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# THE HAMILTONIAN OF QED. ZERO MODE

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1. The QED Lagrangian is

$$\mathfrak{L} = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 - \psi^* \gamma_4 [\gamma_\mu(\partial_\mu - ieA_\mu) + m]\psi . \qquad (1)$$

Let us introduce new variables $\vec{B}$, $\phi$:

$$\vec{B} = \vec{A}(x,t) + \mathrm{grad}\,\Lambda,$$

$$\Lambda(x,t,t_0) = \int_{t_0}^{t} A_0(x,\tau)\,d\tau, \qquad (2)$$

$$\phi = e^{-ie\Lambda}\psi .$$

In these variables one has

$$\mathfrak{L} = \frac{1}{2}(\dot{\vec{B}})^2 - \frac{1}{2}(\mathrm{rot}\,\vec{B})^2 -$$

$$- \phi^* \gamma_4 [\sum_1^3 \gamma_j(\partial_j - ieB_j) + \gamma_4 \partial_4 + m]\phi . \qquad (3)$$

This Lagrangian gives the Hamiltonian

$$h = \int [\frac{1}{2}(\vec{\pi})^2 + \frac{1}{2}(\mathrm{rot}\,\vec{B})^2 + \phi^* \gamma_4 (\sum_1^3 \gamma_j(\partial_j - ieB_j) + m)\phi] d^3x, \qquad (4)$$

here

$$\pi_j(x) = -i\delta/\delta B_j(x), \quad j = 1,2,3,$$

$$[\phi^*_\alpha(x), \phi_\beta(y)]_+ = \delta_{\alpha\beta}\,\delta(x-y). \qquad (4a)$$

Let us note that if

$$A_\mu(x,t) \to A'_\mu(x,t) = A_\mu(x,t) + e\,\partial\lambda(x,t)/\partial x_\mu, \quad it = x_4,$$

$$\psi \to \psi' = \psi e^{ie\lambda(x,t)},$$

then

$$\vec{B}(x,t) \to \vec{B}(x,t) + e\,\mathrm{grad}\,\lambda(x,t_0),$$
$$\phi(x,t) \to \phi(x,t)\, e^{ie\lambda(x,t_0)}. \qquad (5)$$

The Hamiltonian h is invariant under transformation (5). Let us introduce a cubic periodicity volume V. Let us also introduce the decomposition

$$\vec{B}(x) = \mathrm{grad}\,\gamma(x) + \vec{C}/\sqrt{V} + \vec{B}^{tr}(x),$$
$$\int \vec{B}^{tr}\,dx = \int \gamma\,dx = 0, \quad \vec{C} = \int \vec{B}(x)\,dx/\sqrt{V}, \text{(c.f. eq.(13))}. \qquad (6)$$

The quantities $\vec{C}$, $\vec{B}^{tr}$ do not change under transformation (5). Operator

$$\eta(x) = \phi(x)\, e^{-ie\gamma(x)} \qquad (7)$$

gets a constant phase under this transformation. Let us consider the Schrödinger equation

$$(h - E)\,\Omega = 0. \qquad (8)$$

Here

$$\Omega = \Omega(\phi, \phi^*, B), \qquad (9)$$

see eqs. (4) and (4a). It is possible to express $\Omega$ as a function of variables $\eta$, $\eta^*$, $\vec{B}^{tr}$, $\vec{C}$, $\gamma$:

$$\Omega(\phi, \phi^*, B) \equiv \bar{\Omega}(\eta, \eta^*, B^{tr}, \vec{C}, \gamma). \qquad (10)$$

Let us denote

$$h \equiv h_1 + \frac{1}{2}\int (\vec{\pi})^2\,dx. \qquad (11)$$

One has

$$h_1 = h_1(\phi, \phi^*, \vec{B}) = h_1(\eta, \eta^*, \vec{B}^{tr} + \vec{C}/\sqrt{V}). \qquad (12)$$

