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THE GENERALIZED FIELD QUANTIZATION AND THE PAULI PRINCIPLE

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1. Introduction. The identical particles and the second quantization

Our experience teaches us that all particles, which can be imagined as point-like objects, obey either Fermi-Dirac or Bose-Einstein statistics. This law can be applied to particles really observed in our laboratories (electrons, photons, nucleons, mesons, and so on) or to only imaginary particles such as quarks and gluons which cannot be outside hadrons in principle. The reason for this Law of Nature, we may expect, lies in deep properties of the Matter such as its identity and reproduction.

There is the famous Pauli theorem on the connection between a particle spin and particle statistics: particles with integer spins obey Bose statistics and particles with half-integer spins obey Fermi statistics.

However, a question arises: are there any other possibilities for particle statistics besides the ordinary Fermi and Bose statistics? For example, can we have a small violation of the Pauli principle (for electrons)? - the question which has been raised recently by Ignatiev and Kuzmin again[1] (For a previous history of this problem see a remarkable review[2]).

In this report I shall try to derive possible statistics of identical particles from their indistinguishability and to answer the question: how high is the price for a very small violation of the Pauli principle?

The usual way of defining the indistinguishability of identical particles consists in the following: "we use the word *identical* to describe particles that can be substituted for each other under the most general possible circumstances with no change in the physical situation"[3]. A shortcoming of this definition is obvious: we are in need of the notion of permutations of identical particles which is meaningless.

A more correct definition of the indistinguishability can be obtained by means of the concept of quantum fields: identical particles can be considered as excitations (quanta) of the same field. The field quantization rules yield to certain symmetrized properties of particle wave functions. But then the problem turns about: we are in need of a determination of quantization rules for the definition of the indistinguishability.

The most self-consistent consideration of the problem of statistics of (massive) fields was done by Doplicher,Haag and Roberts in their works[/4/] within the local algebra of observables. In free words, their conception of a particle can be described as a well-localized system which is completely uncorrelated to the rest of the world in sense that one may take it out of the world or add it without effecting measurements in the space-like complement of its localization region* (More subtle concept of a particle was formulated by Buchholz and Fredenhagen[/5/] for the case when "the particles sitting at the endpoints of a string (flux lines) cannot be treated as isolated system"). The result** consists in the possibility of the existence of only three types of statistics: *para-Fermi* and *para-Bose* statistics of identical particles for which the number of particles in a symmetric or antisymmetric state, respectively, cannot exceed some given integer p, which is called the order of the (finite) parastatistics, and *infinite* statistics without any restrictions

* Such a possibility is tightly connected with a topology of the space and models. For example, it cannot be applied to the (1+1) - space-time models: we cannot take out one particle (along one-dimensional space direction) without disterbing other particles. In these cases, statistics of particles may be very uncommon.

** The same classification of identical particle statistics was obtained by Hartle, Stolt and Taylor[/6/] whose point of departure was the cluster properties of the wave functions of particles.

on the number of particles in the symmetric and antisymmetric states. Obviously, the finite statistics corresponding to p=1 coincides with ordinary Fermi and Bose statistics. Then, in the works[/5/] it has been shown that "in the case of infinite statistics a reasonable conjugate sector cannot be constructed" or, in other words, in this case antiparticles cannot be included. So the case of infinite statistics was excluded from the local theory.

All the same, we need the definition of certain field commutation relations (second-quantized theory) for the local field formulation of parastatistics. Green[/7/] was the first who gave such example in the form of trilinear commutation relations of fields (Independently, the same relations for a special case of para-Fermi statistics of order two have been proposed by Volkov[/8/]). Greenberg and Messiah[/9/] proved the *sufficiency* of the Green paraquantization for the description of parastatistics of order p. Then, the question of the *necessity* of the Green quantization for the description of parastatistics arose[/10/].

At the first step we, declining from an excessive regour, consider a system of the fixed number n of non-relativistic particles. We propose the principle of the indistinguishability of identical particles in the form of the requirement of *symmetry of the density matrix* with respect to all permutations $\mathcal{P}$ of its arguments

$$\rho(\mathbf{x}_1,\ldots,\mathbf{x}_n;\mathbf{x}'_1,\ldots,\mathbf{x}'_n;t) = \rho(\mathbf{x}_{\mathcal{P}_1},\ldots,\mathbf{x}_{\mathcal{P}_n};\mathbf{x}'_{\mathcal{P}_1},\ldots,\mathbf{x}'_{\mathcal{P}_n};t).\quad(1)$$

Recall, the diagonal density matrix when $\mathbf{x}_1=\mathbf{x}'_1,\ldots,\mathbf{x}_n=\mathbf{x}'_n$ represent density of probability to find *any* particle in a position $\mathbf{x}_1$, *any* particle in a position $\mathbf{x}_2$, and so on, at an istant t*. Considering Fermi and Bose statistics Bogolubov[/11/] noted this common property and emphasizing its

* This requirement can be compared with the independence of the order of the product $\rho_1\circ\rho_2\circ\ldots\circ\rho_n$ of morphisms $\rho_1,\ldots,\rho_n$ of the local algebra of observables with mutual space-like supports[/4/].

correspondence to the symmetry property of classical distribution function for nondistinguishable classical particles called it "*the classical symmetry*" of the density matrix in contrast with the more specific "*quantum symmetry*" of wave functions or, which is the same thing, symmetry (or antisymmetry) of the density matrix with respect to the "*primary*" $x_1,\dots,x_n$ and "*secondary*" $x'_1,\dots,x'_n$ arguments separately.

Now we put the classical symmetry of the density matrix in the basis of our consideration of possible statistics of identical particles in a *general* case without any assumptions of symmetry properties of the density matrix with respect to permutations of primary arguments or permutations of secondary arguments performed separately. Then, the density matrix rather than the wave function undergoes the second quantization[10].

