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PARTIAL SUPERSYMMETRY BREAKING IN N = 4 SUPERSYMMETRIC QUANTUM MECHANICS

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1. Introduction

An important problem in supersymmetric field theories is to explore under which conditions supersymmetry can be spontaneously broken. In theories with extended supersymmetry (supergravities, superstrings, supermembranes, ...) it is often desirable to have a partial supersymmetry breaking, when some of the initial supersymmetries remain exact [1]. For a better understanding of this phenomenon, it is instructive to study it in simple models.

Many characteristic features of the spontaneous supersymmetry breaking are revealed already in the simplest of supersymmetric theories, supersymmetric quantum mechanics (SQM) [2]. In the present paper we describe a SQM model where a partial supersymmetry breaking is realized.

A common starting point of all the studies in SQM is the 1D Poincare supertranslation algebra which includes the supercharges $Q^i (i = 1, \ldots, N)$ and the Hamiltonian H [1]

$$\begin{aligned} \left\{Q^i, Q^j\right\} &= 2\delta^{ij} H \\ \left[Q^i, H\right] &= 0 . \end{aligned} \tag{1.1}$$

Interesting examples of the SQM models with d bosonic and r fermionic degrees of freedom were considered in the component approach in [3], under the restrictive assumption that supercharges Q^i are linear in the fermionic operators (correspondingly, the Hamiltonian is at most quadratic in these operators). In [4] the $d = 1$ models with no such a restriction were formulated in the framework of $N = 2$ superfield formalism. Recently, a general $N = 4$ superfield action of the $N = 4$, $d = 1$ SQM model based on superalgebra (1.1) was proposed [5].

In all these models, only full supersymmetry breaking is possible, not the partial one. The reason is that the relevant supercharges satisfy, both on classical and quantum levels, the standard Poincare superalgebra (1.1). The partial supersymmetry breaking in the SQM models of that kind is forbidden by the famous no-go theorem due to Witten [2]. If some supercharge, say Q^1, is nonzero on the vacuum state

$$Q^1|0> \neq 0, \tag{1.2}$$

which means that the corresponding supersymmetry is spontaneously broken, then

$$< 0|H|0 > \neq 0$$

and , as a consequence of the relations (1.1), the property (1.2) should be valid as well for the rest of supercharges. So, either all the supersymmetries are unbroken or all they are broken.

[1] Hereafter N means the number of real supercharges. Normally, one considers the SQM models with *even N, where the supercharges can be divided into complex pairs $(Q, \bar{Q})$.*

On the other hand, in the $N = 4$ case, the simplest case where it makes sense to talk about a partial breaking, there exists a chance to circumvent the arguments of Witten's theorem. Namely, there appears a potential possibility to partially break $N = 4$ supersymmetry provided its algebra is modified by a central charge Z. This new $N = 4$ superalgebra can be written as

$$\begin{aligned} \left\{Q, \bar{Q}\right\} &= H + Z \\ \left\{S, \bar{S}\right\} &= H - Z , \end{aligned} \tag{1.3}$$

all the remaining commutators and anticommutators being zero. Now, the condition

$$Q|0> \neq 0$$

does not necessarily entail a similar one for the second supercharge S [2]. As a matter of fact, (1.3) means that the genuine $N = 4$ superalgebra is replaced by a direct sum of two $N = 2$ ones.

The main purpose of the present paper is to demonstrate that this phenomenon occurs already in the simplest $N = 4$ SQM model with one physical bosonic and four fermionic degrees of freedom ($d = 1$, $r = 4$). For this model we construct the most general $N = 4$ superfield action which incorporates explicit breaking of symmetry with respect to both $SU(2)$ groups constituting the full $SO(4)$ automorphism group of $N = 4$ superalgebra (1.1). In earlier consideration [4], the breaking of only one of these $SU(2)$'s has been taken into account. It turns out that in the general case the algebra of $N = 4$ supercharges still coincides with the standard one (1.1) while applied to field operators, however, becomes precisely of the form (1.3) on the states: it involves a constant central charge proportional to the product of two $SU(2)$ breaking parameters (both on the classical and quantum mechanical levels). Thus, in this SQM model there arises an opportunity to realize a partial supersymmetry breaking $N = 4 \to N = 2$ and we show that this is indeed the case for a certain class of potentials of the scalar field. For the existence of the phase with a partial breaking it is also crucial that the general Hamiltonian we deal with involves the terms quartic in fermionic operators (thus giving rise to the terms trilinear in fermions in the supercharges).

The paper is organized as follows. In Sec.2 we present the general superfield action of $N = 4$ SQM. Its interesting peculiarity is that it admits two equivalent forms related by a duality transformation. The kinematic off-shell constraint on the basic superfield and the equation of motion of the latter turn out to be dual to each other. In Sec.3 we go to components and give the explicit expressions for the supercharges and Hamiltonian via the physical fields. We start with the classical description and then carry out quantization. Sec.4 is devoted to a general characterization of the phases with broken and unbroken supersymmetry which are present in our model. We deduce the general conditions on the potential of the scalar field under which total or partial breakings of supersymmetry

[2] Analogous reasonings were given in [6].

occur. The cases with the partial breaking ($N = 4 \to N = 2$) are treated in Sec.5. An example of the potential giving rise to such a breaking is worked out in some detail. In Sec.6 we perform a duality transformation to a complex chiral $N = 4$ superfield and demonstrate that our $d = 1$ model can be embedded into the arising $N = 4$, $d = 2$ SQM model as a closed sector corresponding to a fixed value of certain extra $U(1)$ charge which in the dual formulation becomes the central charge of $N = 4$ 1D superalgebra. In turn, this $d = 2$ model proves to follow, by the Scherk-Schwarz type dimensional reduction, from some $N = 2$ 2D Kahler sigma model.

2. Superfield action of $N = 4$, $d = 1$ SQM

In this Section we deduce the most general superfield action of $N = 4$, $d = 1$ SQM at the classical level.

Our starting point is the structure relations of $N = 4$ 1D Poincare supersymmetry in the isospinor notation

$$\begin{aligned} \{Q_{\alpha a}, Q_{\beta b}\} &= 2\epsilon_{\alpha\beta}\epsilon_{ab}H \\ [H, Q_{\alpha a}] &= 0 \quad , \end{aligned} \tag{2.1}$$

where $\alpha = 1,2; a = 1,2$ are the doublet indices of two $SU(2)$ groups which form the $SO(4) \sim SU_I(2) \times SU_{II}(2)$ automorphism group of $N = 4$ superalgebra. The supercharges $Q_{\alpha a}$ can be realized as differential operators in $N = 4$ 1D superspace $R^{1|4} = \{z\} = \{t, \theta^{\alpha a}\}$

$$\begin{aligned} z &= (t, \theta^{\alpha a}) \\ \delta t &= -i\mu^{\alpha a}\theta_{\alpha a} \equiv i\,(\mu^{\alpha a}Q_{\alpha a})\,t \\ \delta\theta^{\alpha a} &= \mu^{\alpha a} \equiv i\left(\mu^{\beta b}Q_{\beta b}\right)\theta^{\alpha a} \end{aligned} \tag{2.2}$$

$$Q_{\alpha a} = -i\left(\frac{\partial}{\partial\theta^{\alpha a}} - i\theta_{\alpha a}\frac{\partial}{\partial t}\right) . \tag{2.3}$$

Let us now consider a real scalar superfield $\phi(z)$ in this superspace. It transforms under $N = 4$ supersymmetry as

$$\delta\phi(z) = \phi'(z) - \phi(z) = -i\,(\mu^{\alpha a}Q_{\alpha a})\,\phi = -\mu^{\alpha a}D_{\alpha a}\phi + 2i\mu^{\alpha a}\theta_{\alpha a}\frac{\partial}{\partial t}\phi \ , \tag{2.4}$$

where

$$D_{\alpha a} = \frac{\partial}{\partial\theta^{\alpha a}} + i\theta_{\alpha a}\frac{\partial}{\partial t}\,, \qquad \{D_{\alpha a}, D_{\beta b}\} = 2i\epsilon_{\alpha\beta}\epsilon_{ab}\frac{\partial}{\partial t}$$

are covariant spinor derivatives. The detailed analysis of the component structure of $\phi(z)$ shows that this superfield is reducible. An irreducible representation of $N = 4$ 1D supersymmetry involving the scalar field $\phi(t) = \phi(z)|_{\theta=0}$ may be singled out from $\phi(z)$ either by the constraint

$$D_{(\alpha a}D^{a}_{\beta)}\phi = m_{\alpha\beta} \tag{2.5a}$$

or by

$$D_{\alpha(a}D^{\alpha}_{b)}\phi = \lambda_{ab}, \tag{2.5b}$$

where $m_{\alpha\beta}$ and λ_{ab} are some arbitrary constant vectors in the group spaces of $SU_I(2)$ and $SU_{II}(2)$, respectively. Note that the constraints (2.5a) and (2.5b) actually lead to the supermultiplets equivalent on shell. As we shall see later, these are related to each other by a duality transformation. So, we may restrict ourselves to considering the constraint (2.5a). When $m_{\alpha\beta} \neq 0$, it breaks $SU_I(2)$ subgroup of the $SO(4)$ automorphism group down to $U_I(1) \subset SU_I(2)$ and leaves in the superfield $\phi(z)$ 4+4 independent components:

$$\phi|_{\theta=0}, \quad A_{ab} \equiv D_{(\alpha a}D^{\alpha}_{b)}\phi\Big|_{\theta=0}, \quad i\psi_{\alpha a} \equiv D_{\alpha a}\phi|_{\theta=0}. \tag{2.6}$$