Equation (6) entails the decomposition

$$\vec{B}(x) = \frac{1}{\sqrt{V}}\Big(\sum_{\substack{k\neq 0\\ \lambda=1,2}} \vec{e}(k,\lambda)\,q_\lambda(k)\,e^{ikx} + \sum_{k\neq 0} i\vec{k}\,q_3(\vec{k})\,e^{ikx} + \vec{C}\Big). \qquad (13)$$



We chose

$$\vec{e}(k,\lambda) = \vec{e}(-k,\lambda), \quad (\vec{e}(k,\lambda)\vec{k}) = 0. \qquad (14)$$

Then

$$-\int (\vec{\pi})^2\,dx = \int \Big(\frac{\delta}{\delta \vec{B}(x)}\Big)^2 dx = \Big(\frac{\partial}{\partial \vec{C}}\Big)^2 +$$
$$+ \sum_{k\neq 0}\Big(\frac{1}{k^2}\,\frac{\partial^2}{\partial q_3(k)\,\partial q_3(-k)} + \sum_{\lambda=1,2}\frac{\partial^2}{\partial q_\lambda(k)\,\partial q_\lambda(-\lambda)}\Big). \qquad (15)$$

The derivative $\partial/\partial q_3(k)$ acts on the functional $\bar{\Omega}(\eta, \eta^*, \vec{B}^{tr}, \vec{C}, \gamma)$ twice: on $\gamma$ function and on $\eta$, $\eta^*$ operators (7):

$$\frac{\partial}{\partial q_3(k)} = \Big(\frac{\partial}{\partial q_3(k)}\Big) - ie\int ds\,\frac{\partial\gamma(s)}{\partial q_3(k)}\,z(s) =$$
$$= \Big(\frac{\partial}{\partial q_3(k)}\Big) - ie\int ds\, e^{iks}\, z(s)/\sqrt{V}, \qquad (16)$$

$$z(s) = \eta(s)\,\frac{\delta}{\delta\eta(s)} - \eta^*(s)\,\frac{\delta}{\partial\eta^*(s)}. \qquad (17)$$

Let us consider, along with the operator $z(s)$, the operator $\rho(s)$,

$$\rho(s) = \eta^*(s)\,\eta(s). \qquad (18)$$

One has

$$[\rho(s), \eta(y)] = -\eta(s)\,\delta(s-y),$$
$$[\rho(s), \eta^*(y)] = \eta^*(s)\,\delta(s-y),$$
$$[z(s), \eta(y)] = \eta(y)\,\delta(s-y),$$
$$[z(s), \eta^*(y)] = -\eta^*(y)\,\delta(s-y). \qquad (19)$$

Equation (19) provokes to expect

$$\rho(s) = -z(s). \qquad (20)$$

Thus we have got <u>new expression</u> for the QED Hamiltonian:

$$h = \frac{1}{2} \sum_{\substack{k \neq 0 \\ \lambda=1,2}} \left( - \frac{\partial^2}{\partial q_\lambda(k)\, \partial q_\lambda(-k)} + k^2 q_\lambda(k)\, q_\lambda(-k) \right) +$$

$$+ \int \eta^* \gamma_4 \left( \sum_1^3 \gamma_j (\partial_j - ie B_j^{tr}) + m \right) \eta(x)\, d^3x \; -$$

$$- \frac{1}{2} \left( \frac{\partial}{\partial \vec{C}} \right)^2 - ie \frac{\vec{C}}{\sqrt{V}} \int \eta^*(x)\, \gamma_4\, \vec{\gamma}\, \eta(x)\, d^3x \; - \tag{21}$$

$$- \frac{1}{2} \sum_{k \neq 0} \frac{1}{k^2} \left[ \frac{\partial^2}{\partial q_3(k)\, \partial q_3(-k)} + \right.$$

$$\left. + \frac{2ie}{\sqrt{V}} \bar{\rho}(-k) \frac{\partial}{\partial q_3(k)} - \frac{e^2}{V} \bar{\rho}(k)\, \bar{\rho}(-k) \right],$$

$$\bar{\rho}(x) = \int \rho(x)\, e^{ikx}\, d^3x . \tag{22}$$

Equation (21) essentially coincides with the standard QED Hamiltonian. It contains, however, an important new (zero mode) term,

$$-\frac{1}{2} (\partial / \partial \vec{C})^2 - e \vec{C} \int \eta^*(x)\, \vec{\alpha}\, \eta(x)\, d^3x / \sqrt{V} .$$