Although we cannot introduce occupation numbers for each of these sets of arguments of density matrix separately, this can be done for the two sets together. One can define the number n_{ij} of particles occupying the given state $k^{(i)}$ (for instant, momentum and spin state) among all the primary states and the state $k^{(j)}$ among all the secondary states. Then, we can consider the "*space of double occupation number* n_{ij}" and characterize the state of the system by specifying the density matrix in this space: $\rho(n_{ij})$, $\sum\limits_{n_{ii}=1}^{\infty} \rho(n_{ii}) = 1$.

After this, we can introduce operators of transition of a particle from one primary state, say s, to another primary state, say r, without any transitions in the secondary states

$$N_{sr}|n^o_{ij}> = \sum_{q=1}^{\infty} [n^o_{sq}(n^o_{rq} - \delta_{rs}+1)]^{1/2}|n^o_{rq}+1, n^o_{sq}-1>, \qquad (2)$$

where

$$|n^o_{ij}> = \prod_{i,j}^{\infty} \delta_{n_{ij}n^o_{ij}}$$

are basis vectors with fixed double number. The operator $N_r \equiv N_{rr}$ is the operator of the total number of particles in the primary state r. The Hernitian-conjugate operator is

$$N^+_{rs} = N_{sr} . \qquad (3)$$

For the commutator of two operators (2) we have

$$[N_{ij}, N_{rs}]_- = \delta_{is} N_{rj} - \delta_{jr} N_{is} . \qquad (4)$$

In these relations, we can readily recognize the Lie algebra of the generators of the unitary group SU(∞) in the space of all possible single-particle states whose number goes to infinity. Note that relations (4) have been obtained by Bogolubov when quantizing the density matrix in Bose and Fermi statistics[11]. Our analysis[10] shows that they must be satisfied for any generalization of the statistics of identical particles.

For the advancement to the system with a variable number of particles we need to introduce the operators of creation and annihilation of particles. Here, we must formulate a number of propositions*

Proposition 1. The transition operator N_{ij} can be represented as a product of two operators

$$N_{ij} = \alpha^{-1} [a_i, a^+_j]_\varepsilon + c_{ij} , \qquad (5)$$

where the bracket is defined as $[a_i, a^+_j]_\varepsilon \equiv a_i a^+_j + \varepsilon\, a^+_j a_i$, and $\alpha, \varepsilon, c_{ij}$ are numbers. By vertue of (3), the numbers α and ε must be real and $c^*_{ij} = c_{ji}$.

Proposition 2. The operators a_i and a^+_i lower and raise, respectively, the number of particles in a state i by unity, i.e. we have the following relation with the particle-number operator

$$[N_j, a_i]_- = - \delta_{ij} a_i, \qquad [N_j, a^+_i]_- = \delta_{ij} a^+_i . \qquad (6)$$

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* Our consideration is very close to that presented in the book[12] for the special cases $\varepsilon = \pm 1$. In our case ε and α are left arbitrary real numbers. The parameter α may be excluded by renormalization of operators but we conserve it arbitrary for attaching him values which are met in the literature.Note also that our notation must be changed as $\varepsilon \Rightarrow 1/\varepsilon$ and $\alpha \Rightarrow \alpha/\varepsilon$ for a conformity with the notation in papers[10,14].

Proposition 3. The theory does not depend on the choice of basis vectors of representations, i.e. it must be invariant under undegenerate transformations

$$a'_i = \sum_j u_{ij} a_j , \qquad (a'_j)^+ = \sum_j u^*_{ij} a^+_j . \tag{7}$$

The importance of this assumption was first pointed out by Bialynicki-Birula[12].

Substitution of (5) into (4) and the requirement of independence of the result under transformations (7) lead to a condition of unitarity of these trasformations

$$\sum_m u_{im} u^*_{jm} = \delta_{ij} \tag{8}$$

and a condition for the constants

$$\sum_{m,n} u^*_{im} u_{jn} c_{mn} = c_{ij} . \tag{9}$$

The latter gives

$$c_{ij} = c\, \delta_{ij} \tag{10}$$

where c is some real constant. Substitution (5) into (6) gives

$$[[a_j, a^+_j]_\varepsilon, a_i]_- = -\alpha\, \delta_{ij} a_i . \tag{11}$$

Performing the infinitesimal trasformation of the operators

$$a'_i = a_i + \sum_j \omega_{ij} a_j , \qquad \omega_{ij} = -\omega^*_{ji}$$

and equating the small quantities of the first order in ω on both sides of (11), we arrive at the *initial trilinear relation* for the creation and annihilation operators[10]

$$[[a_i, a^+_j]_\varepsilon, a_k]_- = -\alpha\, \delta_{jk} a_i . \tag{12}$$

plus its Hermitian-conjugate relation. We can readily verify that (12) transforms (4) into an identity (with allowance for (5) and (10)).

However, now we order the positive definiteness of diagonal elements of the density matrix, or the fulfilment of the requirement that the squares of norms of state vectors in the Fock representation of (12) be positive.

2. The Fock representation and allowed quantization schemes

We postulate the existence of a unique vacuum state $|0\rangle$ such that

$$a_i |0\rangle = 0 \quad \text{for all } i. \tag{13}$$

The action of (11) on the vacuum state due to the uniquiness of this state leads to a proportionality*

$$a_i a^+_j |0\rangle = f_{ij} |0\rangle , \tag{14}$$

where f_{ij} are some numbers. The action of (4) on the vacuum state (taking into account (5) and (10)) gives

$$f_{ij} = p\, \delta_{ij} . \tag{15}$$

For the norm of any single-particle state we have

$$\| \sum_j \Psi_j a^+_j |0\rangle \|^2 = p \sum_j |\Psi_j|^2 \geq 0,$$

so, it is $p \geq 0$.

The following theorem was proved[14]: If $\varepsilon \neq 0$, then from (A) the condition of positive definiteness of the norm of the state vectors in the Fock space and (B) the requirement that the number of particles in either a symmetric or an antisymmetric state cannot exceed a given number $M \geq 2$ (the condition of *finite* parastatistics**) it follows that $\varepsilon=-1$ and $\varepsilon=+1$, respectively.

The main line of the proof is as follows: The norms of symmetrical (characterized by the parameter $\lambda=+1$) or antisymmetrical ($\lambda=-1$) in $M+1$ particles vector (in the presence

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* The following proof is analogous to the proof for the Green quantization in the special cases $\varepsilon = \pm 1$ which was given by Greenberg and Messiah[9].

