the functions $f(\tau)$, $g(\tau)$ and $g(\tau - 2\pi)$ are related by the Moebius transformation:

$$g(\tau) = \frac{\alpha_1 f(\tau) + \beta_1}{\gamma_1 f(\tau) + \delta_1} = \frac{\alpha_2 g(\tau - 2\pi) + \beta_2}{\gamma_2 g(\tau - 2\pi) + \delta_2} . \qquad (3.20)$$

The constant coefficients in (3.20) α_i, β_i, γ_i, δ_i obey the normalization condition $\alpha_i \delta_i - \beta_i \gamma_i = 1$ and two relations following from the boundary conditions (3.1) and (3.2′). The world surface M_1^1 of a relativistic string with massive ends turns out to be a helicoid in the space E_2^1 [6].

4. CONCLUSION

It has been shown that the world surface of a relativistic string with massive ends is completely defined by trajectories of the ends. In a three-dimensional Minkowski space E_2^1 these trajectories are characterized by two geometric invariants, a constant geodesic curvature and torsion that is generally a function of the evolution parameter τ on the string surface. When the torsions are constant, our approach allows us to obtain a well-known particular solution describing the rotation of straight string with massive ends in a given plane[5,6] . In this case the trajectories of motion of the masses are helices in E_2^1 and the surface is a helicoid. The minimal surface is just the helicoid that represents a ruled surface generated by the motion of a straight line, therefore there are no transverse excitations of the string and this solution cannot be used for determining corrections to a linearly growing potential[11] .

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