Let us turn to constructing the most general action of $N = 4$, $d = 1$ SQM. We will deal with only one superfield $\phi(z)$, because the adding of more superfields would increase the dimensionality d of the manifold of scalar fields. Thus, the most general $d = 1$ action can be written as

$$S = \frac{\gamma}{16}\int dt d^4\theta \left\{A(\phi) + \theta^{\alpha a}\theta^{b}_{\alpha}\lambda_{ab}(\phi) + \theta^{\alpha a}\theta^{\beta}_{a}B_{\alpha\beta}(\phi) + \theta^{\alpha a}\theta^{\beta}_{a}\theta_{\beta c}\theta^{c}_{\alpha}C(\phi)\right\}, \tag{2.7}$$

where $A(\phi), \lambda_{ab}(\phi), B_{\alpha\beta}(\phi)$ and $C(\phi)$ are some functions of the above superfield ϕ, arbitrary for the moment[3]. Requiring the action (2.7) to be invariant under $N = 4$ supersymmetry transformation (2.4) puts severe restrictions on the functions $\lambda_{ab}(\phi), B_{\alpha\beta}(\phi)$ and $C(\phi)$:

$$\lambda''_{ab}(\phi) = B''_{\alpha\beta}(\phi) = C'(\phi) = 0 \Rightarrow \tag{2.8}$$

$$\lambda_{ab}(\phi) = \lambda_{ab}\cdot\phi,\ B_{\alpha\beta}(\phi) = B_{\alpha\beta}\cdot\phi,\ C(\phi) = C \tag{2.9}$$

with $\lambda_{ab}, B_{\alpha\beta}, C$ being constants. We can omit in (2.7) the constant term $\theta^4 \cdot C$ and the term $\theta^{\alpha a}\theta^{\beta}_{a}B_{\alpha\beta}\cdot\phi$, which, after integration over $d^4\theta$ and making use of the constraint (2.5a), is reduced to the shift of the component Lagrangian by a constant $B_{\alpha\beta}m^{\alpha\beta}$. So, the action (2.7) takes the following most general form

$$S = \frac{\gamma}{16}\int dt d^4\theta \left\{A(\phi) - \frac{1}{8}\theta^{\alpha a}\theta^{b}_{\alpha}\lambda_{(ab)}\cdot\phi\right\}, \tag{2.10}$$

where $\lambda_{(ab)}$ is a constant vector in the group space of $SU_{II}(2) \subset SO(4)$.

The action (2.10) supplemented with the constraint (2.5a) reveals interesting peculiarities. First, it includes the term which contains explicit θ's. This term actually does not break $N = 4$ supersymmetry as it could seem. After passing to the prepotential V^{bc} which solves the constraint (2.5a)

$$\phi = D_{\alpha(b}D^{\alpha}_{c)}V^{bc} + \frac{1}{8}\theta^{\beta c}\theta^{\alpha}_{c}m_{\beta\alpha} \tag{2.11}$$

[3] The terms containing odd degrees of Grassmann variables $\theta_{\alpha a}$ necessarily include the spinor derivatives $D_{\alpha a}\phi$ and so can be reduced to the form (2.7) after integration by part.

it can be written as a kind of the Fayet-Iliopoulos term

$$S = \frac{\gamma}{16}\int dt d^4\theta \left\{ A(\phi) - \frac{1}{2}\lambda_{ab}V^{ab} \right\}. \tag{2.12}$$

Now prepotential V^{ab} is unconstrained in the action (2.12), so one may vary it to get the equation of motion for $N = 4$, $d = 1$ SQM:

$$D_{\alpha(a}D^{\alpha}_{b)}A'(\phi) = \lambda_{ab}\,. \tag{2.13}$$

One more peculiarity of the action (2.10) and the constraint (2.5a) is the presence of two constant sets of parameters $(m_{\alpha\beta}, \lambda_{ab})$, which in general (when $m_{\alpha\beta} \neq 0, \lambda_{ab} \neq 0$) break both $SU_{I,II}(2)$ subgroups of the $SO(4)$ automorphism group

$$SO(4) \sim SU_I(2) \times SU_{II}(2) \to U_I(1) \times U_{II}(1)\,.$$

Note that in our approach, when we start with constraint (2.5a), the meaning of these two constant vectors is essentially different. The first constants $(m_{\alpha\beta})$ are purely kinematic (as it may be easily seen from (2.5a), $(m_{\alpha\beta})$ enters into $\phi(z)$ as a dimension 1 constant component contracted with the θ-monominal $\theta^{\alpha a}\theta^{\beta}_{a}$), but the second ones (λ_{ab}) are dynamical, because they appear in the action (2.10).

The roles of parameters $m_{\alpha\beta}$ and λ_{ab} are reversed after passing to the dual form of the action (2.10). To perform the duality transformation, let us insert the constraint (2.5a) into the action with the help of a Lagrange multiplier superfield $\rho^{\alpha\beta}(z)$:

$$S = \frac{\gamma}{16}\int dt d^4\theta \left\{ A(\phi) - \frac{1}{8}\theta^{\alpha a}\theta^{b}_{\alpha}\lambda_{ab}\phi + \rho^{\alpha\beta}\left(D_{(\alpha a}D^{a}_{\beta)}\phi - m_{\alpha\beta}\right) \right\}. \tag{2.14}$$

Varying $\rho^{\alpha\beta}$, we come back to (2.5a) and (2.10). On the other hand, ϕ is unconstrained in the action (2.14) and so one may vary it before varying $\rho^{\alpha\beta}$. As a result, one gets the following equation

$$A'(\phi) - \frac{1}{8}\theta^{\alpha a}\theta^{b}_{\alpha}\lambda_{(ab)} - D^{a}_{(\alpha}D_{\beta)a}\rho^{\alpha\beta} = 0\,. \tag{2.15}$$

Defining the new $N = 4$ superfield

$$u(z) \equiv A'(\phi(z)) \tag{2.16}$$

it is easy to check that $u(z)$ defined by (2.15),(2.16) satisfies the following constraint

$$D_{\alpha(a}D^{\alpha}_{b)}u(z) = \lambda_{ab}\,. \tag{2.17}$$

After expressing ϕ through u from eq.(2.16) and substituting (2.15) back into (2.14), we arrive at the dual form of the $N = 4$, $d = 1$ SQM action

$$S_{dual} = \frac{\gamma}{16}\int dt d^4\theta \left\{ \tilde{A}(u) - \frac{1}{8}\theta^{\alpha a}\theta^{\beta}_{a}m_{\alpha\beta}u \right\}, \tag{2.18}$$

where

$$\tilde{A}(u) = A(\phi(u)) - u\phi(u) \qquad (2.19)$$

and $u(z)$ is constrained by eq.(2.17). Evidently, λ_{ab} acquire now a status of kinematical constants, while $m_{\alpha\beta}$ become dynamical. In the next Sect. we shall see that $m_{\alpha\beta}$ and λ_{ab} enter into the physical component action on equal footing as the coupling constants.

We close this Section with several comments.

First, the basic peculiarity of our $N = 4$, $d = 1$ SQM action is the explicit breaking of symmetries with respect to both $SU_{I,II}(2)$ subgroups of the $SO(4)$ automorphism group. This opens up a way to realize a partial ($N = 4 \to N = 2$) supersymmetry breaking, which is forbidden in ordinary scheme, owing to the possibility of central extension of the $N = 4$ 1D Poincare superalgebra in the present SQM model. We closely inspect this situation in Sec.3 and 4.

Second, our action and constraints are very simple when written in terms of $N = 4$ superfields; on the contrary, finding and checking the invariances of the component action or the action written in terms of $N = 2$ superfields represent a more difficult task. The $N = 2$ superfield action of $N = 4$, $d = 1$ SQM was constructed in [4], but the possibility to simultaneously break two $SU(2)$ automorphism symmetries was not noticed there. The $N = 4$ superfield $u(z)$ in explicitly θ expanded form subjected to the $\lambda_{ab} = 0$ version of the constraint (1.1) was used in [5] for setting up an action of $N = 4$ SQM. However, no manifestly supersymmetric superfield form of the Fayet-Iliopoulos term was given.