** As it has been mentioned befor, the infinite statistics was excluded from the local theory[5]. Below, we shall consider a nonlocal theory of infinite statistics.

of other nonsymmetrized particles) were calculated by means of the (Hermitian cojugate) relations (12) and conditions (13)-(15) and equated to zero. Then, $z \equiv \lambda\varepsilon = -1$ turns out to be the only real root of this equation (of the 2M-2 degree). From this it follows that $\varepsilon=-1$ for para-Fermi statistics ($\lambda=+1$) and $\varepsilon=+1$ for para-Bose statistics ($\lambda=-1$). At the same time, it was proved that $\alpha\varepsilon>0$ and for the parameter $p=\alpha\varepsilon M/2$. If we put $\alpha\varepsilon=2$, then p=M and p coincides merely with the order of parastatistics. In this case relations (12) become the Green ones[/7/].

As we have mentioned befor, Greenberg and Messiah[/9/] proved the sufficiency of the Green paraquantization for the description of parastatistics on the same basic assumptions of the existence of a unique vacuum state (13) and the requirement of the positive definiteness of norms of symmetrical (for $\varepsilon=-1$) or antisymmetrical (for $\varepsilon=+1$) vectors. Greenberg and Messiah proved that the parameter p in (15) must be a positive integer number (1,2,...) which determines the order of parastatistics.

So, if there is no exclusion $\varepsilon=0$, we could assert the necessity and sufficiency of the Green paraquantization for the description of parastatistics. However, a recent investigation[/15/] shows that the case $\varepsilon=0$ is also convenient for describing parastatistics. In this case, relation (12) becomes

$$[a_i a_j^+, a_k]_- = -\alpha\, \delta_{jk} a_i \qquad (16)$$

plus its Hermitian conjugate relation (In [/15/] the sign of α was opposite: $\alpha \Rightarrow -\alpha$)*. The same relations (13)-(15) are valid and the requirement of positive definiteness of norms of symmetrical for $\alpha\le 0$ or antisymmetrical for $\alpha\ge 0$ vectors gives p = M$|\alpha|$, where M is a maximal number of particles in these states. If one puts $|\alpha|=1$, then p = M and defines the order of para-Fermi($\alpha=-1$) or para-Bose($\alpha=+1$) statistics.

Now, we have *two* local field theories which are convenient for the description of parastatistics! What is the difference between them?

The basis vectors of the Fock space are obtained by applying polynomials in a^+ to the vacuum vector $|0\rangle$. In the case of the Green paraquantization, these vectors possess some additional symmetry and additional commutation relations take place within the Fock space*

$$[[a_i, a_j]_\varepsilon, a_k]_- = 0, \quad \varepsilon = \pm 1 \qquad (17)$$

plus Hermitian-conjugate relations. Due to (17) some symmetrized multiparticle states disappear and the Fock space contains one and only one irreducible subspace corresponding to each irreducible representation of the particle permutation group (Young diagram) admitted by a given order of parastatistics[/12/].

In the case of the new paraquantization (16) there are no such additional restrictions[/15/]. In this sense, the new quantization is more adequate to the particle parastatistics than the Green one.

However, we shall see below that the new field theory, corresponding to eqs.(16), carries an inherent asymmetry with respect to particles and antiparticles. At present time, we cannot decide whether it is a virtue or a shortcoming of the new theory.

Other peculiarity of the new quantization (16) consists in the limiting behaviour as $p \Rightarrow \infty$. In this limiting case the relations

$$a_i a_j^+ = \delta_{ij} \qquad (18)$$

hold for all state vectors in the Fock space[/15/]. This is

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* There is another exclusion from the above theorem corresponding to the limiting case in (12): $\varepsilon\Rightarrow\infty, \alpha\Rightarrow\infty$ and ε/α-finite. However, this case can be reduced to the case $\varepsilon=0$ by means of a redefinition of creation and annihilation operators, $a_i \Rightarrow a_i^+$ [/15/].

* For comparision recall that the relations $[a_i, a_j]_\varepsilon = 0$ in the case of ordinary Fermi or Bose statistics ($\varepsilon=\pm 1$) are consequences of the relations $[a_i, a_j^+]_\varepsilon = \delta_{ij}$ within the Fock representation.

just the equation which Greenberg (as he noted, by suggestion of R. Hegstrom)[16] has recently assumed for the description of infinite statistics. It can be suggested that this quantization should correspond to the statistics of identical particles with an infinite number of internal hidden degrees of freedom, which is equivalent to the statistics of nonidentical particles since they are distinguishable by their internal states. The particles corresponding to eqs.(18), as Greenberg showed, obey the Maxwell-Boltzmann statistics[16]. In this limiting case, the corresponding field theory becomes nonlocal[15,16] in accordance with the general prediction[5].

We are prepared now to analyze the possibility (or impossibility) of a small violation of the Pauli principle. As it has been mentioned in the beginning, Ignatiev and Kuzmin[1] have recently raised again the problem of verification of the accuracy and possible small violation of the Pauli principle, for instance, when electrons occupy the atomic levels. Greenberg and Mohapatra[17] have formulated trilinear relations for field operators that should be a generalization of the one-level Ignatiev-Kuzmin simple model for the local quantum field theory of violation of the Pauli principle. It is important that this theory is really a local theory. However, it is not rather unexpected that the Greenberg-Mohapatra relations coincide with (12) written in some other parametrization[18,19]:

$$\varepsilon = (1-2\beta^2)/(2-\beta^2), \qquad \alpha = (1-\beta^2+\beta^4)/(-2+\beta^2), \tag{19}$$

where β^2 is a real positive parameter characterizing the small violation of the Pauli principle. In this parametrization, more than two identical particles cannot occupy the same level (when parameter in (15) $p = 1$) and $\beta=0$ corresponds to ordinary Fermi statistics, $\beta^2=1$ ($\varepsilon=-1$) corresponds to Green's para-Fermi statistics of order 2, and $\beta^2=1/2$ ($\varepsilon=0$) corresponds to the new para-Fermi statistics of order 2. However, according to our preceding discussion of the theorem[14], only these descrete values of the parameter ε (and β^2) are admissible. So the parameter β^2 cannot take arbitrary continuous values and, particularly, small values $\beta^2 \ll 1$. The direct calculations[18,19] in the framework of the Greenberg-Mohapatra scheme lead to the negative norm of some (four-) particle state vector (with three particles in the same quantum state), and this negative norm is proportional to $-\beta^2$. Thus, this scheme cannot be the theory of small violation of the Pauli principle if the norms of all particle state vectors must be positive. In its turn, the latter requirement is connected with the probabilistic interpretation of quantum theory. Thus, we can conclude that our world was created so (in accordance with the Pauli principle) that it could be measured by us!

