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дубна

E2-90-379

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EMBEDDING OF LEPTONS AND QUARKS IN OCTONIONIC STRUCTURES

Submitted to Fifth International Conference
on Hadronic Mechanics and Nonpotential Interaction,
USA

1990

Lately, the idea of succession of the lepton-quark generations (or families) becomes very popular in the elementary particle physics. At present time, we have two total and third almost accomplished (now only the t-quark needs confirmation) generations

$$\nu_e, e, u, d; \quad \nu_\mu, \mu, c, s; \quad \nu_\tau, \tau, t?, b.$$

The most remarkable property of these generations is the recurrence of the interaction characteristics: The corresponding leptons and quarks have the same electric charges (0,-1, 2/3, -1/3) and are submitted to the same electroweak interaction, all quarks participate in the strong interaction with the same (universal) gluons, and the corresponding $SU(3)_c$ symmetry is perfect. It is very likely that leptons and quarks from different generations differ only by their masses (Here we digress from the problem of the quark-mixing in the weak interaction).

In this report, I propose to compare the repetition of the quark-lepton generations with some mathematical scheme of post-octonions obtained by means of the Cayley-Dickson procedure of doubling of hypernumber systems/1/. Maybe, the adequate mathematical description can help us to answer the physical sacramental questions?

It is well known that this doubling procedure* yields to the recurrence: real number ($\mathcal{R}$) ⇒ complex numbers ($\mathcal{C}$) ⇒ quaternions (Q) ⇒ octonions (O), with the loss of commutativity for Q and the associativity for O, successively. These hypernumber systems contain 0,1,3,7 imaginary units, respectively. This series may be continued by the doubling of

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* This procedure is taken out into Appendix A

octonions. Then, we receive *di*-octonions with the number of imaginary units equal to 15. Further, we obtain *di*-dioctonions with 31 imaginary units and so on. We name these hypernumber systems obtained from octonions by the doubling procedure *postoctonions*. All these systems are nonassociative*.

Now we compose *the hyperfield*

$$\psi(x) = \sum_{\alpha=1}^{p} e_\alpha \Psi^\alpha(x), \qquad (1)$$

where $\Psi^\alpha(x)$ are the charge-self-conjugated Majorana fields obeying ordinary Fermi statistics with normal (Fermi) relative anticommutation relations, and quantities e_α are the *imeginary* units of a given hypernumber system. Thus, the hyperfield (1) satisfies the Dirac equation

$$(\hat{\gamma}^\mu \partial_\mu - m)\psi(x) = 0, \qquad (2)$$

where $\hat{\gamma}^\mu \equiv i\gamma^\mu$. We make use of the Majorana basis for the Dirac gamma matrices and thus matrices $\hat{\gamma}^\mu$ are real. This condition is essential because the hypernumbers are defined over real numbers, and we have no right to use imaginary numbers. In the Majorana basis the charge conjugation involves only Hermitian conjugation for the fields:

$$\psi_c(x) = \psi^+(x), \qquad (3)$$

and for the Majorana field (1) (and its components) we have

$$\psi^+(x) = \psi(x). \qquad (4)$$

As the imaginary units satisfy the Clifford algebra (with the nonessential change of the sign in the left-hand side of equation)

* The octonions are nonassociative but they remain alternative and power-associative. Already dioctonions lose the latter properties. The flexibility is the only property remaining valid for all postoctonions.

$$e_\alpha e_\beta + e_\beta e_\alpha = -2\,\delta_{\alpha\beta}, \qquad (5)$$

the hyperfield (1) is the Green parafield satisfying the Green trilinear commutation relations[2]. Thus, we can consider the hyperfield (1) as nonassociative algebraic realization of the para-Fermi statistics of an order p of identical particles.

Günaydin and Gürsey[3] were the first who adopted octonions for the description of colour quarks. Then, Gürsey[4] proposed to use octonions for the unified description of leptons and coloured quarks.

In the case of octonions, we have seven imaginary units. The octofield (octonionic hyperfield) (1) can be represented as a column

$$\psi(x) = \begin{pmatrix} l(x) \\ q(x) \end{pmatrix}, \qquad (6)$$

where a scalar component $l(x)$ coincides with the seventh component $\psi^7(x)$, and it is the Majorana field $\nu(x)$ times i which we called *a lepton* (Majorana neutrino) component of the octofield; the vector components are the Dirac fields*

$$q_k = \Psi^k + i\Psi^{k+3}, \quad k=1,2,3, \qquad (7)$$

which we named *colour quark* components of octofield.