Finally, as a simple example, we recall the case of $N = 4$, $d = 1$ superconformal quantum mechanics [7]. This system corresponds to the specific choice $\lambda_{ab} = 0$ and $A(\phi) = \phi \log \phi$ in (2.12),(2.5a)

$$S_{SCQM} = \frac{\gamma}{16} \int dt d^4\theta \;\; \phi \log \phi$$

$$D_{(\alpha a} D^{a}_{\beta)} \phi = m_{\alpha\beta} \; . \qquad (2.20)$$

In this case, the equation for the dual-transformed superfield $u(z)$ (2.16) can be explicitly solved

$$u(z) \equiv A'(\phi) = \log \phi + 1 \Rightarrow \phi = \exp(u - 1) \qquad (2.21)$$

Thus, the dual form of the action for $N = 4$ superconformal quantum mechanics is as follows

$$S^{dual}_{SCQM} = \frac{\gamma}{16} \int dt d^4\theta \left\{ \exp(\tilde{u}) - \frac{1}{8} \theta^{a}_{\alpha} \theta_{\beta a} m^{\alpha\beta} \tilde{u} \right\} \qquad (2.22)$$

$$D_{\alpha(a} D^{\alpha}_{b)} \tilde{u} = 0 \, ,$$

where

$$\tilde{u} = u - 1 \; .$$

By this we finish the superfield description of $N = 4$, $d = 1$ SQM and turn to the component consideration.

3. Component action, Hamiltonian and SUSY algebra of $N = 4$ SQM

The action for $N = 4$, $d = 1$ SQM in terms of the physical component fields ϕ and $\psi_{\alpha a}$ (2.6) can be easily obtained from the superfield action (2.12), exploiting constraint (2.5a) and eliminating the auxiliary field $A_{(ab)}$ by its equation of motion

$$
\begin{aligned}
S &= \gamma \int dt \left[A'' \frac{\dot{\phi}^2}{2} + \frac{i}{2} A'' \psi_{\alpha a} \dot{\psi}^{\alpha a} + \frac{1}{4} \left(\frac{\lambda^2}{A''} + m^2 A'' \right) + A''' m_{\alpha\beta} \psi^{\alpha a} \psi^{\beta}_{a} + \right. \\
&+ \left. \frac{A'''}{A''} \lambda_{ab} \psi^{\alpha a} \psi^{b}_{\alpha} + \frac{1}{24} \left(A^{IV} - \frac{(3A''')^2}{2A''} \right) \psi_{\alpha a} \psi^{\alpha}_{b} \psi^{\beta b} \psi^{a}_{\beta} \right] .
\end{aligned} \tag{3.1}
$$

For bringing the kinetic terms of the fields $\phi, \psi_{\alpha a}$ into the standard form we pass to the new field variables $x, \chi_{\alpha a}$ and new potential $W(x)$:

$$
\begin{aligned}
\chi_{\alpha a} &= \sqrt{A''(\phi)} \psi_{\alpha a} \\
\frac{d\phi}{dx} &= \frac{1}{\sqrt{A''(\phi)}} \\
W(x) &\equiv \phi(x)
\end{aligned} \tag{3.2}
$$

after that (3.1) is rewritten as

$$
\begin{aligned}
S &= \gamma \int dt \left[\frac{\dot{x}^2}{2} + \frac{i}{2} \chi_{\alpha a} \dot{\chi}^{\alpha a} + \frac{1}{4} \left(\lambda^2 (W')^2 + \frac{m^2}{(W')^2} \right) + \frac{W''}{2(W')^2} m_{\alpha\beta} \chi^{\alpha a} \chi^{\beta}_{a} - \right. \\
& \left. - \frac{W''}{2} \chi^{\alpha a} \chi^{b}_{\alpha} \lambda_{ab} - \frac{1}{12} \left(\frac{W''}{W'} \right)' \chi_{\alpha a} \chi^{\alpha}_{b} \chi^{\beta b} \chi^{a}_{\beta} \right] .
\end{aligned} \tag{3.3}
$$

The physical component action (3.3) is invariant under the following $N = 4$ supersymmetry transformations

$$
\begin{aligned}
\delta \chi_{\alpha a} &= i \frac{W''}{W'} \mu^{\beta b} \chi_{\alpha b} \chi_{\beta a} + i \mu_{\alpha b} \lambda^{b}_{a} W' + \frac{i}{W'} \mu_{\beta a} m^{\beta}_{\alpha} - \dot{x} \mu_{\alpha a} , \\
\delta x &= -i \mu^{\alpha a} \chi_{\alpha a}
\end{aligned} \tag{3.4}
$$

which directly stem from the superfield transformation law (2.4).

From now on, to make further formulas more readable, we set the coupling constant γ equal to 1, keeping in mind that the dependence on γ can be restored at any step by dimensionality arguments.

Being aware of the transformation law (3.4) of the component fields $x, \chi_{\alpha a}$, one may compute, by the standard Noether procedure, the classical supercharges $Q^{cl}_{\alpha a}$ which generate $N = 4$ SUSY:

$$
Q^{cl}_{\alpha a} = ip \chi_{\alpha a} - W' \lambda^{b}_{a} \chi_{\alpha b} - m^{\beta}_{\alpha} \chi_{\beta a} \frac{1}{W'} + \frac{W''}{3W'} \chi^{b}_{\alpha} \chi_{\gamma b} \chi^{\gamma}_{a} . \tag{3.5}
$$

With respect to the Poisson brackets the supercharges (3.5) form the following $N = 4$ superalgebra

$$\begin{aligned} \left\{Q^{cl}_{\alpha a}, Q^{cl}_{\beta b}\right\} &= 2\epsilon_{\alpha\beta}\epsilon_{ab}H^{cl} + 2m_{\alpha\beta}\lambda_{ab} \\ \left[H^{cl}, Q^{cl}_{\alpha a}\right] &= 0 \ , \end{aligned} \tag{3.6}$$

where the classical Hamiltonian H^{cl} is given by

$$\begin{aligned} H^{cl} &= \frac{p^2}{2} - \frac{1}{4}\left(\lambda^2(W')^2 + \frac{m^2}{(W')^2}\right) - \frac{W''}{2(W')^2} m_{\alpha\beta}\chi^{\alpha a}\chi^{\beta}_{a} \\ &+ \frac{W''}{2}\lambda_{ab}\chi^{\alpha a}\chi^{b}_{\alpha} + \frac{1}{12}\left(\frac{W''}{W'}\right)' \chi_{\alpha a}\chi^{\alpha}_{b}\chi^{\beta b}\chi^{a}_{\beta} \ . \end{aligned} \tag{3.7}$$

The most exciting feature of the superalgebra (3.6) is the presence of the central charge $\sim m\lambda$. This central charge appears already at the classical level and is proportional to the product of two $SU(2)$ breaking parameters $m_{\alpha\beta}, \lambda_{ab}$. So, it is not zero only provided both $SU_{I,II}(2)$ subgroups of the SUSY automorphism group $SO(4)$ are simultaneously broken.

Let us proceed to quantization. We follow the standard routine and replace the Poisson brackets by the Dirac ones

$$\begin{aligned} [p, x] &= i \\ \{\chi_{\alpha a}, \chi_{\beta b}\} &= \epsilon_{\alpha\beta}\epsilon_{ab} \ . \end{aligned} \tag{3.8}$$

The further steps are to put the products of spinors in supercharges $Q_{\alpha a}$ (3.5) and Hamiltonian H (3.7) into the normally ordered form, so that the original $N = 4$ SQM superalgebra (3.6) is reproduced. It is straightforward to find that appropriate quantum supercharges and Hamiltonian are given by the following expressions

$$Q^{q}_{\alpha a} = ip\chi_{\alpha a} - W'\lambda^{b}_{a}\chi_{\alpha b} - \frac{1}{W'}m^{\beta}_{\alpha}\chi_{\beta a} + \frac{W''}{3W'}\chi^{b}_{\alpha}\chi_{\gamma b}\chi^{\gamma}_{a} - \frac{5W''}{6W'}\chi_{\alpha a} \tag{3.9}$$

$$\begin{aligned} H^{q} &= \frac{p^2}{2} - \frac{1}{4}\left(\lambda^2(W')^2 + \frac{m^2}{(W')^2}\right) + \frac{1}{8}\left(\frac{W''}{W'}\right)^2 - \frac{W''}{2(W')^2}m_{\alpha\beta}\chi^{\alpha a}\chi^{\beta}_{a} + \frac{W''}{2}\lambda_{ab}\chi^{\alpha a}\chi^{b}_{\alpha} \\ &+ \frac{1}{12}\left(\frac{W''}{W'}\right)'\left[\chi_{\alpha a}\chi^{\alpha}_{b}\chi^{\beta b}\chi^{a}_{\beta} - 5\right] . \end{aligned} \tag{3.10}$$

It should be stressed that the central charges in (3.6) are not renormalized upon quantization. Actually, all the freedom in $Q_{\alpha a}$ and H associated with passing to normal ordering is completely fixed by requiring the second of eqs.(3.6) to hold in the quantum case.

Before closing this Section we briefly discuss the main peculiarities of the model (3.9),(3.10).