There were other attempts to violate the Pauli principle[20,21]. In addition to being nonlocal these theories have, in my opinion, a more serious shortcoming: they are tied a priori to a definite representation and do not permit superpositions of states (in contradiction with our Proposition 3).

3. Parafields

We now turn to the field operators $\psi(x)$ depending on space-time x. We can consider fields of any spin but for definiteness and taking into account the following applications we shall write all expressions as if we were dealing with the Dirac fields $\psi(x)$ and $\bar\psi(x)=\psi^+(x)\gamma_0$. We consider free fields which can be expanded with respect to negative- and positive-frequency solutions

$$\psi(x)=(2\pi)^{-3/2}\int d^3p(m/E(\mathbf{p}))^{1/2}\sum_{\sigma=\pm 1/2}[a(\sigma,\mathbf{p})u(\sigma,\mathbf{p})e^{-ipx}+ + b^+(\sigma,\mathbf{p})v(\sigma,\mathbf{p})e^{ipx}], \tag{20}$$

where $\mathbf{p}$ is a momentum and σ is a spin state (now the Kronecker symbol δ_{ij} means $\delta_{\sigma_i\sigma_j}\delta^{(3)}(\mathbf{p}_i-\mathbf{p}_j)$). The operator $a(\sigma,\mathbf{p})$ is the annihilation operator of a particle , and $b^+(\sigma,\mathbf{p})$ is the creation operator of an antiparticle.

In accordance with the previous discussion we can propose two possible generalized schemes for the field quantization: either old Green's paraquantization corresponding to $\varepsilon=\pm1(\alpha=\pm2)$ or the new one corresponding to $\varepsilon=0(\alpha=\pm1)$.

In the case of the Green paraquantization we have the following trilinear commutation relations:

$$[[\psi(x),\bar{\psi}(y)]_{\varepsilon},\psi(z)]_{-}= - 2iS(y-z)\psi(x) \qquad (21)$$

and their Hermitian-conjugate relations. Here, for the half-integer spin field $\varepsilon=-1$ and $S(x)$ is the well-known singular (on the light-cone) function (for the Dirac field), and for the integer spin field (for example, scalar field) $\varepsilon=+1$ (and it is necessary to exchange $\psi\Rightarrow\phi,\bar{\psi}\Rightarrow\phi^{+}$, and $S(x)\Rightarrow\Delta(x)$, where $\Delta(x)$ is a singular function for the scalar field). Due to (17) we have also

$$[[\psi(x),\psi(y)]_{\varepsilon},\psi(z)]_{-} = 0. \qquad (22)$$

It is easy to prove that the Green theory is local if currents and other observables take the form of a commutator for $\varepsilon=-1$ or an anticommutator for $\varepsilon=+1$. The commutation relations (21) and (22) are also invariant under the charge conjugate transformation which we write out for the Dirac field

$$\psi(x) \Rightarrow \psi_{c}(x) = C\,\bar{\psi}^{T}(x) \qquad (23)$$

where C is the charge-conjugation matrix and T means the transposition matrix and spinor.

For the Green (free) parafield the spin-statistics theorem was proved[22]: the half-integer spin corresponds to para-Fermi statistics ($\varepsilon=-1$) and integer spin corresponds to para-Bose statistics ($\varepsilon=+1$).

For the new parquantization corresponding to (16) we have

$$[\psi(x)\bar{\psi}(y),\psi(z)]_{-} =- iS(z-y)\psi(x) \qquad (24)$$

with its Hermitian-cojugate relation. A remarkable property of this quantization is its universality for different

fields. This property may be useful for further supersymmetrecal generalization. All distinction consists in the spin structure of the field and singular function in the right-hand side of eq.(24). It can be proved[15] that owing to the properties of singular functions and *the existence of antiparticles* the parameter α in (16) being equal to +1 for the scalar field and -1 for the Dirac spinor field corresponds to the correct spin-statistics connection: the scalar field obeys para-Bose statistics and the Dirac spinor field obeys para-Fermi statistics.

Relations (24) ensure the locality of any observables which are taken in the form $\psi(x)\bar{\psi}(x)$. However, these relations are not invariant under the charge conjugate transformation (23). In these circumstances there are some restrictions on the number of antiparticles in the system. The number of particles in any state can be arbitrary. In more detail we shall discuss this situation in the next section.

We remark that due to the locality this theory is invariant under the *CPT*-transformation.

Now I make the common remark about the locality of both the quantization schemes. Both the relations (21) and (24) ensure the locality of the corresponding observables. But the corresponding particles do not possess the simple commutation or anticommutation properties under particle permutations. The multiparticle states obey more complex permutation rules generating IR's(irreducible representations) of the permutation group. The probability of finding the system of particles in either IR is determined by the trace of this IR which is equal to the sum of squares of modules of orthonormalized basis vectors of this IR. We can think of this expression as a usual probability for fermions or bosons avaraged over some auxiliary internal degree of freedom like an isospin and so on which takes the number of internal states equal to an order of parastatistics. The important property of the description of a multiparticle state by means of IR's is the cluster law. According to this law, one may take one particle out of the system without any influence on the IR's of the remaining particles. More accurately, such movig off consists rather in nonoverlapping of one wave function with other wave functions of particles than in sending away a certain numbered particle from others[6,28]. We should not assume the permutations of nonidentical numbered particles but we should imagine the identical particles (quanta of the same parafield) in different internal states determined by different order of particle operators and the corresponding IR's. Due to the fulfilment of the cluster law I do not believe in a possibility of any "kinematic" explanation of confinement of paraparticles, as it was proposed in[23,24] (In such a manner we could forbid the existence of fermions in free states as well, due to their anticommutation). Parastatistics is the usual Fermi or Bose statistics plus some hidden internal degree of freedom[25].