The product of two octofields is defined in the same manner as a product of two octonions (see, Appendix A) with a substitution of the complex conjugation by the charge- or Hermitian-conjugation (without an alternation of the order of initial anticommuting field operators).

* To avoid misunderstanding, one should emphasize that the appearance of the imaginary unit i in these components is a consequence of our special representation of octonionic units (see, Appendix A). The octonionic algebra remains to be defined over real numbers.

Let us extend this scheme and pass to dioctonions with fifteen imaginary units[1]. Then, we can include the full first lepton-quark generation into the dioctofield which can be represented by two columns

$$\psi(x) = \begin{pmatrix} i\nu(x) & e(x) \\ u(x) & d(x) \end{pmatrix}. \tag{8}$$

The second step gives us *two* lepton-quark generations within the di-dioctofield

$$\psi(x) = \begin{pmatrix} i\nu_e & e & \nu_\mu & \mu \\ u & d & c & s \end{pmatrix}. \tag{9}$$

One ought to emphasize that in the construction (9) only the electron neutrino ν_e is a Majorana particle, whereas the muon neutrino ν_μ is a Dirac particle.

If we continue this process, we shall achieve *four* (not three!) lepton-quark generations at the next step of this procedure. Thus, the existence of four lepton-quark generations is an inevitable result of this construction.

At the next step we have eight generations, and our mind gives up to estimate the finish of this avalanche-like generation doubling process.

Now we consider a gauging of theories of that kind.

It is remarkable that beginning from octonions the group of automorphisms of hypernumber systems is always the same Cartan exceptional G_2 group[5] (14 parameters, rank 2)*. The G_2 group transforms separate columns of postoctonions (8), (9) by the vertical independently.

For the gauging of this automorphism group we ought to introduce gauge vector fields and change the derivative to

* The automorphism group of complex numbers is complex conjugation Z_2 . The automorphism group of quaternions is *SO(3)* whose gauging was presented in[6].

the gauge-covariant derivative

$$D_\mu \begin{pmatrix} l \\ q \end{pmatrix} = \begin{pmatrix} \partial_\mu l - ig(y_\mu^+ \cdot q + y_\mu \cdot q^+) \\ \partial_\mu q - ig[y_\mu(l - l^+) - y_\mu^+ \times q^+ + \sum_{j=1}^{3} w_\mu^j q_j] \end{pmatrix}, \tag{10}$$

where gauge fields y_μ are responsible for the lepton-quark transitions, and w_μ^j are gluon fields mixing up colour components of quarks. The components of the gluon fields obey the following conditions:

$$w_\mu^{ij} = (w_\mu^{ji})^+, \qquad w_\mu^{11} + w_\mu^{22} + w_\mu^{33} = 0. \tag{11}$$

Naturally, we need to suppress the lepton-quark transitions. This suppression can be obtained by the Higgs mechanism.

We introduce the scalar octofield with only first non-vanishing column

$$\phi = \begin{pmatrix} i\phi_o & o \\ f & o \end{pmatrix}, \quad \phi_o^+ = \phi_o. \tag{12}$$

The Lagrangian of this scalar octofield has the Higgs form

$$\mathcal{L}_\phi = -(D_\mu\phi)(D^\mu\phi) - \mu^2\phi^2 - \lambda\phi^4 = (D_\mu\phi_o)(D^\mu\phi_o)^+ + (D_\mu f)(D^\mu f)^+ +$$
$$+ \mu^2(\phi_o^2 + f\cdot f^+) - \lambda(\phi_o^2 + f\cdot f^+)^2. \tag{13}$$

The minimum of the scalar potential is achieved at

$$-\langle\phi^2\rangle_o = \langle\phi_o^2\rangle_o + \langle f\cdot f^+\rangle_o = v^2/2, \qquad v^2 \equiv \mu^2/\lambda. \tag{14}$$

Using the G_2- gauge invariance we can choose the gauging with $f \equiv 0$. Then, decomposing $\phi_o = \langle\phi_o\rangle_o + \theta(x)$ we take the Higgs scalar θ with the mass equal to $\sqrt{2}\,\mu$, vector bosons y_μ with the masses equal to $\sqrt{2}\,gv$, and massless gluons w_μ^j. We can choose the value of the parameter v sufficiently large for the strong suppression of lepton-quark transitions. Thus, the perfect *SU(3)* symmetry of colour quarks belonging to different generations can be explained as an unbroken subgroup of the spontaneously broken automorphism group G_2.