First, as was already mentioned, the presence of a non-zero central charge in superalgebra (3.6) opens up a possibility to realize a partial spontaneous breaking of $N = 4$

SUSY. Indeed, the structure of (3.6) implies that the basic condition for the existence of $N = 4$ supersymmetric ground state

$$Q_{\alpha a}|0>=0 \qquad (3.11)$$

can never be satisfied if $m_{(\alpha\beta)} \neq 0$ and $\lambda_{(ab)} \neq 0$ simultaneously. So, in this case $N = 4$ supersymmetry is necessarily broken and, as will be shown in Sec.4 , one can find the potentials $W(x)$ for which $N = 2$ supersymmetry is still exact. The reason why no contradiction arises with Witten's no-go theorem mentioned in Introduction is just the non-zero central charges in superalgebra (3.6) (see discussion of this point in another context in [6]).

Second, it is amusing that at the classical level the bosonic self-interaction $\sim \lambda^2(W')^2 + \frac{m^2}{(W')^2}$ is possible only if $\lambda \neq 0$ and/or $m \neq 0$, i.e. when at least one of two $SU(2)$'s is explicitly broken. This is not so for the quantum Hamiltonian (3.10) which contains an effective bosonic self-interaction $\left(\frac{W''}{W'}\right)^2$. Note that the expressions (3.9),(3.10) are form-invariant under the change $W' \to \frac{1}{W'}, \lambda \leftrightarrow m$ and the simultaneous permutation of $SU(2)$ indices $a \leftrightarrow \alpha$, thus indicating that the two $SU(2)$'s actually enter into the game on equal footing.

Finally, it turns out very essential that our general supercharges $Q_{\alpha a}$ (3.5),(3.9) are nonlinear in fermions $\chi_{\alpha a}$, containing the terms trilinear in the latter. Correspondingly, the Hamiltonian involves a term quartic in fermionic operators. In previous considerations[3], the supercharges were as a rule assumed to be linear in fermions that corresponds to limiting to the Hamiltonians quadratic in fermions. For the case $N = 4$ this limitation places very strong restrictions on the possible potentials W : $\frac{W''}{W'} = const \quad (W = c_1 e^{c_2 x} + c_3)$. No such restrictions emerge in the general situation we deal with: the potential W can be an arbitrary function of x. Just due to this freedom we may choose W so as to ensure partial supersymmetry breaking (see next Section).

4. Phases with exact and spontaneously broken supersymmetry

In this Section we study which phases exist in our model and formulate the conditions under which one or another phase comes out. As usual, these phases are characterized by different symmetry of the ground state. We are not going to analyze in full the spectrum of states in each case; our consideration will be limited to the classification of the ground states.

To simplify the analysis, it will be convenient to pass from the manifestly $SO(4)$ covariant notation to the notation with only one of two $SU(2)$'s being manifest, e.g. $SU_{II}(2)$. Correspondingly, we represent the $SO(4)$ vector $\chi_{\alpha a}$ as a pair consisting of the $SU_{II}(2)$ spinor χ_a and its conjugate $\bar{\chi}_a$

$$\chi_{\alpha a} \equiv (\chi_a, \bar{\chi}_a) \ . \qquad (4.1)$$

Also, without loss of generality, we choose the basis in the group spaces of $SU_{I,II}(2)$ in such a way that $m_{(\alpha\beta)}$, $\lambda_{(ab)}$ are diagonal

$$m^{\beta}_{\alpha} = m \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = m(\sigma^3)^{\beta}_{\alpha}, \quad \lambda^{b}_{a} = \lambda(\sigma^3)^{b}_{a} \,. \tag{4.2}$$

In this basis, the supercharges (3.9), Hamiltonian (3.10) and the (anti)commutation relations (3.6) are as follows [4]

$$\begin{aligned} Q_a &= ip\chi_a - \lambda(\sigma^3)^b_a\chi_b W' - \frac{m}{W'}\chi_a - \frac{W''}{2W'}\chi^2\bar{\chi}_a - \frac{W''}{2W'}\chi_a \\ \bar{Q}^a &= -ip\bar{\chi}^a - \lambda\bar{\chi}^b(\sigma^3)^a_b W' - \frac{m}{W'}\bar{\chi}^a + \frac{W''}{2W'}\chi^a\bar{\chi}^2 - \frac{W''}{2W'}\bar{\chi}^a \\ H &= \frac{p^2}{2} + \frac{m^2}{2(W')^2} + \frac{\lambda^2(W')^2}{2} + \frac{1}{8}\left(\frac{W''}{W'}\right)^2 - \frac{mW''}{(W')^2}(\chi\bar{\chi} - 1) + W''\lambda(\chi\sigma^3\bar{\chi}) - \\ &\quad -\frac{1}{4}\left(\frac{W''}{W'}\right)'\left[\chi^2\bar{\chi}^2 - 2\chi\bar{\chi} + 1\right] \end{aligned} \tag{4.3, 4.4}$$

$$\begin{aligned} \left\{Q_a, \bar{Q}^b\right\} &= 2\delta^b_a H + 2m\lambda(\sigma^3)^b_a \\ \left\{Q_a, Q_b\right\} &= \left\{\bar{Q}^a, \bar{Q}^b\right\} = [H, Q_a] = \left[H, \bar{Q}^a\right] = 0 \,. \end{aligned} \tag{4.5}$$

Note that (4.5) coincides with (1.3) upon identification $Q_1 \equiv Q, Q_2 \equiv S$.

For the SQM model (4.3)-(4.5), in accord with the general consideration (see, e.g. [3]), exact $N = 4$ supersymmetry implies the existence of a square integrable wave function of the ground state $|0>$, subjected to the condition

$$Q_a|0>= \bar{Q}^a|0>= 0 \,. \tag{4.6}$$

By inspection of the relations (4.5), one immediately concludes that in the case $m\lambda \neq 0$ it is impossible to obey (4.6) simultaneously for Q_1 and Q_2. As was already mentioned, the reason is the appearance of a non-zero constant central charge in (4.5). Thus, in the present model we have two radically different situations depending on whether $m\lambda$ equals zero or does not. Further, we will characterize both these cases.

I. $m\lambda \neq 0$

In this case $N = 4$ supersymmetry is spontaneously broken. There are two different patterns of such a breaking

Ia. *Partial breaking* $N = 4 \to N = 2$. It occurs provided either the equation

$$Q_1|0>= \bar{Q}^1|0>= 0 \quad (m\lambda < 0) \tag{4.7a}$$

---

[4] The natural position of indices is as follows: $\chi_a, \bar{\chi}^a = (\chi_a)^\dagger$. Indices are raised and lowered with the help of antisymmetric tensor $\epsilon^{ab}(\epsilon^{12} = -\epsilon_{12} = 1)$; in the bilinear forms of spinors the first index is always assumed to be in the natural position, e.g. $\chi^2 = \chi_a\chi^a, \bar{\chi}^2 = \bar{\chi}^a\bar{\chi}_a$, etc.

or

$$Q_2|0>=\bar{Q}^2|0>=0 \quad (m\lambda>0) \tag{4.7b}$$

have at least one solution[5] (different signs of $m\lambda$ in (4.7a) and (4.7b) are selected by the standard positiveness arguments). In this case one of two $N=2$ supersymmetries ($Q_1, \bar{Q}^1$ for (4.7a) and $Q_2, \bar{Q}^2$ for (4.7b)) is still exact, while the remaining one is spontaneously broken.

Ib. *Total supersymmetry breaking* It comes out if no solutions of eqs. (4.7a),(4.7b) exist.

II. $m\lambda=0$

No central charges are present in (4.5) in this case. So, in accordance with Witten's theorem [2], supersymmetry is exact if the equations

$$Q_a|0>=\bar{Q}^a|0>=0 \tag{4.8}$$

are solvable, otherwise it is totally broken.

In the next Section, using the explicit expressions (4.3) for the supercharges $Q_a, \bar{Q}^a$, we deduce the conditions on the SQM scalar potential W under which eqs. (4.7),(4.8) can be solved.

In the rest of this Section we dwell in brief on the peculiarities of realization of the SUSY automorphism group $SO(4) \sim SU_I(2) \times SU_{II}(2)$ in the present model.