4. Algebraic realization of parafields

Let us compose the following combinations:

$$\psi(x)=\sum_{\alpha=1}^{p} e_\alpha \psi^\alpha(x), \qquad \bar\psi(x)=\sum_{\alpha=1}^{p} e^+_\alpha \bar\psi^\alpha(x). \tag{25}$$

Here, $\psi^\alpha(x)$ are the Dirac fields obeying ordinary Fermi statistics with normal (Fermi) relative anticommutation relations (For scalar parafield we take all the corresponding scalar fields $\phi^\alpha(x)$ to be Bose fields with the normal (Bose) relative commutation relations). Quantities e_α are the basis elements of some determinative algebra. We shall call elements of this algebra the *hypernumbers* and the field (25) constructed by means of these hypernumbers the *hyperfield**.

* Here, we shall take the basis elements e_α to be independent of space-time. One can develop a formulation, motivated by the ideas of differential geometry, in which the basis elements depend on space-time[26].

For the Green paraquantization (21),(22) ($\varepsilon=\pm1$) the realization (25) was introduced by Greenberg and Macrae[26], and by the author[27]. In this case, basis elements form either a *real Clifford algebra* with the anticommutation relations

$$[e_\alpha, e_\beta]_+ = 2\,\delta_{\alpha\beta}, \qquad e^+_\alpha = e_\alpha \tag{26}$$

or a *complex Clifford algebra* with the anticommutation relations

$$[e_\alpha, e^+_\beta]_+ = 2\,\delta_{\alpha\beta}, \tag{27}$$

and

$$[e_\alpha, e_\beta]_+ = 0. \tag{28}$$

As a consequence of (28) there is a property of nilpotence

$$\prod_{i=1}^{p+1} e_{\alpha_i} = 0 \tag{29}$$

and $(p+1)$st power of the field vanishes identically. So hyperfields constructed by means of the complex Clifford algebra can be applied only to the system with the maximal number of particle equal to p. For the consideration of the system with more than p particles, we are compeled to go out of proper field theory and have recourse to some contrivance in the form of a trace-operation over states of the groups of p particles or particle-antiparticle pairs[26].

For the new paraquantization (24) the basis elements form a *Greenberg algebra**, with the relations

$$e_\alpha e^+_\beta = \delta_{\alpha\beta}. \tag{30}$$

In this case for the charge-conjugate field we have

$$\psi\ (x) = \sum_{\alpha=1}^{p} e^+_\alpha \psi^\alpha_c(x), \qquad \bar\psi\ (x) = \sum_{\alpha=1}^{p} e_\alpha \bar\psi^\alpha_c(x) \tag{31}$$

where $\psi^\alpha_c = C\ \bar\psi^\alpha$, $\bar\psi_c = [C^{-1}\psi^\alpha]^T$. Therefore, we really have

* As it has been mentioned befor, Greenberg[16] proposed this algebra for the description of the infinite statistics. We consider the *finite* Greenberg algebra as an auxiliary fabric for the construction of (25). The necessary information on the Greenberg algebra is presented in Appendix A.

the charge-asimmetrical theory since the original field (25) contains e_α whereas its charge-conjugate field (31) contains e^+_α and relation (30) is asymmetrical with respect to e_α and e^+_α.

We have to introduce the vacuum state vector for the construction of the Fock representation of the new field. We choose this vector in the form

$$|vac.\rangle = p^{-1/2} \sum_{\alpha=1}^{p} e^+_\alpha |0\rangle . \qquad (32)$$

Then, for the action of antiparticle operators on the vacuum state we have (for any p)

$$b_r |vac.\rangle = 0, \qquad (33)$$

$$b_r b^+_s |vac.\rangle = \delta_{rs} |vac.\rangle, \qquad (34)$$

which can be compared with analogous relations for particle operators (13)-(15) and which reveals different vacuum conditions for particles and antiparticles unless $p=1$ (usual statistics).

Owing to (33) and (34) we have the same Fock representation with a unique vacuum vector (32) which has to be done in /15/ without applying the algebraic realization (25). The antiparticles obey the same parastatistics as particles do. However, the number of antiparticles is limited by the number of particles: the former can exceed the latter not more than by unity. The number of particles can be arbitrary.

In principle, we can choose the vacuum vector in a more general form

$$|vac.,N_a\rangle = p^{-N_a/2} \sum_{\alpha_1,\ldots,\alpha_{N_a}=1}^{p} e^+_{\alpha_1} \ldots e^+_{\alpha_{N_a}} |0\rangle \qquad (35)$$

with an arbitrary $N_a \geq 1$. Then, the number of "superfluous" antiparticles is limited by N_a. However, these superfluous antiparticles obey the usual (Fermi or Bose) statistics between themselves whereas the "normal" antiparticles which are included into particle-antiparticle pairs obey the parastatistics of order p between themselves and superfluous

antiparticles. It is a curious phenomenon of the dependence of statistics of antiparticles on their states.

It is interesting also that if we try to describe the usual Fermi (or Bose) statistics by means of these theories with $p=1$, then the particle-antiparticle asymmetry is remainded and the number of antifermions (antibosons) can exceed the number of fermions (bosons) not more than by N_a. Thus, the number of initial antifermions (antibosons) cannot be arbitrary large ($>N_a$) whereas the number of initial fermions (bosons) can be arbitrary. I am not going to assert that this limiting on the number of antiparticles could explain the world asymmetry but it is not in contradiction with this asymmetry.