The gauge $SU(2)_L \times U(1)$-symmetry of electroweak interactions mixes up the (left) components of two columns of di-octofields by the horizontal. As usual, left components (ν_L, e_L), (u_L, d_L) etc. compose doublets and right components e_R, u_R, and d_R etc. compose singlets.

For the spontaneous breakdown of the electroweak symmetry we introduce another Higgs scalar hyperfield *including the usual unit*:

$$\xi(x) = \begin{pmatrix} \xi_o^-(x) & X_o^-(x) \\ x(x) & X^-(x) \end{pmatrix}, \tag{15}$$

so $\xi_o^+ \neq -\xi_o$. We can get rid of colour components x and X^- via G_2- gauge transformation. However, this gauge could not coincide with the previous one eliminating fields **f**. If we pass to this latter gauge, the colour components in (15) could appear again:

$$x = iy(\xi_o - \xi_o^+), \quad X^- = iy[X_o^- - (X_o^-)^+], \tag{16}$$

where the parameters y characterize the transition from one gauge to another. But now we can apply the $SU(2)_L \times U(1)$ gauge invariance and shake off the component X_o^- from the beginning and turn the ξ_o-component into the proper Hermitian (real) form: $\xi_o = \xi_o^+$ (to compare with the proper imaginary $i\phi_o$ in (12)). Thus, we achieve the gauge when the dioctofield $\xi(x)$ turns into the usual Higgs scalar field, and our theory becomes the usual Glashow-Salam-Weinberg theory.

Spinor lepton-quark fields and Higgs-scalar fields can be embedded into the post-octonionic parafields. However, gauge fields cannot be formulated within proper octofields. For their formulation we need to transform the separate components of spinor and scalar hyperfields.

The only self-consistent gauge hyperfield is the quaternionic one which is equivalent to the $SO(3)$-gauge theory. In this case, gauge fields become "*quaterfiels*" too[6].

Now we consider the possibility of determining charges of leptons and quarks in the framework of the postoctonionic hyperfield theory.

We propose, following Günaydin and Gürsey[3,4], that only *associative* combinations of nonassociative hyperfields can be involved into the consistent theory*. Then, we shall consider the transformations which leave these combinations invariant.

For the inclusion of two quark-lepton generations we at once consider di-dioctofield (9).

The bilinear associative combination has the form of a commutator

$$-\tfrac{1}{2}[\psi_\alpha, \psi_\beta]_- = \nu_\alpha^e \nu_\beta^e + \frac{1}{2} \sum_{l=e,\nu_\mu,\mu} (l_\alpha^+ l_\beta + l_\alpha l_\beta^+) + \frac{1}{2} \sum_{q=u,d,c,s} (q_\alpha^+ q_\beta + q_\alpha q_\beta^+), \tag{17}$$

where only spinor indices α and β are kept in arguments.

Now we consider the phase-trasformation

$$l \Rightarrow l\, exp(i\omega\, Q_l), \quad q \Rightarrow q\, exp(i\omega\, Q_q), \tag{18}$$

where ω is an arbitrary phase, and Q_l and Q_q are electric charges of leptons and quarks, respectively. The invariance of the form (17) under transformations (18) implies only that the electron neutrino ν_e being the Majorana particle has no electric charge. Other charges of leptons and quarks remain arbitrary.

Remark, the Lagrangian contains the commutator of free fields in the form (17) which is antisymmetric under the exchange of α and β.

Further, we consider the only trilinear associative combination in the form**

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* One may consider this requirement as the demand of the c-number character of the Wightman vacuum distributions.