The groups $SU_{I,II}(2)$ are realized only on spinors $\chi_a, \bar{\chi}^a$. Assuming the standard Dirac brackets for χ_a

$$\left\{\chi_a, \bar{\chi}^b\right\} = \delta_a^b$$

the generators of these $SU(2)$ groups are

$$\begin{array}{ll} SU_I(2) & SU_{II}(2) \\ & \\ B = \frac{1}{2}\left[\chi_a, \bar{\chi}^a\right] & B_i = \frac{1}{2}\chi_a(\sigma_i)\bar{\chi}^b \\ B_+ = \frac{1}{2}\chi^2 & \\ B_- = \frac{1}{2}\bar{\chi}^2 & \\ & \\ \left\{\begin{array}{l} [B, B_\pm] = \pm 2B_\pm \\ [B_+, B_-] = B \end{array}\right. & [B_i, B_j] = \epsilon_{ijk}B_k \ . \end{array} \tag{4.9}$$

From the expression for the quantum Hamiltonian (4.4) it follows that B and B_3 always commute with H (irrespective of values of the breaking parameters λ and m).Thus, these three operators B, B_3 and H can be simultaneously diagonalized. Correspondingly, the

[5]Henceforth, under the "solution" we always assume a square integrable one.

eigenfunctions of H have definite $U(1)$ charges b and b_3. However, it is easy to find from the explicit expressions (4.9)

$$(B)^3 - B = 0, \quad B_i B_i = \frac{3}{4}\left[1 - (B)^2\right]. \tag{4.10}$$

The first of these equations implies that B has as the eigenvalues only the numbers $0, \pm 1$. The Casimir operator $C_I = \frac{1}{4}(B^2 + B_+ B_- - 2B)$ for B, B_+, B_- given by (4.9) is reduced to $C_I = \frac{3}{4}B^2$, so it takes only the values $0, 3/4$ that correspond to the singlet and doublet representations of $SU_I(2)$. On the other hand, on the same states the Casimir operator $C_I I = B_i B_i$ of $SU_{II}(2)$, in virtue of (4.10), takes the following values

$$\begin{array}{ccc} B|\ >= 0 & \Rightarrow & (B_i B_i)|\ >= \frac{3}{4}|\ > \\ SU_I(2) \ \ singlet & & SU_{II}(2) \ \ doublet \\ & & \\ B|\ >= \pm|\ > & \Rightarrow & (B_i B_i)|\ >= 0 \\ SU_I(2) \ \ doublet & & SU_{II}(2) \ \ singlet \ . \end{array} \tag{4.11}$$

Thus, in our $N = 4$ SQM any state with a definite energy (including the ground state) transforms as a doublet with respect to one of two $SU(2)$ automorphism groups and as a singlet with respect to another $SU(2)$ group. In other words, one of these $SU(2)$'s is always broken (if $m = \lambda = 0$, this breakdown is purely spontaneous, because in this case H commutes with all the $SU(2)$ generators).

5. Partial supersymmetry breaking

In this Section we inquire under which choice of the SQM potential $W(x)$ our model exhibits a partial supersymmetry breaking.

As we know, this phenomenon occurs only if $m\lambda \neq 0$ in (4.5). Then, from the $N = 4$ superalgebra (4.5)

$$\begin{aligned} \left\{Q_1, \bar{Q}^1\right\} &= 2H + 2m\lambda \\ \left\{Q_2, \bar{Q}^2\right\} &= 2H - 2m\lambda \end{aligned} \tag{5.1}$$

it follows that one may keep exact only one of two $N = 2$ supersymmetries, by requiring either

$$Q_1|0_1> = \bar{Q}^1|0_1> = 0 \quad (m\lambda < 0) \tag{5.2}$$

or

$$Q_2|0_2> = \bar{Q}^2|0_2> = 0 \quad (m\lambda > 0)\,. \tag{5.3}$$

Without loss of generality, we shall deal with the conditions (5.2) because the solutions of eq.(5.3) are obtained from those of eq.(5.2) (if exist) by changing $\lambda \to -\lambda$. Representing the supercharges (4.3) as

$$
\begin{aligned}
Q_a &= ip\chi_a - \lambda(\sigma^3)_a^b\chi_b W' - \frac{m}{W'}\chi_a + \frac{W''}{2W'}\chi_a + \frac{W''}{W'}\chi_a B \\
\bar{Q}^a &= -ip\bar{\chi}^a - \lambda\bar{\chi}^b(\sigma^3)_b^a W' - \frac{m}{W'}\bar{\chi}^a - \frac{W''}{2W'}\bar{\chi}^a + \frac{W''}{W'}\bar{\chi}^a B
\end{aligned} \tag{5.4}
$$

we get the following convenient expressions for Q_1 and $\bar{Q}^1$

$$
\begin{aligned}
Q_1 &= \chi_1\left(ip - \lambda W' - \frac{m}{W'} + \frac{W''}{2W'} + \frac{W''}{W'}B\right) \\
\bar{Q}^1 &= \bar{\chi}^1\left(-ip - \lambda W' - \frac{m}{W'} - \frac{W''}{2W'} + \frac{W''}{W'}B\right) .
\end{aligned} \tag{5.5}
$$

Further, looking at (5.1), we conclude that $|0_1>$ is the eigenstate of H with the eigenvalue $-m\lambda$

$$
H|0_1>= -m\lambda|0_1> \text{ or } \tilde{H}|0_1>= 0, \;\; \tilde{H} = H + m\lambda . \tag{5.6}
$$

It appeared convenient to change the scale of energy in (5.1) by defining the new Hamiltonian $\tilde{H} = H + m\lambda$. Then $|0_1>$ subjected to (5.2) has a zero energy with respect to $\tilde{H}$. The generators B and B_3 commute with the newly defined $\tilde{H}$ as well as with H, so $\tilde{H}, B, B_3$ still form a complete set of mutually commuting operators and $|0_1>$ should be simultaneously an eigenstate of B and B_3.

Now we shall closely follow Affleck [8]. Let us introduce two types of the states: $|E_1>$ which are totally empty with respect to the fermions χ_1, i.e.

$$
\chi_1|E_1>= 0 \tag{5.7}
$$

and the totally filled ones $|F_1>$

$$
\bar{\chi}^1|F_1>= 0 . \tag{5.8}
$$

In the end of previous Section we have shown that any state which is an eigenfunction of the $U(1)$ charges B and B_3 has the structure $0 \otimes \frac{1}{2}$ or $\frac{1}{2} \otimes 0$ with respect to the group $SU_I(2) \times SU_{II}(2)$. Let us list, in accord with this property, all the possible states $|E_1>$ and $|F_1>$, denoting by the subscripts I and II the singlets of $SU_I(2)$ and $SU_{II}(2)$, respectively.

$$
|E_1>_I : \quad \begin{cases} \chi_1|E_1>_I = 0 \\ \bar{\chi}^2\chi_2|E_1>_I = |E_1>_I \end{cases} \Rightarrow \quad \begin{array}{l} B|E_1>_I = 0 \\ B_3|E_1>_I = -\frac{1}{2}|E_1>_I \end{array} \tag{5.9}
$$

$$
|F_1>_I : \quad \begin{cases} \bar{\chi}^1|F_1>_I = 0 \\ \chi_2\bar{\chi}^2|F_1>_I = |F_1>_I \end{cases} \Rightarrow \quad \begin{array}{l} B|F_1>_I = 0 \\ B_3|F_1>_I = \frac{1}{2}|F_1>_I \end{array} \tag{5.10}
$$

$$
|E_1>_{II} : \quad \begin{cases} \chi_1|E_1>_{II} = 0 \\ \chi_2|E_1>_{II} = 0 \end{cases} \Rightarrow \quad \begin{array}{l} B|E_1>_{II} = |E_1>_{II} \\ B_3|E_1>_{II} = 0 \end{array} \tag{5.11}
$$

$$
|F_1>_{II} : \quad \begin{cases} \bar{\chi}^1|F_1>_{II} = 0 \\ \bar{\chi}^2|F_1>_{II} = 0 \end{cases} \Rightarrow \quad \begin{array}{l} B|F_1>_{II} = -|F_1>_{II} \\ B_3|F_1>_{II} = 0 \end{array} . \tag{5.12}
$$

What remains to do is to specify the x-dependence of the states $|E_1>_{I,II}, |F_1>_{I,II}$ under which the equations

$$Q_1|E_1,F_1>_{I,II}=0,\quad \bar{Q}^1|E_1,F_1>_{I,II}=0$$

are satisfied. Using the expressions (5.5) for the supercharges $Q_1, \bar{Q}^1$ and the definitions (5.9)-(5.12) it is easy to establish the relevant differential equations and to indicate their solutions

$\underline{|E_1>_I}$:

$$\begin{aligned}
Q_1|E_1>_I &\equiv 0 \\
\bar{Q}^1|E_1>_I &= 0 = \bar{\chi}^1\left(\frac{d}{dx} - \lambda W' - \frac{m}{W'} - \frac{W''}{2W'}\right)|E_1>_I \Rightarrow \\
|E_1>_I &= \sqrt{W'}e^{\lambda W + mV}\,|\tilde{E_1}>_I
\end{aligned} \tag{5.13}$$

$\underline{|F_1>_I}$:

$$\begin{aligned}
\bar{Q}^1|F_1>_I &\equiv 0 \\
Q_1|F_1>_I &= 0 = \chi_1\left(-\frac{d}{dx} - \lambda W' - \frac{m}{W'} + \frac{W''}{2W'}\right)|F_1>_I \Rightarrow \\
|F_1>_I &= \sqrt{W'}e^{-\lambda W - mV}\,|\tilde{F_1}>_I
\end{aligned} \tag{5.14}$$

$\underline{|E_1>_{II}}$:

$$\begin{aligned}
Q_1|E_1>_{II} &\equiv 0 \\
\bar{Q}^1|E_1>_{II} &= 0 = \bar{\chi}^1\left(\frac{d}{dx} - \lambda W' - \frac{m}{W'} + \frac{W''}{2W'}\right)|E_1>_{II} \Rightarrow \\
|E_1>_{II} &= \frac{1}{\sqrt{W'}}e^{\lambda W + mV}\,|\tilde{E_1}>_{II}
\end{aligned} \tag{5.15}$$

$\underline{|F_1>_{II}}$:

$$\begin{aligned}
\bar{Q}^1|F_1>_{II} &\equiv 0 \\
Q_1|F_1>_{II} &= 0 = \chi_1\left(-\frac{d}{dx} - \lambda W' + \frac{m}{W'} - \frac{W''}{2W'}\right)|F_1>_{II} \Rightarrow
\end{aligned}$$

$$|F_1>_{II} \ = \ \frac{1}{\sqrt{W'}} e^{-\lambda W - mV} \, |\tilde{F}_1>_{II}. \tag{5.16}$$

Here

$$V'(x) \equiv \frac{1}{W'(x)} \tag{5.17}$$

and the states with tilde do not depend on x (these are the corresponding integration constants).