5. The gauging of hyperfields

The algebraic realization of parafields in the form of hyperfields (25) allows us to fulfil hypernumber gauge trasformations on the hyperfield

$$\psi'(x) = e^{-i\chi} \psi(x) e^{i\chi} = e_\alpha M^{\alpha\beta} \psi_\beta(x) \qquad (36)$$

where the parameters χ are hypernumbers defined by

$$\chi = n_{\alpha\beta} x^{\beta\alpha} \qquad (37)$$

in the framework of a given algebra (see, Table I). Here and in what follows the summation sign is not exhibit; we use the convention that repeated indices are summed over, $x_{\beta\alpha}$ are number parameters also presented in Table I. Matrix M entering in (36) is the orthogonal matrix in the case of the real Clifford algebra and the unitary matrix in both the complex Clifford algebra and the Greenberg one.

Now, supposing these parameters depending on the space-time x, we can develop a full local Yang and Mills gauge theory.

For the Green paraquantization such a gauging of the Clifford hyperfields has been accomplished by Greenberg and Macrae/26/, and by the author/27/. In this case, for the real Clifford algebra (26) we have the $SO(p)$ gauge theory. For the complex Clifford algebra we have the SU(p) gauge theory with the above-mentioned nilpotence property of parafield[26].

For the new paraquantization we have the hyperfield realization (25) founded on the Greenberg algebra (30). In this case, we have also the SU(p) gauge theory this time without any additional restrictions on the fields but with the property of the charge-asymmetry discussed befor.

Now we can show how the gauge-symmetry theories can be formulated in terms of hyperfields.

The gauge vector fields are introduced in the form

$$\mathcal{B}_\mu(x) = n_{\alpha\beta}B_\mu^{\beta\alpha}(x), \quad \mathcal{B}_\mu^+(x) = \mathcal{B}_\mu(x) \tag{38}$$

where $B_\mu^{\beta\alpha}(x)$ are the usual bosonic vector fields. Their properties are indicated in Table I too. Then, the derivative changes to the gauge-covariant derivative

$$\mathcal{D}_\mu(x) = \partial_\mu + \iota g\mathcal{B}_\mu(x) \tag{39}$$

where g is the coupling constant and ι is an auxiliary number factor indicated in Table I. Note that in this formulation, the gauge-covariant derivative must enter always into the commutator with the expression subjected to its action.

The gauge-field tensor is defined in the commutator form

$$\mathcal{G}_{\mu\nu}(x) = \iota^*/g\ [\mathcal{D}_\mu,\mathcal{D}_\nu]_- = n_{\alpha\beta}G_{\mu\nu}^{\beta\alpha}. \tag{40}$$

The components of this tensor compose the matrix

$$G_{\mu\nu} = \partial_\mu B_\nu - \partial_\mu B_\nu - ig[B_\mu,B_\nu]_- . \tag{41}$$

Finally, we write the equation of motion in terms of hyperfields. The Dirac equation is given by

$$i[\mathcal{D}_\mu,\gamma^\mu\psi(x)]_- - m\ \psi(x) = 0 \tag{42}$$

or, in components,

$$(i\gamma^\mu\partial_\mu\psi^\alpha - m\ \psi^\alpha + gB_\mu^{\alpha\beta}\gamma^\mu\psi^\beta)e_\alpha = 0. \tag{43}$$

For the gauge fields we have the equation of motion

in the form

$$[\mathcal{D}^\mu,\mathcal{G}_{\mu\nu}(x)]_- - g/2\ j_\nu(x) = 0, \tag{44}$$

or, in components,

$$(\partial^\mu G_{\mu\nu}^{\alpha\beta} - ig[B^\mu,G_{\mu\nu}]^{\alpha\beta} + g/2\ J_\nu^{\beta\alpha})n_{\beta\alpha} = 0. \tag{45}$$

The current

$$j_\nu(x) = n_{\alpha\beta}J_\nu^{\alpha\beta}(x) \tag{46}$$

has components indicated in Table I.

Thus, we accomplish the formulation of the local gauge symmetries in terms of hyperfields.

It should be taken into account that the gauge vector fields become parafields, i.e. satisfy the trilinear Green's commutation relations (21), only for the hyperfield realization by means of the real Clifford algebra in the exceptional case $p=3$[27] (Remark that in this case, the Clifford algebra is isomorphic to the algebra of quaternions[28]). In this case, we have the SO(3) gauge symmetry which can be formulated utterly in the framework of parafields independing of their algebraic realization[29,30]. In the remaining cases the gauge fields are not proper parafields and for their formulation we are in need of a certain algebraic realization. Thus, starting from a proper parafield theory we arrive at a more broad interpretation of it as a hyperfield theory. This expansion of the meaning of a parafield theory was the reason for the difference between my approach [27,30] to the gauging of parastatistics and Greenberg and Macrae's one[26].

The last remark concerns the possibility of the formulation of the Abelian U(1) gauge symmetry within the hyperfield theory in both the nilpotent comlex Clifford algebra and the Greenberg algebra. We can take the parameter of a gauge transformation

$$\chi^\circ = n_{\alpha\alpha}x^\circ = (\chi^\circ)^+ \tag{47}$$

where x° is a real number and the sum $n_{\alpha\alpha}$ commutes with

T a b l e I. Gauge symmetries in terms of hypernumbers

SO(p)	SU(p)	
the real Clifford algebra (26)	the nilpotent complex Clifford algebra (27,28)	the Greengerg algebra (30)
$\iota=1$	$\iota=i$	
$n_{\alpha\beta}=(e_\alpha e_\beta-e_\beta e_\alpha)/8i$	$n_{\alpha\beta}=e^+_\alpha e_\beta/2$	$n_{\alpha\beta}$- Greenberg operator
$(x^{\alpha\beta})^*=x^{\alpha\beta}=-x^{\beta\alpha}$	$x^{\alpha\beta}=x^a\lambda_a^{\alpha\beta}/2$, sum over $a=1,\dots,p^2-1$ $(x^a)^*=x^a$, λ_a- generalized Gell-Mann matrices: $[\lambda_a,\lambda_b]_-=2if_{abc}\lambda_c$, $\lambda_a^+=\lambda_a$, $tr\lambda_a=0$, $tr(\lambda_a\lambda_b)=2\delta_{ab}$	
$(B_\mu^{\alpha\beta})^+=B_\mu^{\alpha\beta}=-B_\mu^{\beta\alpha}$	$B_\mu^{\alpha\beta}=B^a_\mu\lambda_a^{\alpha\beta}/2$, $(B^a_\mu)^+=B^a_\mu$	
$J_\mu^{\alpha\beta}=\bar\psi^\beta\gamma_\mu\psi^\alpha-\bar\psi^\alpha\gamma_\mu\psi^\beta$	$J_\mu^{\alpha\beta}=\bar\psi^\alpha\gamma_\mu\psi^\beta=[\bar\psi^\kappa(\lambda_a^{\kappa\eta}/2)\gamma_\mu\psi^\eta]\lambda_a^{\beta\alpha}$	