** The analogous trilinear form was considered by Gamba in the investigation of peculiarities of the eight-dimensional space[8].

$$
\begin{aligned}
\psi_\alpha(\psi_\beta\psi_\gamma)+(\psi_\gamma\psi_\beta)\psi_\alpha = {} & [-i\nu^e_\alpha(\nu^\mu_\beta\nu^{\mu+}_\gamma+\mathbf{u}_\beta\cdot\mathbf{u}^+_\gamma+\mathbf{c}^+_\beta\cdot\mathbf{c}_\gamma)+\nu^\mu_\alpha\mathbf{u}_\beta\cdot\mathbf{c}^+_\gamma+\nu^{\mu+}_\alpha\mathbf{u}^+_\beta\cdot\mathbf{c}_\gamma \\
& -\tfrac{1}{6}\mathbf{u}_\alpha\cdot\mathbf{u}_\beta\times\mathbf{u}_\gamma-\tfrac{1}{6}\mathbf{u}^+_\alpha\cdot\mathbf{u}^+_\beta\times\mathbf{u}^+_\gamma+\tfrac{1}{2}\mathbf{u}_\alpha\cdot\mathbf{c}_\beta\times\mathbf{c}_\gamma+\tfrac{1}{2}\mathbf{u}^+_\alpha\cdot\mathbf{c}^+_\beta\times\mathbf{c}^+_\gamma] + \\
& +[-i\nu^e_\alpha(e_\beta e^+_\gamma+\mu^+_\beta\mu_\gamma+\mathbf{d}^+_\beta\cdot\mathbf{d}_\gamma+\mathbf{s}_\beta\cdot\mathbf{s}^+_\gamma)+\nu^\mu_\alpha(e_\beta\mu^+_\gamma+\mathbf{d}^+_\beta\cdot\mathbf{s}_\gamma)+\nu^{\mu+}_\alpha(e^+_\beta\mu_\gamma+\mathbf{d}_\beta\cdot\mathbf{s}^+_\gamma) \\
& +e_\alpha(\mathbf{u}_\beta\cdot\mathbf{d}^+_\gamma+\mathbf{s}^+_\beta\cdot\mathbf{c}_\gamma)+e^+_\alpha(\mathbf{u}^+_\beta\cdot\mathbf{d}_\gamma+\mathbf{s}_\beta\cdot\mathbf{c}^+_\gamma)+ \\
& +\mu_\alpha(\mathbf{s}^+_\beta\cdot\mathbf{u}_\gamma+\mathbf{d}^+_\beta\cdot\mathbf{c}_\gamma)+\mu^+_\alpha(\mathbf{s}_\beta\cdot\mathbf{u}^+_\gamma+\mathbf{d}_\beta\cdot\mathbf{c}^+_\gamma)+ \\
& +\tfrac{1}{2}\mathbf{u}_\alpha\cdot\mathbf{d}_\beta\times\mathbf{d}_\gamma+\tfrac{1}{2}\mathbf{u}^+_\alpha\cdot\mathbf{d}^+_\beta\times\mathbf{d}^+_\gamma-\tfrac{1}{2}\mathbf{u}_\alpha\cdot\mathbf{s}_\beta\times\mathbf{s}_\gamma-\tfrac{1}{2}\mathbf{u}^+_\alpha\cdot\mathbf{s}^+_\beta\times\mathbf{s}^+_\gamma-\mathbf{d}_\alpha\cdot\mathbf{s}_\beta\times\mathbf{c}_\gamma-\mathbf{d}^+_\alpha\cdot\mathbf{s}^+_\beta\times\mathbf{c}^+_\gamma] + \\
& + \text{all permutations of } (\alpha,\beta,\gamma). \qquad (19)
\end{aligned}
$$

The terms in the first square brackets are obtained when second and fourth columns of di-dioctofields (9) vanish, i.e. di-dioctofields (9) contain only two half-generations. Any phase-transformation (18) is forbidden by these terms. To avoid such prohibition of any phase-transformation we are compeled to subtract the form (19) with first half-generations from the whole form (19) at the onset. Then, only terms standing in the second square brackets remain. The demand of the invariance of these terms under the transformation (18) leads to the following relations for the electric charges of leptons and quarks (of two generations)

$$Q_{\nu_\mu}+Q_e-Q_\mu=0 \quad (20a) \qquad Q_{\nu_\mu}+Q_d-Q_s=0 \quad (20b)$$

$$Q_e+Q_u-Q_d=0 \quad (20c) \qquad Q_e+Q_s-Q_c=0 \quad (20d)$$

$$Q_\mu+Q_s-Q_u=0 \quad (20e) \qquad Q_\mu+Q_d-Q_c=0 \quad (20f)$$

$$Q_u+2Q_d=0 \quad (20g)$$

$$Q_u+2Q_s=0 \quad (20h)$$

$$Q_d+Q_s+Q_c=0 \quad (20i).$$

The solution of these equations gives the following relations: $Q_u=-2Q_d=-2Q_s$ due to (20g) and (20h), $Q_c=Q_u$ due to (20i), $Q_{\nu_\mu}=0$ due to (20b), $Q_e=Q_\mu$ due to (20a), and finally, $Q_e=3Q_d=-(3/2)Q_u$ due to (20c). Thus, we obtain the right eigenvalues of the electric charges of leptons and quarks