Note that we have defined an auxiliary potential $V(x)$ in (5.17) by reasons of convenience, in order to obtain the solutions of the above equations in a generic form. In each specific case, for getting the explicit form of the solution one needs to solve (5.17) for $V(x)$.

So, the formal solutions of eqs.(5.13)-(5.16) exist for any potential $W(x)$. However, the important requirement to be satisfied by the ground state is its square integrability

$$\int_{-\infty}^{\infty} dx < A|A > \neq \infty \, , \tag{5.18}$$

where $|A\ >$ stands for any of the states (5.13)-(5.16). This condition places the following restrictions on the admissible potentials $W(x)$

$$\int_{-\infty}^{\infty} dx |e^{\pm 2\lambda W \pm 2mV} W'| \neq \infty \tag{5.19}$$

and/or

$$\int_{-\infty}^{\infty} dx |e^{\pm 2\lambda W \pm 2mV} V'| \neq \infty \ . \tag{5.20}$$

Since the solutions (5.3) follows from (5.2) by the simple change $\lambda \to -\lambda$ in the latter, the conditions on $W(x)$ can be eventually written as

$$\int_{-\infty}^{\infty} dx |e^{2\lambda W + 2mV} W'| \neq \infty \tag{5.21}$$

and/or

$$\int_{-\infty}^{\infty} dx |e^{2\lambda W + 2mV} V'| \neq \infty \ . \tag{5.22}$$

For the potentials satisfying (5.21)-(5.22) (with $m \neq 0$ and $\lambda \neq 0$) there is a partial supersymmetry breaking phase in the $N = 4$ SQM model we consider.

An analogous study of the restrictions on $W(x)$ which allow $N = 4$ supersymmetry to be exact (with $m\lambda = 0$) leads to the following constraints

$$\underline{m = 0, \lambda = 0}:$$

$$\int_{-\infty}^{\infty} \frac{dx}{|W'(x)|} \neq \infty$$

or (5.23)

$$\int_{-\infty}^{\infty} dx |W'(x)| \neq \infty$$

$\underline{m = 0, \lambda \neq 0}$:

$$\int_{-\infty}^{\infty} dx |W'(x) e^{2\lambda W}| \neq \infty$$

or (5.24)

$$\int_{-\infty}^{\infty} dx |W'(x) e^{-2\lambda W}| \neq \infty$$

$\underline{m \neq 0, \lambda = 0}$:

$$\int_{-\infty}^{\infty} dx |V'(x) e^{2mV}| \neq \infty$$

or (5.25)

$$\int_{-\infty}^{\infty} dx |V'(x) e^{-2mV}| \neq \infty \ ,$$

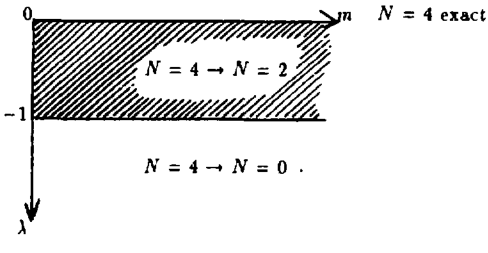
where $V(x)$ is as before defined by eq.(5.17). Otherwise, $N = 4$ supersymmetry is fully broken.

By this we end the analysis of different patterns of $N = 4$ supersymmetry breaking in the model under consideration.

To close this Section, we present an example of the potential $W(x)$ for which, depending on the values of the parameters m and λ, all the possible phases described above exist

$$W(x) = \frac{x^4}{8} + \frac{x^2}{2} + \frac{\log x^2}{4} \ . \qquad (5.26)$$

Without entering into details, we depict the "phase diagram" for $W(x)$ (5.26) in the range $m\lambda \leq 0$ ($m \geq 0, \lambda \leq 0$)



At $\lambda = 0$ one has the phase with exact $N = 4$ SUSY; on the axe $m = 0, \lambda < 0$ and within the strip $\lambda < -1$ the total supersymmetry breaking $N = 4 \rightarrow N = 0$ is realized;

within the strip $-1 \leq \lambda < 0$ $N = 4$ SUSY is partially broken down to $N = 2$ SUSY. Note that the potential (5.26) was chosen mainly in order to have the auxiliary potential $V(x)$ (5.17) as simple as possible. Of course, one may also find other potentials $W(x)$ giving rise to the partial supersymmetry breaking.

6. Duality transformation to $N = 4$, $d = 2$ SQM

The central charge in superalgebra (3.6) is constant and therefore it does not manifest itself as far as the transformation properties of the involved fields ($\phi(t)$ and $\chi_{\alpha a}(t)$) are concerned: supersymmetry transformations of the latter constitute the standard $N = 4$ 1D Poincare superalgebra. So the difference between (3.6) and (2.1) is actually observable only when studying how the supercharges act on the states. In this Section we show that the central charge in (3.6) can be made active (i.e., giving rise to nontrivial transformations of the fields) after passing to one more description of our model, that time in terms of a chiral $N = 4$ 1D superfield. Like in going from (2.10), (2.5a) to (2.18), (2.17) we exploit the duality transformation similar, e.g., to the one relating tensor and chiral $N = 1$ 4D multiplets [9]. The chiral field representation will allow us to reveal an interesting correspondence between the $N = 4$, $d = 1$ SQM and some class of $N = 2$ 2D Kahler sigma models.

To begin with, it will be convenient for us to pass to the complex notation

$$\theta^{\alpha a} = (\theta^a, -\bar{\theta}^a), \quad D_{\alpha a} = (D_a, -\bar{D}_a)$$

and to choose at once $m_{\alpha\beta} = m(\sigma^3)_{\alpha\beta}, \lambda_{ab} = \lambda(\sigma^3)_{ab}$. We will restrict our discussion here entirely to the superfield level, without presenting the component results. Doing this way will be sufficient for learning most characteristic features of the dual description.

With the above notation the SQM action (2.10) and constraint (2.5a) are rewritten as

$$S = \frac{\gamma}{16} \int dt d^4\theta \left\{ A(\phi) + \frac{\lambda}{4}(\theta\sigma^3\bar{\theta})\phi \right\} \tag{6.1}$$

$$(D)^2\phi(z) = 0 \quad (\bar{D})^2\phi(z) = 0 \tag{6.2a}$$

$$\left[D, \bar{D}\right] \phi(z) = -2m, \tag{6.2b}$$

where

$$(D)^2 = D_a D^a, \ (\bar{D})^2 = \bar{D}^a \bar{D}_a, \ \left[D, \bar{D}\right] = \left[D_a, \bar{D}^a\right].$$

It is a simple exercise [7] to check that (6.2a) already imply

$$\frac{\partial}{\partial t}\left[D, \bar{D}\right] \phi(z) = 0 \quad \Rightarrow \quad \left[D, \bar{D}\right] \phi(z) = const, \tag{6.3}$$

so the role of (6.2b) is reduced to fixing a constant appearing in (6.3).