For obtaining separate components one needs to take "the trace-operation" for a given algebra: for the Clifford algebra by means of the formulae

$$1/2\ [e_\alpha,e^+_\beta]_+=\delta_{\alpha\beta},\qquad 1/2\ [[e_\gamma,n_{\alpha\beta}]_-,e^+_\kappa]_+=\delta_{\alpha\gamma}\delta_{\beta\kappa},$$

and for the Greenberg algebra by means of the formulae

$$e_\alpha e^+_\beta=\delta_{\alpha\beta},\qquad [e_\gamma,n_{\alpha\beta}]_-e^+_\kappa=\delta_{\alpha\gamma}\delta_{\beta\kappa}.$$

all $n_{\alpha\beta}$. The corresponding Abelian gauge field is

$$\mathcal{A}_\mu(x)=A_\mu(x)n_{\alpha\alpha}=\mathcal{A}^+_\mu(x)\qquad(48)$$

where $A_\mu(x)=A^+_\mu(x)$ is the usual bosonic Hermitian self-conjugate vector field. Thus, in the framework of such hyperfield theories we have rather the SU(p)×U(1)-gauge symmetry than merely SU(p). The additional usual phase-transformation for the Dirac fields and the corresponding Abelian gauge vector field (which could coincide with A_μ) can occur too.

6. The conclusion

Let us sum up our discussion and decide what we can and what we cannot do within the Local Quantum Field Theory

1) We cannot introduce "a small violation" of the Pauli principle (i.e., Fermi statistics) or Bose statistics within the local QFT without a violation of the positive definiteness of the state vector norms. The appearance of negative norm states is not compatible with the probability interpretation of quantum mechanics.

The formulation of a small violation of the Pauli principle in the framework of infinite statistics ought to be nonlocal and very likely infringes the principle of superposition.

2) We can introduce "a big violation" of the Pauli principle in the form of parastatistics. We can give a local formulation of parastatistics. In fact, we have *two* such theories: "the old" Green paraquantization and "the new" one which is charge-asymmetrical. Both of them have own advantages and presently we cannot discriminate these two possibilities.

3) In the framework of hyperfield realizations of parafields we can formulate gauge-invariant theories. In the case of the Green paraquantization based on the real or nilpotent complex Clifford algebra, we can formulate the SO(p) or SU(p)(×U(1)) gauge symmetries, respectively. In the case of the new (charge-asymmetrical) paraquantization based on the Greenberg algebra we also can formulate the SU(p)(×U(1)) gauge symmetry.

Now a question arises: what is the connection of these open possibilities for the description of the local gauge symmetries with the physical symmetries such as colour, electroweak and, maybe, flavour symmetries? Can these possibilities help us to understand the reason of these physical symmetries?

Hitherto, we consider a realization of parafields by means of the associative hypernumbers. There is a very enticing possibility to draw the nonassociative hypernumbers, octonions and postoctonions for the construction of nonassociative parafields[31,32]. I plan to study this possibility further.

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Appendix A. The Greenberg algebra

The basic elements of the Greenberg algebra obey the relations

$$e_\alpha e^+_\beta = \delta_{\alpha\beta}, \qquad \alpha,\beta = 1,\dots,p. \tag{A.1}$$

The Hermitian-conjugation is defined as usually

$$(e_\alpha e_\beta)^+ = e^+_\beta e^+_\alpha, \qquad (e^+_\alpha)^+ = e_\alpha . \tag{A.2}$$

These elements can be presented by infinite matrices. For example, in the most simple case $p=1$ the only element e is presented by the matrix with the nonvanishing matrix elements equal to unity up the diagonal.

Following Greenberg[16], one can define the operator in the form of an infinite series

$$n_{\alpha\beta} = e^+_\alpha e_\beta + e^+_\gamma e^+_\alpha e_\beta e_\gamma + e^+_{\gamma_1} e^+_{\gamma_2} e^+_\alpha e_\beta e_{\gamma_2} e_{\gamma_1} + \dots \tag{A.3}$$

(summation over repeated indices is implied) which obey the following properties:

$$n^+_{\alpha\beta} = n_{\beta\alpha}, \tag{A.4}$$

$$[n_{\alpha\beta}, e_\gamma]_- = -\,\delta_{\alpha\gamma} e_\beta, \qquad [n_{\alpha\beta}, e^+_\gamma]_- = \delta_{\beta\gamma} e^+_\alpha, \tag{A.5}$$

$$[n_{\alpha\beta}, n_{\gamma\kappa}] = \delta_{\beta\gamma} n_{\alpha\kappa} - \delta_{\alpha\kappa} n_{\gamma\beta}. \tag{A.6}$$

The sum $n = n_{\alpha\alpha}$ possesses the number operator properties

$$[n, e_\alpha]_- = -\,e_\alpha, \qquad [n, e^+_\alpha]_- = e^+_\alpha, \qquad [n, n_{\alpha\beta}]_- = 0. \tag{A.7}$$