This term makes the Hamiltonian (21) unbounded from below (see a subsequent paper[5].

We have got also a new derivation of the QED Hamiltonian. This derivation may be applied to the QCD.

Let us note that introduction of straight-forward cut-off in the Hamiltonian (21)

$$(\eta(x) = \frac{1}{\sqrt{V}} \sum_{|p|<L} u_{p\sigma\lambda}\, e^{ipx}\, a_{p\sigma\lambda} ,$$

$$\vec{B}^{tr}(x) = \frac{1}{\sqrt{V}} \sum_{\substack{|k| < L , \\ k \neq 0,\ \lambda=1,2}} q_\lambda(k)\, \vec{e}_\lambda(k)\, e^{ikx} \;)$$

does not contradict the Gauge invariance (see the remarks after eqs.(6) and (7)).

2. Now let me comment on the consideration of item 1. As a matter of fact, one can find the derivation of the QED Hamiltonian (mainly) in old textbooks (by Heitler [1], Schiff [2], Dirac [3], Wentzel[4]). All these derivations use, instead of eq. (6), the representation

$$\vec{B}(x) = \text{grad}\, \gamma + \text{rot}\, \vec{\delta} , \tag{6a}$$

$\text{rot}\,\vec{\delta} = \vec{B}^{tr}$, of the vector field $\vec{B}(x)$, thus omitting the term $\vec{C}/\sqrt{V}$ which gives rise to new terms in the QED Hamiltonian.

This term was neglected in order to ensure the property $B(x) \to 0$ as $x \to \infty$ of the field $\vec{B}(x)$. Which of the representations, eq.(6) or eq.(6a) is correct? As much as I do understand, eq.(6) gives the standard commutation rule

$$[\, \pi_j(x) ,\, B_\ell(y) \,] = -i\, \delta(x-y)\, \delta_{j\ell} , \tag{4b}$$

while eq.(6a) gives a wrong commutator

$$[\, \pi_j(x) ,\, B_\ell(y) \,] = -i\, \delta_{j\ell}\, [\, \delta(x-y) - 1/V \,] , \tag{4c}$$

for eq.(6a) implies the decomposition

$$B_j(y) = V^{-1/2} \sum_{k \neq 0} q_j(k)\, e^{ikx} .$$

2.1. Thus I dare think that all past derivations of the QED Hamiltonian possess an essential mistake.

2.2. Transformation (2) enabled me to avoid using a supplementary condition for elimination of the field $A_0(x, t)$. I am indebted to Dr. V.L.Ljuboshits who has explained me that transformation (2) is connected with the De Witt gauge.

2.3. I do understand that my proof of eq. (20) is not exhaustive.

2.4. Let me stress once again that it seems to be promising to apply the method of the present work to the problem of the QCD Hamiltonian derivation.

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REFERENCES

1. Heitler W. - Quantum Theory of Radiation, Oxford, Clarendon Press, 1955.
2. Schiff L.I. - Quantum Mechanics, Mc.Graw-Hill Company INC., New-York - Toronto - London, 1955.
3. Dirac P.A.M. - The Principles of Quantum Mechanics, Oxford, Clarendon Press, 1958.
4. Wentzel G. - Quantum Theory of Fields, Interscience Publ. Inc., New York, 1949.
5. Zastavenko L.G. - JINR E2-90-281, Dubna, 1990.

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