In Sec.2 we solved constraints (6.2) via an unconstrained prepotential. Now we change the strategy and insert the basic constraints (6.2a) into the action with the help of superfield Lagrange multiplier

$$(6.1) \Rightarrow S = \frac{\gamma}{16}\int dtd^4\theta \left\{ A(\phi) + \frac{\lambda}{4}(\theta\sigma^3\bar{\theta})\phi + B(D)^2\phi + \bar{B}(\bar{D}^2)\phi \right\} . \qquad (6.4)$$

Varying (6.4) with respect to B, $\bar{B}$ one recovers the original theory. On the other hand, varying $\phi(z)$ (which is unconstrained in (6.4)) yields the algebraic equation for $\phi(z)$

$$A'(\phi) + \frac{\lambda}{4}(\theta\sigma^3\bar{\theta}) + [\Omega(\zeta) + \bar{\Omega}(\bar{\zeta})] = 0\,, \qquad (6.5)$$

where Ω and $\bar{\Omega}$ are mutually conjugated chiral $N = 4$ 1D superfields

$$\begin{aligned} \Omega(\zeta) &= (D)^2 B \Rightarrow D_a\Omega = 0 \\ \bar{\Omega}(\bar{\zeta}) &= (\bar{D})^2\bar{B} \Rightarrow \bar{D}_a\bar{\Omega} = 0 \\ \zeta &= \left\{ t_R = t - i\theta\bar{\theta}, \bar{\theta}^a \right\}, \\ \bar{\zeta} &= \left\{ t_L = t + i\theta\bar{\theta}, \theta^a \right\} . \end{aligned} \qquad (6.6)$$

Solving eq.(6.5) for $\phi(z)$

$$\phi = \phi\left[\Omega + \bar{\Omega} + \frac{\lambda}{4}(\theta\sigma^3\bar{\theta})\right] \qquad (6.7)$$

and substituting (6.7) into (6.4) we arrive at the dual-transformed action of $N = 4$ SQM in the form

$$\begin{aligned} S_{dual} &= \frac{\gamma}{16}\int dtd^4\theta \left\{ \hat{A}(\Omega + \bar{\Omega}) + \hat{A}'(\Omega + \bar{\Omega})\frac{\lambda}{4}(\theta\sigma^3\bar{\theta}) - \right. \\ &\left. - \hat{A}''(\Omega + \bar{\Omega})\frac{\lambda^2}{64}(\theta)^2(\bar{\theta})^2 \right\} \qquad (6.8) \\ \hat{A}(\Omega + \bar{\Omega}) &\equiv A\left[\phi(\Omega + \bar{\Omega})\right] + (\Omega + \bar{\Omega})\phi(\Omega + \bar{\Omega}). \qquad (6.9) \end{aligned}$$

We observe three basic peculiarities of the action (6.8).

First, it involves two physical bosonic degrees of freedom $\Omega(t) = \Omega(\zeta)|_{\theta=0}$ while in the initial superfield $\phi(z)$ only one such a degree is present. So, the above duality transformation introduces a new physical bosonic field $Im\Omega(t) \equiv \omega(t)$ and essentially differs in that aspect from the previously utilised transformation (Sec.2) which preserved the on-shell content of the supermultiplet. We shall explain below in which precise sense the model with the action (6.8) is equivalent to the one we started with.

The second peculiarity is that (6.8) exhibits a new $U(1)$ symmetry realized as a shift of the additional field $\omega(t)$

$$\Omega'(\zeta) = \Omega(\zeta) + i\alpha\,, \quad \bar{\Omega}'(\bar{\zeta}) = \bar{\Omega}(\bar{\zeta}) - i\alpha\,. \qquad (6.10)$$

The appearance of such isometries is a generic feature of the dual transformations of that sort [9].

Third, because of the explicit presence of θ's in (6.8), this action is by no means invariant under the standard $N = 4$ 1D supertranslations acting on $\Omega, \bar{\Omega}$ as

$$\delta\Omega(\zeta) = \Omega'(\zeta') - \Omega(\zeta), \quad \delta\bar{\Omega}(\bar{\zeta}) = \bar{\Omega}'(\bar{\zeta}') - \bar{\Omega}(\bar{\zeta}) \qquad (6.11)$$
$$\delta t_R = -2i\mu^a\bar{\theta}_a, \quad \delta\bar{\theta}^a = \bar{\mu}^a,$$
$$\delta t_L = (\delta\bar{t}_R), \quad \delta\theta^a = \mu^a.$$

It is easy to find modified $N = 4$ transformations which leave (6.8) invariant. In achieving this, the crucial role belongs to $U(1)$ symmetry (6.10).

Let us denote the generator of this $U(1)$ as J

$$\delta\Omega = i\alpha J\Omega, \quad \delta\bar{\Omega} = i\alpha J\bar{\Omega} \qquad (6.12)$$
$$J\Omega = 1, \quad J\bar{\Omega} = -1.$$

and define new supersymmetry transformations by

$$\tilde{\delta}\Omega = \delta\Omega - \frac{\lambda}{4}(\mu\sigma^3\bar{\theta})J\Omega = \delta\Omega - \frac{\lambda}{4}(\mu\sigma^3\bar{\theta}) \qquad (6.13)$$
$$\tilde{\delta}\bar{\Omega} = \delta\bar{\Omega} + \frac{\lambda}{4}(\theta\sigma^3\bar{\mu})J\bar{\Omega} = \delta\bar{\Omega} - \frac{\lambda}{4}(\theta\sigma^3\bar{\mu}),$$

where $\delta\Omega, \delta\bar{\Omega}$ are the conventional variations given by (6.11). It is straightforward to check invariance of (6.8) under (6.13).

The supercharges corresponding to (6.13) are

$$\tilde{Q}_a = Q_a + i\frac{\lambda}{4}(\sigma^3)_a^b\bar{\theta}_b J \qquad (6.14)$$
$$\tilde{\bar{Q}}_a = \bar{Q}_a + i\frac{\lambda}{4}(\sigma^3)_a^b\theta_b J.$$

They satisfy the following (anti)commutation relations

$$\left\{\tilde{Q}_a, \tilde{\bar{Q}}^b\right\} = 2\delta_a^b H - \frac{\lambda}{2}(\sigma^3)_a^b J. \qquad (6.15)$$
$$\left[H, \tilde{Q}_a\right] = \left[J, \tilde{Q}_a\right] = 0.$$

The superalgebra $\tilde{Q}, \tilde{\bar{Q}}, H, J$ is recognized as a direct sum of two $N = 2$ 1D superalgebras with the "Hamiltonians" $H - \frac{\lambda}{4}J$ and $H + \frac{\lambda}{4}J$. It coincides with the previously considered superalgebra (4.5) on the subspace $\{| >_m\}$ of the whole space of states, such that

$$J| >_m = -4m| >_m . \qquad (6.16)$$

We conclude that the initial $N = 4$, $d = 1$ SQM model is embedded into the model with the action (6.8) as a closed sector corresponding to the fixed value (6.16) of the $U(1)$

central charge operator J. The whole phase space of the above $N = 4$, $d = 2$ SQM model can be viewed as a collection of the $N = 4$, $d = 1$ SQM spaces labelled by the eigenvalue of J as a parameter. For each fixed m, eq. (6.16) can be regarded as a constraint which reduces, in a manifestly supersymmetric way, the number of independent on-shell bosonic degrees of freedom from 2 to 1, ensuring the agreement with the on-shell field content of $N = 4$, $d = 1$ SQM.

This phenomenon can be well understood already at the classical level. The conserved Noether current generating $U(1)$ symmetry (6.10)

$$J(t) = \frac{\partial L(t)}{\partial \dot{\omega}(t)}, \quad S_{dual} \equiv \int dt L(t) \tag{6.17}$$

coincides, up to a numerical factor, with the lowest component of the superfield $\left[D, \bar{D}\right]\phi$ where ϕ is assumed to be expressed through $\Omega + \bar{\Omega}$ according to (6.7)

$$J(t) = 2\left[D, \bar{D}\right]\phi\Big|_{\theta=0} = 4\left[4\phi'(Re\Omega)\dot{\omega} + \phi''(Re\Omega)D_a\bar{\Omega}\bar{D}^a\Omega\right]\Big|_{\theta=0}. \tag{6.18}$$

In the dual formulation, constraints (6.2a) become the equations of motion, so eq. (6.3) is now fulfilled dynamically, as a consequence of these equations, and it is simply the conservation law for the $U(1)$ current (6.18))

$$\dot{J}(t) = 0 \quad on \ shell. \tag{6.19}$$

Then the condition

$$J(t) = -4m \quad on \ shell \ . \tag{6.20}$$

can be regarded as selecting a particular constant in the variety of solutions of (6.19). Eq.(6.20) can be explicitly solved for $\dot{\omega}(t)$, after that the remaining equations of motion for $Re\Omega(t)$ and four physical fermions $D_a\bar{\Omega}(\zeta)\Big|_{\theta=0}, \bar{D}^a\Omega(\zeta)\Big|_{\theta=0}$ coincide, up to a field redefinition, with those following from the $N = 4$, $d = 1$ SQM action (2.10). Thus, the $d = 1$ and $d = 2$ models in question are equivalent classically and quantum-mechanically provided the constraint (6.20) or its quantum version (6.16) are imposed.

The form of the action (6.8) suggests that it could be obtained via the Scherk-Schwarz type dimensional reduction [10] from a $U(1)$ invariant action of some $N = 2$ supersymmetric Kahler 2D sigma model

$$S_{2D} = \int dt dx d^4\theta \hat{A}\left(\Omega_2 + \bar{\Omega}_2\right), \tag{6.21}$$

where Ω_2 is a 2D chiral superfield, $\Omega_2 = \Omega_2\left(t_R, x_R, \bar{\theta}\right)$, $x_R = x - i\theta\sigma^3\bar{\theta}$.