The trace-operation is defined as

$$tr(e^+_{\alpha_1} \dots e^+_{\alpha_N} e_{\beta_M} \dots e_{\beta_1}) \equiv \delta_{NM}\, e_{\gamma_1} \dots e_{\gamma_N} (e^+_{\alpha_1} \dots e^+_{\alpha_N} e_{\beta_N} \dots e_{\beta_1}) \times$$
$$\times e^+_{\gamma_N} \dots e^+_{\gamma_1} = \delta_{NM}\, \delta_{\alpha_1\beta_1} \dots \delta_{\alpha_N\beta_N}, \tag{A.8}$$

and

$$tr(n_{\alpha_1\beta_1} n_{\alpha_2\beta_2} \dots n_{\alpha_N\beta_N}) \equiv (-1)^N [n_{\alpha_1\beta_1}, e_{\gamma_1}]_- e^+_{\gamma_2} [n_{\alpha_2\beta_2}, e_{\gamma_2}]_- e^+_{\gamma_3} \times$$
$$\dots \times e^+_{\gamma_N} [n_{\alpha_N\beta_N}, e_{\gamma_N}]_- e^+_{\gamma_1} = \delta_{\alpha_1\beta_N} \delta_{\alpha_2\beta_1} \dots \delta_{\alpha_N\beta_{N-1}}. \tag{A.9}$$

For instance, $tr\, n_{\alpha\beta} = \delta_{\alpha\beta}$, $tr\, n = p$, $tr\, n^2 = p$.

References

1. Ignatiev A. Yu., Kuzmin V. A. - Yad. Fiz., 1987, v. 46, p. 786 [Sov. J. Nucl. Phys., 1987, v. 46, p. 444].
2. Okun L. B. - Usp. Fiz. Nauk, 1989, v. 158, p. 293; Comm. Nuc. and Part. Physics, 1989, v. 19, p. 99.
3. Schiff L. I. Quantum Mechanics. McGraw-Hill, New York - Toronto - London, 1968; p. 364.
4. Doplicher S., Haag R., Roberts J. E. - Commun. Math. Phys., 1971, v. 23, p. 199; ibid., 1974, v. 35, p. 49; Haag R. - In: Progress in Quantum Field Theory, North-Holland, Amsterdam - Oxford - New York - Tokyo, 1986, p. 1.
5. Fredenhagen K., - Commun. Math. Phys., 1981, v. 79, p. 141; Buchholz D., Fredenhagen K., ibid., 1982, v. 84, p. 1.
6. Hartle J. B , Taylor J. R. - Phys. Rev., 1969, v. 178, p. 2043;

Hartle J. B., Stolt R. H., Taylor J. R., ibid., 1970, v. 2D, p. 1759;
Stolt R. H., Taylor J. R. - Nucl. Phys., 1970, v. 19B, p.1; Nuovo Cim., 1971, v. 5A, p.185.

7. Green H. S. - Phys. Rev., 1953, v. 90, p. 170.
8. Volkov D. V. - Zh. Eksp. Teor. Fiz., 1959, v.36, p. 1560; ibid., 1960, v. 38, p. 519 [Sov. Phys. JETP, 1959, v. 9, p. 1107; ibid., 1960, v. 11, p. 375].
9. Greenberg O. W., Messiah A. M. L., - Phys. Rev., 1965, v. 138B, p. 1155.
10. Govorkov A. B. - J. Phys., 1980, v. 13A, p. 1673.
11. Bogolubov N. N. Izbrannye trudy (Collected Works). Naukova dumka, Kiev, 1970, v. 2, pp. 303-333.
12. Ohnuki Y., Kamefuchi S. Quantum Field Theory and Parastatistics. Springer - Verlag, Berlin - Heidelberg - New York, 1982.
13. Bialynicki-Birula I. - Nucl. Phys., 1963, v. 49, p. 605.
14. Govorkov A. B. - Teor. Mat. Fiz., 1983, v. 54, p. 361 [Theor. Math. Phys., 1983, v.54, p. 234].
15. Govorkov A. B. JINR Preprint E2-90-28, Dubna, 1990.
16. Greenberg O. W. - Phys. Rev. Lett., 1990, v. 64, p. 705.
17. Greenberg O. W., Mohapatra R. N. - Phys. Rev. Lett., 1987, v. 59, p. 2507.
18. Greenberg O. W., Mohapatra R. N. - Phys. Rev. Lett., 1989, v. 62, p. 712.
19. Govorkov A. B. - Phys. Lett., 1989, v. A137, p. 7.
20. Okun L. B. - Pis'ma Zh. Eksp. Teor. Fiz., 1987, v. 46, p. 420; Yad. Fiz., 1988, v. 47, p. 1182.
21. Mohapatra R. N. University of Maryland Preprint 90-120, 1990.

22. Dell'Antonio G. F., Greenberg O. W., Sudarshan E. C. G. - In: Group Theoretical Concepts and Methods in Elementary Particle Physics, Gordon and Breach, New York, 1964.
23. Cattani M., Fernandes N. C. - Nuovo Cimento, 1984, v. A79, p. 107; ibid., 1985, v. B87, p. 70; Phys. Lett., 1987, v. A124, p 229;
Cattani M. - Acta Phys. Pol., 1989, v. B20, p. 983.
24. Flato M., Fronsdal C. - J. Geom. Phys., 1989, v. 6, p. 293.
25. Govorkov A. B. - Fiz. Ehlem. Chast. Atom. Yad., 1983, v. 14, p. 1229 [Sov. J. Particles and Nuclei, 1983, v. 14, p. 520].
26. Greenberg O.W., Macrae K. I. - Nucl. Phys., 1983, v. B219, p. 358.
27. Govorkov A. B. - Teor. Mat. Fiz., 1983, v. 55, p. 3.
28. Govorkov A. B. - Teor. Mat. Fiz., 1986, v. 68, p. 381; ibid., 1986, v. 69, p. 69.
29. Greenberg O. W., Nelson C. A. - Phys. Rep., 1977, v. 32C, p.69.
30. Govorkov A. B. - Teor. Mat. Fiz., 1982, v. 53, p. 283.
31. Gunaydin M., Gursey F.- Phys. Rev.,1974, v. 9D, p. 3387; Gursey F. - In: Proc. of the Johns Hopkins Workshop on Current Problems in High-Energy Particle Theory, Johns Hopkins, Baltimore, 1974, p. 15.
32. Govorkov A. B. - In: JINR Rapid Comm., No. 7-85, Dubna: JINR, 1985, p. 17.

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