$$
\begin{gathered}
Q_{\nu_e}=Q_{\nu_\mu}=0, \qquad Q_\mu=Q_e, \\
Q_u=Q_c=-(2/3)Q_e, \qquad Q_d=Q_s=(1/3)Q_e.
\end{gathered}
\qquad (21)
$$

We emphasize that these values are obtained automatically from the requirement of the invariance of the modified tri-linear form (19) under the phase-transformation (18).

Now, the main open question of this scheme is: what is the reason for the increase of lepton quark masses for successive generations and where the limit of the doubling generation process lies?

Appendix A. The Cayley-Dickson doubling procedure

Let there be an algebra with unity and with the number p of imaginary units such that their products are

$$e_a e_b = -\delta_{ab}+f_{abc}e_c \ , \quad a,b=1,\ldots,p \ , \qquad (A.1)$$

where δ_{ab} is the Kronecker symbol and f_{abc} are antisymmetrical structure constants.

One introduces a new imaginary unit e, $e^2=-1$ and composes other new imaginary units by means of multiplication of initial imaginary units on e

$$E_a=e_a e=-ee_a . \qquad (A.2)$$

Then, the algebra consisting of $1, e_a, e, E_a$ $(a=1,\ldots,p)$ is accomplished by the multiplication rules

$$
\begin{gathered}
eE_a = -E_a e = e_a , \\
e_a E_b = -E_b e_a = -\delta_{ab} e - f_{abc} E_c , \\
E_a E_b = -\delta_{ab} - f_{abc} e_c .
\end{gathered}
\tag{A.3}
$$

Therefore, the antisymmetry of the product of both initial and new imaginary units is conserved.

For the octonions we have the following multiplication table:

	1	e_j	e_{j+3}	e_7
e_i		$-\delta_{ij} + \varepsilon_{ijk} e_k$	$\delta_{ij} e_7 - \varepsilon_{ijk} e_{k+3}$	$-e_{i+3}$
e_{i+3}		$-\delta_{ij} - \varepsilon_{ijk} e_{k+3}$	$-\delta_{ij} - \varepsilon_{ijk} e_k$	e_i
e_7		e_{j+3}	$-e_j$	-1 ,

where $i,j=1,2,3,$ and ε_{ijk} is the antisymmetric tensor with $\varepsilon_{123}=1$.

Any octonion can be presented by a column

$$
a = \begin{pmatrix} a_o \\ \mathbf{a} \end{pmatrix} ,
\tag{A.4}
$$

where a_o is a complex scalar, and $\mathbf{a}$ is a comlex vector. The product of two octonions is defined as*

$$
ab = \begin{pmatrix} a_o \\ \mathbf{a} \end{pmatrix} \begin{pmatrix} b_o \\ \mathbf{b} \end{pmatrix} = \begin{pmatrix} a_o b_o - \mathbf{a} \cdot \mathbf{b}^* \\ a_o \mathbf{b} + \mathbf{a} b_o^* + \mathbf{a}^* \times \mathbf{b}^* \end{pmatrix} ,
\tag{A.5}
$$

where a star denotes the comlex conjugation (or the Hermitian conjugation for the field operators *without* an alternation of the order of operators). In this representation octonionic units have the forms

* The same multiplication rule for octonions has been applied by de Alfaro, Fubini and Furlan/8/ for the description of instantons in the eight-dimensional space.

$$
1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad e_7 = \begin{pmatrix} i \\ 0 \end{pmatrix}, \quad e_j = \begin{pmatrix} 0 \\ \mathbf{e}_j \end{pmatrix}, \quad e_{j+3} = \begin{pmatrix} 0 \\ i\mathbf{e}_j \end{pmatrix} ,
\tag{A.6}
$$

where $\mathbf{e}_j$ are three orthonormal basis vectors:

$$
\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} , \quad \mathbf{e}_i \times \mathbf{e}_j = \varepsilon_{ijk} \mathbf{e}_k .
$$

The conjugate octonion (and any conjugate postoctonion as well) marked by a tilda is determined by a change of signs of all imaginary units, i.e.