This is indeed so. Factoring out the θ dependence associated with the shift of x

$$\Omega_2\left(t_R, x_R, \bar{\theta}\right) = \exp\left\{-i\theta\sigma^3\bar{\theta}\frac{\partial}{\partial x}\right\}\Omega\left(t_R, x, \bar{\theta}\right) \tag{6.22}$$

$$\bar{\Omega}_2\left(t_L, x_L, \theta\right) = \exp\left\{i\theta\sigma^3\theta\frac{\partial}{\partial x}\right\}\bar{\Omega}\left(t_L, x, \theta\right)$$

and neglecting the x dependence by identifying

$$\frac{1}{i}\frac{\partial}{\partial x} = \frac{\lambda}{8}J \tag{6.23}$$

(*i.e. x is assumed to be compactified on a circle of the radius $\sim \lambda^{-1}$*), we get

$$\Omega_2 + \bar{\Omega}_2 \Rightarrow \Omega(\zeta) + \bar{\Omega}(\bar{\zeta}) + \frac{\lambda}{4}\theta\sigma^3\bar{\theta}. \tag{6.24}$$

Substituting (6.24) into (6.20) and setting $\int dx \sim \gamma$, $[\gamma] = cm^1$, we arrive at the action (6.8).

It seems surprising that, starting from the most general $N = 4$, $d = 1$ action in one dimension, we have eventually found that it can be equally obtained via a dimensional reduction from the action of a supersymmetric 2D Kahler sigma model. The parameters m and λ introduced originally as the parameters of explicit breaking of two automorphism $SU(2)$ symmetries acquire an interesting interpretation as the inverse compactification radius and momentum associated with the compactified extra coordinate.

In the above discussion we have started with the action (2.10) and constraint (2.5a). However, we could equally choose to start with the equivalent description of our $d = 1$ SQM model given by eqs.(2.18), (2.17). Rearranging $D_{\alpha a}$ as $\left(D_\alpha, -\bar{D}_\alpha\right)$ and performing the duality transformation in this second superfield formulation, we would arrive at the action of the type (6.7) with the Kahler potential

$$\tilde{A}\left(\Omega + \bar{\Omega}\right) = \tilde{A}\left[u\left(\Omega + \bar{\Omega}\right)\right] + \left(\Omega + \bar{\Omega}\right)u\left(\Omega + \bar{\Omega}\right), \tag{6.25}$$

$\theta_\alpha, \bar{\theta}_\beta$ replacing $\theta_a, \bar{\theta}_b$ and m and λ interchanged. In this alternative dual description the meaning of parameters m and λ is reversed: λ becomes an eigenvalue of the corresponding extra momentum generator while m is recognized as the inverse compactification radius. Since both m and λ can be interpreted as eigenvalues of some compact $U(1)$ generators, quantum self-consistency of the theory seems to require them to be quantised. Hence, the central charge in the $N = 4$ superalgebra (4.5) should also be quantised in proper mass units

$$\gamma(m\lambda) = k\mu_0, \quad k \ \ integer\ . \tag{6.26}$$

We end with several comments.

The central charge modified superalgebra (6.15) is reminiscent of the one found by Olive and Witten [11] in some 2D models possessing topologically nontrivial soliton solutions. There, the central charge is proportional to the topological charge and it is quantised on the topologocal grounds. It would be of interest to inquire whether the central charges in the models presented here admit a topological re-interpretation.

Let us also mention that an analogous effect of the dynamical creation of an operator central charge by the duality transformation in $N = 4$ 1D superconformal algebra $su(1,1|2)$ has been revealed when studying an interplay between real and complex superfield formulations of $N = 4$, $d = 1$ superconformal mechanics [7]. As distinct from the case considered here, in [7] the central charge emerged in the anticommutator of supertranslations and superconformal boosts while the Poincare superalgebra itself remained intact (as is seen from eqs.(2.20) and (2.22), only one of two automorphism $SU(2)$'s is broken in the superconformal case).

Finally, we briefly discuss the relation to tensor $N = 1$ 4D supermultiplet [9]. Constraints (6.2a) are precisely the 1D reduction of those defining the above multiplet in $N = 1$ 4D superspace, eq.(6.3) being a 1D analog of the notoph field strength constraint $\partial^{\mu} A_{\mu} = 0$. So the off-shell field content of our $d = 1$ superfield $\phi(z)$ (or $u(z)$) could be directly obtained from $N = 1$ 4D tensor multiplet via the reduction 4D $\rightarrow$ 1D. Three components of the notoph field strength $A_i, i = 1,2,3$ become unconstrained auxiliary fields while $A_0(t)$ turns into a constant as a consequence of eq. (6.3). This way, one is left with one physical boson and four physical fermions. The specificity of $N = 4$ 1D case manifests itself in the appearance of the $SO(4) = SU_I(2) \times SU_{II}(2)$ automorphism group of spinor charges while $N = 1$ 4D Poincare superalegbra possesses the $\Gamma_5 \times SL(2,C)$ automorphism group (γ_5 invariance times Lorentz invariance). As a result, the parameter m in (6.2b) can be interpreted as a component of some constant $SU(2)$ vector and constraints (6.2) admit a manifestly $SO(4)$ invariant form (2.5a). Also, the FI term giving rise to the breaking of $SU_{II}(2)$ can be defined, with a constant $SU_{II}(2)$ vector λ_{ab} as the coupling constant. The existence of two dual-equivalent descriptions of the same $N = 4$, $d = 1$ system (through superfields $\phi(z)$ or $u(z)$) is also a pure 1D phenomenon. Since upon the reduction 4D $\rightarrow$ 1D, one of the automorphism $SU(2)$'s comes from $SL(2,C)$, one may expect that the general $N = 4$, $d = 1$ action, e.g. (2.10), is obtainable from the general $N = 1$ 4D tensor multiplet action modified by terms which explicitly break Lorentz invariance. We did not examine this possibility in detail.

In the dual $d = 2$ description through chiral superfields $\Omega, \bar{\Omega}$, the $N = 1$ 4D tensor multiplet origin of the actions (6.8), (6.21) is expressed in that Ω and $\bar{\Omega}$ always appear in the fixed $U(1)$ invariant combination $\Omega + \bar{\Omega}$. This is a generic feature of the dual-transformed tensor multiplet actions [9]. The 2D action (6.21) can be obtained by a direct reduction from the corresponding 4D dual-transformed action while further reduction 2D $\rightarrow$ 1D goes nontrivially: it involves the identification (6.23) which breaks 2D lorentz group $SO(1,1)$ (a remnant of 4D Lorentz group).

7. Conclusions

In the present paper we have described the simplest example of $N = 4$ SQM model where the arguments of Witten's no-go theorem [2] fail and, as a result, the partial spontaneous breaking of supersymmetry becomes possible. The crucial property allowing to circumvent the theorem just mentioned is the appearance of a non-zero constant central charge in the anticommutator of $N = 4$ 1D supercharges. This central charge is proportional to the product of two parameters measuring the strength of explicit breaking of two automorphism $SU(2)$ symmetries. Thus, the necessary condition for $N = 4$ supersymmetry to be partially broken in the model under consideration is that both automorphism $SU(2)$'s are explicitly broken

We have also found an interesting relation between our 1D models and $N = 2$ 2D Kahler sigma models with $U(1)$ isometry. The former models follow from the latter ones via a dimensional reduction of the Scherk-Schwarz type with further restriction of the relevant space of states to a subspace spanned by the states having the same value of the $U(1)$ charge. As a result of dimensional reduction, the $U(1)$ generator becomes the central charge generator of $N = 4$ 1D Poincare superalgebra.

There exist several conceivable ways of extending these results. E.g., it would be interesting to construct multicomponent $N = 4$ SQM models involving more superfields $\phi(z)$ and to study the phenomenon of partial supersymmetry breaking in these general models. They may bear a tight relation to higher dimensional gauge theories, such as super Yang-Mills and supergravity. It is known that in these theories supersymmetry may happen to be partially broken on account of appropriate classical solutions (instantons, monopoles, ...) [1] [6] [12].

A separate intriguing question raised by the above consideration is as follows. As was already discussed, in the formulations of $N = 4$, $d = 1$ SQM via real superfields $\phi(z)$ or $u(z)$ the central charge of $N = 4$ superalgebra is a constant producing no transformations of the involved fields. However, after performing a duality transformation the central charge becomes active and generates a nontrivial $U(1)$ symmetry. One may wonder whether a similar phenomenon can be revealed in 2D models based on infinite-dimensional algebras of the Kac-Moody or Virasoro types, which also involve constant central charges. In other words, may such models be embedded into more general ones in which the relevant central charges are generators of some symmetries having a nontrivial action on the fields?[6] E.g., the constant central charges characterizing various conformally invariant 2D systems (in particular, the minimal models) could come out as different eigenvalues of a single central charge operator in some more general embracing theory. The spaces of states of these specific systems could then be identified with appropriate subspaces of the Hilbert space of the general theory. One may think about the Chern-Simons and topological field theories as possible candidates for such a theory [14].

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[6]In a recent preprint [13] the central charge of affine Lie algebra sl_2 was treated on equal footing with other generators in order to construct a zero curvature representation leading to a new type Toda theory. This is also in the spirit of the idea of geometric quantisation.

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