$$
\tilde{a} = \begin{pmatrix} a_o^* \\ -\mathbf{a} \end{pmatrix} .
\tag{A.7}
$$

Any dioctonions can be presented by *two* columns

$$
a = \begin{pmatrix} a_o & A_o \\ \mathbf{a} & \mathbf{A} \end{pmatrix}
\tag{A.8}
$$

and the product of two dioctonions is

$$
ab = \begin{pmatrix} a_o & A_o \\ \mathbf{a} & \mathbf{A} \end{pmatrix} \begin{pmatrix} b_o & B_o \\ \mathbf{b} & \mathbf{B} \end{pmatrix} =
$$

$$
\begin{pmatrix} a\,b - \mathbf{a} \cdot \mathbf{b}^* - A\,B^* - \mathbf{A}^* \cdot \mathbf{B} & a\,B - \mathbf{a}^* \cdot \mathbf{B} + A\,b^* + \mathbf{A} \cdot \mathbf{b}^* \\ a_o \mathbf{b} + \mathbf{a}\, b_o^* + \mathbf{a}^* \times \mathbf{b}^* + & a_o^* \mathbf{B} + \mathbf{a}\, B_o - \mathbf{a}^* \times \mathbf{B}^* + \\ \quad + A_o^* \mathbf{B} - \mathbf{A}\, B_o^* - \mathbf{A}^* \times \mathbf{B}^* & \quad + \mathbf{A}\, b_o - A_o \mathbf{b} - \mathbf{A}^* \times \mathbf{b}^* \end{pmatrix} .
\tag{A.9}
$$

The dioctonionic units have the forms

$$
1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_7 = \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix}, \quad e_j = \begin{pmatrix} 0 & 0 \\ \mathbf{e}_j & 0 \end{pmatrix}, \quad e_{j+3} = \begin{pmatrix} 0 & 0 \\ i\mathbf{e}_j & 0 \end{pmatrix}
$$

$$
e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_7 = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix}, \quad E_j = \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{e}_j \end{pmatrix}, \quad E_{j+3} = \begin{pmatrix} 0 & 0 \\ 0 & i\mathbf{e}_j \end{pmatrix} ,
$$

where $j=1,2,3$.

Any di-dioctonions can be presented by *four* colums

$$
a = \begin{pmatrix} a_o & A_o & a'_o & A'_o \\ \mathbf{a} & \mathbf{A} & \mathbf{a}' & \mathbf{A}' \end{pmatrix} .
\tag{A.10}
$$

The product of two di-dioctonions is

$$\begin{pmatrix} a_o & A_o & a'_o & A'_o \\ \mathbf{a} & \mathbf{A} & \mathbf{a}' & \mathbf{A}' \end{pmatrix}\begin{pmatrix} b_o & B_o & b'_o & B'_o \\ \mathbf{b} & \mathbf{B} & \mathbf{b}' & \mathbf{B}' \end{pmatrix} =$$

$$\left(\begin{array}{l}
a_o b_o-\mathbf{a}\cdot\mathbf{b}^*-A_o B_o^*-\mathbf{A}^*\cdot\mathbf{B}-a'_o b'^*_o-\mathbf{a}'^*\cdot\mathbf{b}'-A'^*_o B'_o-\mathbf{A}'\cdot\mathbf{B}'^* \\
a_o\mathbf{b}+\mathbf{a}\,b_o^*+\mathbf{a}^*\times\mathbf{b}^*+A_o^*\mathbf{B}-\mathbf{A}B_o^*-\mathbf{A}^*\times\mathbf{B}^*+a'^*_o\mathbf{b}'-\mathbf{a}'b'^*_o-\mathbf{a}'^*\times\mathbf{b}'^*- \\
\qquad\qquad -A'^*_o\mathbf{B}'+\mathbf{A}'B'^*_o+\mathbf{A}'^*\times\mathbf{B}'^* \\[1em]
a_o B_o-\mathbf{a}^*\cdot\mathbf{B}+A_o b_o^*+\mathbf{A}\cdot\mathbf{b}^*+a'^*_o B'_o+\mathbf{a}'^*\cdot\mathbf{B}'-A'_o b'^*_o-\mathbf{A}'\cdot\mathbf{b}'^* \\
a_o^*\mathbf{B}+\mathbf{a}\,B_o-\mathbf{a}^*\times\mathbf{B}^*-A_o\mathbf{b}+\mathbf{A}b_o-\mathbf{A}^*\times\mathbf{b}^*+a'_o\mathbf{B}'-\mathbf{a}'B'_o+\mathbf{a}'^*\times\mathbf{B}'^*+ \\
\qquad\qquad +A'_o\mathbf{b}'-\mathbf{A}'b'_o+\mathbf{A}'^*\times\mathbf{b}'^* \\[1em]
a_o b'_o-\mathbf{a}^*\cdot\mathbf{b}'+a'_o b_o^*+\mathbf{a}'\cdot\mathbf{b}^*-A_o^* B'_o+A'_o B_o^*-\mathbf{A}\cdot\mathbf{B}'^*+\mathbf{A}'^*\cdot\mathbf{B} \\
a_o^*\mathbf{b}'+\mathbf{a}\,b'_o-\mathbf{a}^*\times\mathbf{b}'^*-A_o^*\mathbf{B}'+\mathbf{A}B'^*_o+\mathbf{A}^*\times\mathbf{B}'^*-a'_o\mathbf{b}+\mathbf{a}'b_o-\mathbf{a}'^*\times\mathbf{b}^* \\
\qquad\qquad -A'^*_o\mathbf{B}+\mathbf{A}'B_o^*+\mathbf{A}'^*\times\mathbf{B}^* \\[1em]
a_o^* B'+\mathbf{a}^*\cdot\mathbf{B}'+A_o b'_o-\mathbf{A}\cdot\mathbf{b}'^*-a'_o B_o+\mathbf{a}'^*\cdot\mathbf{B}+A'_o b_o-\mathbf{A}'\cdot\mathbf{b}^* \\
a_o\mathbf{B}'-\mathbf{a}\,B'_o+\mathbf{a}^*\times\mathbf{B}'^*+A_o\mathbf{b}'+\mathbf{A}b'^*_o+\mathbf{A}^*\times\mathbf{b}'^*-a'^*_o\mathbf{B}-\mathbf{a}'B_o+\mathbf{a}'^*\times\mathbf{B}^*+ \\
\qquad\qquad +A'_o\mathbf{b}+\mathbf{A}'b_o^*+\mathbf{A}'^*\times\mathbf{b}^*
\end{array}\right).$$

The di-dioctonionic units have the forms

$$1=\begin{pmatrix} 1&0&0&0\\0&0&0&0 \end{pmatrix},\ e_7=\begin{pmatrix} i&0&0&0\\0&0&0&0 \end{pmatrix},\ e_j=\begin{pmatrix} 0&0&0&0\\ \mathbf{e}_j&0&0&0 \end{pmatrix},\ e_{j+3}=\begin{pmatrix} 0&0&0&0\\ i\mathbf{e}_j&0&0&0 \end{pmatrix}$$

$$e=\begin{pmatrix} 0&1&0&0\\0&0&0&0 \end{pmatrix},\ E_7=\begin{pmatrix} 0&i&0&0\\0&0&0&0 \end{pmatrix},\ E_j=\begin{pmatrix} 0&0&0&0\\ 0&\mathbf{e}_j&0&0 \end{pmatrix},\ E_{j+3}=\begin{pmatrix} 0&0&0&0\\ 0&i\mathbf{e}_j&0&0 \end{pmatrix}$$

$$e'=\begin{pmatrix} 0&0&1&0\\0&0&0&0 \end{pmatrix},\ e'_7=\begin{pmatrix} 0&0&i&0\\0&0&0&0 \end{pmatrix},\ e'_j=\begin{pmatrix} 0&0&0&0\\ 0&0&\mathbf{e}_j&0 \end{pmatrix},\ e'_{j+3}=\begin{pmatrix} 0&0&0&0\\ 0&0&i\mathbf{e}_j&0 \end{pmatrix}$$

$$E'=\begin{pmatrix} 0&0&0&1\\0&0&0&0 \end{pmatrix},\ E'_7=\begin{pmatrix} 0&0&0&i\\0&0&0&0 \end{pmatrix},\ E'_j=\begin{pmatrix} 0&0&0&0\\ 0&0&0&\mathbf{e}_j \end{pmatrix},\ E'_{j+1}=\begin{pmatrix} 0&0&0&0\\ 0&0&0&i\mathbf{e}_j \end{pmatrix}$$

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Received by Publishing Department
on June 4, 1990.